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BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

A Historical and Critical Review
of Mathematical Science

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PARTICIPATION IN THE PUBLICATION OF THE AMERICAN JOURNAL OF MATHEMATICS

An event which may prove to be of the highest importance is the consummation of the agreement entered into by the American Mathematical Society and the Johns Hopkins University, by which, beginning with 1927, they share in the conduct of the American Journal of Mathematics. The Society will participate in the editorial responsibility for the Journal and will contribute very substantially to its financial support. The size will be practically doubled.

The editorial board for 1927 will consist of Professors G. D. Birkhoff, A. B. Coble, Abraham Cohen, G. C. Evans, and Frank Morley (chairman). It has been agreed that the character of the Journal shall continue as in the past, the editors being given a free hand to develop the publication along traditional lines. Members of the Society are invited to contribute of their important papers. With the renewed vigor possible under the cooperation arranged, the Journal and Transactions should stand together in the front rank of the world's mathematical periodicals.

With the publication of the Colloquium Lectures, the Bulletin, and the Transactions, and the new venture in supporting the Journal, the Society expects through its efforts to provide for some twenty-two hundred pages annually. This is possible only because of the Endowment Fund, the generous support of the Sustaining Members, and the splendid assistance of the General Education Board, which has entrusted the National Academy with a sum of money for the support of scientific publication.

R. G. D. RICHARDSON,
Secretary.

THE FIFTIETH REGULAR MEETING OF THE SAN FRANCISCO SECTION

The fiftieth regular meeting of the San Francisco Section of the Society was held at the University of California on Saturday, October 30, 1926. Professor Cajori acted as temporary chairman till the arrival of the chairman, Professor Blichfeldt. The total attendance was forty, including the following twenty-two members of the Society:

Alderton, Allardice, A. B. D. Andrews, Bernstein, Blichfeldt, Buck, Cajori, Corbin, R. L. Green, M. W. Haskell, E. R. Hedrick, Hotelling, Frank Irwin, D. H. Lehmer, D. N. Lehmer, Sophia Levy, W. A. Manning, J. H. McDonald, F. R. Morris, Pauline Sperry, A. R. Williams, Wong.

The following officers were elected for the year: Chairman, Professor R. E. Allardice; Secretary, Professor B. A. Bernstein; Program Committee, Professors E. T. Bell, Daniel Buchanan, Frank Irwin, B. A. Bernstein (ex officio).

The Section accepted the invitation of President Klinck of the University of British Columbia that the 1927 Summer meeting be held in June at the University of British Columbia. It was decided to hold the next fall meeting on October 29, 1927, at the University of California.

Titles and abstracts of papers read at the meeting follow. Professor McDonald's paper was delivered at the request of the Program Committee. The papers of Professors Bell and Wear were read by title.

1. Professor J. H. McDonald: *The relations between the algebraic and transcendental solution of certain problems.*

The problems referred to belong to the integral calculus. The determination of integrals requires us to decide whether two elliptic integrals are transformable into each other, whether a given radical admits an Abel-pseudo-elliptic integral, whether a given algebraic relation has an associated elliptic integral. These problems admit enunciations which are arithmetic, algebraic, or function-theoretic. It is the object of the colloquium to consider the different aspects of these problems and to present contributions to their solution.

2. Professor E. T. Bell: *Reduction formulas for the number of representations of integers in certain quadratic forms.*

The algebraic method of the paper enables us to pass from formulas for numbers of representations in a quadratic form in $r \geqq 2$ indeterminates to the like for rs indeterminates, $s > 1$. It is applied to obtain complete proofs of the results stated by Liouville (Journal de Mathématiques, vol. 10, pp. 43–49; vol. 11, p. 211).

3. Professor E. T. Bell: *A diophantine automorphism.*

This paper appears in full in the present issue of this Bulletin.

4. Professor B. A. Bernstein: *Note on the dual of a boolean expression.*

The author obtains a new rule for writing down the dual of an expression in boolean algebras, and he makes some application of this rule.

5. Professor Florian Cajori: *Frederick the Great on mathematics and mathematicians.*

The author describes Frederick the Great's relations to Euler, Maupertuis, Lagrange, D'Alembert and Lambert, and points out that, despite his total lack of appreciation of mathematics, Frederick gave a great stimulus to the progress of this science through the patronage extended to mathematicians at the Berlin academy.

6. Professor Florian Cajori: *Madame du Châtelet on fluxions.*

The author gives Madame du Châtelet's interpretation of Newton's lemmas in the *Principia* relating to the concept of a limit.

7. Professor Florian Cajori: *Circuelo on the names "arithmetical" and "geometrical" proportion and progression.*

The author describes and comments on Circuelo's explanation of the choice of the adjectives "arithmetical" and "geometrical" in naming proportions and progressions.

8. Professor E. R. Hedrick: *A necessary and sufficient condition for the Borel theorem in a general type of space.*

In this paper it is shown that a necessary and sufficient condition for the validity of the Borel theorem in very general types of spaces is that if any family F of closed sets has no point in common, then a finite number of sets of F exist which have no point in common. Another form of this statement is useful. If in any family F of closed sets, every finite subfamily has a point in common, the entire family has a point in common. The relation of the Borel theorem to other fundamental ideas, such as the Dedekind cut, which is well known, is thrown into clearer light through the present formulation, which is simpler and more symmetric, in some respects, than the usual statements of the Borel theorem.

9. Dr. Harold Hotelling: *An application of analysis situs to statistics.*

The frequency distribution of a correlation coefficient between two sets of numbers derived by manipulation from a common body of data is important in some kinds of statistical work. These frequency distributions have however been studied only experimentally, if at all, and little is known about them. The author shows that their exact determination would be equivalent to solving certain metrical problems concerning special transformations of a hypersphere into itself. J. W. Alexander's theorems on invariant points reveal properties of the distributions.

10. Professor D. N. Lehmer: *Note on a theorem on factorization of numbers.*

This paper appears in full in the present issue of this Bulletin.

11. Professor L. E. Wear: *A variation in the idea of a line integral.*

Under suitable conditions the line integral $\int_L f(x, y)dx$ is $\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta x_i$, where n is the number of intervals into which the arc of the path has been divided and ξ_i, η_i is a point on the path within the ith interval. The perpendiculars to Ox from the ends of the intervals intercept the lengths Δx_i on the axis of x; they intercept arcs Δs_i on some suitable curve, C. The integral $\int_L f(x, y)ds$ will be defined as $\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta s_i$, where the values $f(\xi_i, \eta_i)$ are to be taken on the path L, and Δs_i are arcs of the curve C. By using the equations of L and C the integral is reduced to an ordinary definite integral. The ideas can be extended to the general case of $\int_L Mdx+Ndy$ and the analog of Green's Theorem can be written down. Similarly for higher spaces.

12. Professor W. A. Manning: *On simply transitive primitive groups.*

Three of the theorems in Rietz's memoir on *Primitive groups of odd order* (American Journal, vol. 26 (1904), p. 1) have to do with simply transitive primitive groups in which all the transitive constituents of a subgroup leaving one letter fixed are primitive. The present author shows that they are all special cases of the following theorem: *If all the transitive constituents of G_1 (the subgroup that leaves fixed one letter of a simply transitive primitive permutation-group) are primitive, G_1 is a simple isomorphism between its transitive constituents.* The author also proves the following theorem: *If one and only one of the transitive constituents of G_1 (defined above) is an imprimitive group, the order of G_1 is the same as that of its imprimitive constituent.*

B. A. BERNSTEIN,
Secretary of the Section.

THE OCTOBER MEETING IN NEW YORK

The two hundred fifty-first regular meeting of the Society was held at Columbia University, on Saturday, October 30, 1926, extending through the usual morning and afternoon sessions. The attendance included the following fifty-five members of the Society:

Alexander, W. L. Ayres, A. A. Bennett, Birkhoff, Bower, B. H. Camp, Carrié, Abraham Cohen, Doermann, Eisenhart, Fiske, Fite, Fort, Gehman, Gronwall, Hedlund, Himwich, Huber, Dunham Jackson, Joffe, R. A. Johnson, O. D. Kellogg, A. L. Kimball, B. F. Kimball, Koopman, Kormes, Lamson, Langman, Lefschetz, Littauer, Michal, Frank Morley, Richard Morris, Mullins, Pepper, Pfeiffer, Pierpont, Raudenbush, Raynor, Reddick, R. G. D. Richardson, Ritt, Seely, Siceloff, Smail, Virgil Snyder, M. H. Stone, Teach, T. Y. Thomas, Veblen, Wedderburn, Weida, Anna Pell Wheeler, Whittemore, W. A. Wilson.

The Secretary announced the election of the following eighteen persons to membership in the Society:

Miss Julia Wells Bower, Vassar College;
Mr. Edward John Braun, Wayland Academy;
Professor Loren G. Butler, Albany College;
Dr. Ralph Douglas Doner, Purdue University;
Mr. Herbert Pulse Evans, University of Wisconsin;
Mr. Joel S. Georges, University of Chicago;
Professor Roy French Graesser, University of Arizona;
Mr. Joshua V. Longenecker, Farmers and Bankers Life Insurance Company, Wichita;
Mr. Marcus Evans Mullings, University of Cincinnati;
Mr. Lôuis John Paradiso, Lehigh University;
Mr. Herbert Armond Perkins, Hampton Institute;
Mr. James Ellis Powell, Michigan Agricultural College;
Mr. Paul Klein Rees, Texas Technological College;
Mr. Warren Alonzo Rees, Texas Agricultural and Mechanical College;
Mr. Davis Payne Richardson, University of Chicago;
Mr. James Holmes Sturdivant, University of Texas;
Mr. Cecil Francis Whitaker, Macon, Ga.;
Professor Herman Zanstra, University of Washington.

Eleven applications for membership in the Society were received.

The Board of Trustees at their meeting revised the budget and transacted other financial business.

At the meeting of the Council it was announced that the committee on the Josiah Willard Gibbs Lecture had selected Dr. H. B. Williams, Professor of Physiology, College of Physicians and Surgeons, Columbia University to give the fourth Gibbs Lecture in Philadelphia at the time of the Annual Meeting and that his subject is *Mathematics and the Biological Sciences*. President Birkhoff appointed as committee on arrangements for this lecture, Professors F. H. Safford (chairman) and J. A. Miller and Miss Marion Reilly.

It was announced that President Birkhoff had appointed Vice-President Evans to represent the Society at the semi-centennial celebration of the Agricultural and Mechanical College of Texas on October 14–16, 1926.

Professor A. B. Coble was invited to give a Colloquium in connection with the Summer Meeting in Amherst in 1928.

A list of nominations for trustees, officers, and other members of the Council was adopted unanimously and ordered printed on the ballot.

President Birkhoff presided at the morning session, relieved in the afternoon by Professor O. D. Kellogg.

Titles and abstracts of the papers read at this meeting follow below. The papers of Douglas, Ettlinger, Franklin, Graves, Hollcroft, Moore, Rice, Shaub, Weisner, Whyburn, and Wilson were read by title. Professor Drach was introduced by Professor Fite, Professor LaMer by Dr. Gronwall, and Mr. Serghiesco by Professor Ritt.

1. Dr. B. O. Koopman: *On analytic solutions of differential equations in the neighborhood of non-analytic singular points.*

This paper will appear in an early issue of this Bulletin.

2. Professor J. F. Ritt: *A factorization theory for functions* $\sum_{i=1}^{n} a_i e^{\alpha_i x}$.

When, in the expression $a_0 e^{\alpha_0 x} + \cdots + a_n e^{\alpha_n x}$, we allow n to assume all positive integral values, and the a's and α's all constant values, we obtain a class of functions which is closed with respect to multiplication. The problem thus arises of determining all representations of a given function

of this class as a product of functions of this class. Simplicity of results is obtained, without loss of generality, by the assumption that $\alpha_0 = 0$, $a_0 = 1$. A function in which the α's all have rational ratios to one another is called simple. A function which is not the product of two functions is called *irreducible*. It is proved that every function can be expressed, in one and only one way, as a product of simple functions and of irreducible functions, the α's of every simple function having irrational ratios to those of every other.

3. Professor J. F. Ritt: *Simplification of Liouville's method in the theory of elementary functions.*

Liouville, about a century ago, studied the possibility of solving differential equations in finite terms. He was followed in this work by certain Scandinavian and Russian mathematicians. The present paper gives a much more powerful method than that of Liouville for handling questions of expressibility in finite terms. With the new method, the proofs of results already in the literature can be shortened from fifty to ninety percent. An illustration of the application of the method is contained in the Comptes Rendus for August 2, 1926.

4. Dr. B. F. Kimball: *Geodesics on a toroid.*

The surface studied is called a toroid, as it resembles a torus; it is, however, more general. The geodesics on this surface are described and classified. The conjugate points on each class of geodesics are also discussed, and some space is given to the description of the envelopes of the geodesics. The envelopes are investigated by calculating the second variation and then applying the theory of Lindeberg (Mathematische Annalen, vol. 59 (1904), p. 321). The method would seem to be a productive one.

5. Mr. S. Serghiesco: *Interpretation of the characteristic (in the Kronecker-Picard sense) of a particular system of four functions.*

Let $F_1(x, y, z)$, $F_2(x, y, z)$, $F_3(x, y, z)$, $F_4(x, y, z)$ be a system of four given continuous functions, $F_4(x, y, z) = 0$ representing a certain closed surface S. It is known that the surface integral $I = (1/4\pi)\iint_S A\,dydz + B\,dzdx + C\,dxdy$, where A, B, C are special functions of $(F_1 \cdots, F_{1x}', \cdots, F_{1y}' \cdots, F_{1z}' \cdots)$ is defined as the characteristic of F_1, F_2, F_3, and F_4. It is shown in this paper that when the four given functions are $F(x, y, z)$, $F_x'(x, y, z)$, $F_y'(x, y, z)$, $F_z'(x, y, x)$, the characteristic of these functions represents the total curvature, divided by 4π, of the whole closed surface S: $F(x, y, z) = 0$. A geometrical verification is given through theorems of Gauss and definitions in differential geometry.

6. Dr. H. M. Gehman: *Concerning homogeneous plane continua.*

Mazurkiewicz has proved (Fundamenta Mathematicae, vol. 5 (1924), p. 137) that the only homogeneous plane continuous curves that exist

are the simple closed curve, the open curve, and the plane. It is not known whether there exist plane continua which are not continuous curves. The following theorems give some information concerning the nature of such continua if they do exist. (1) If a homogeneous plane continuum contains the interior of any circle, it contains all points of the plane. (2) If one point of a homogeneous plane continuum M is a cut-point, then M is an open curve. (3) A homogeneous plane continuum which is an irreducible continuum between two points is indecomposable.

7. Mr. W. L. Ayres: *A new characterization of plane continuous curves.*

This paper will appear in full in an early issue of this Bulletin.

8. Mr. V. B. Teach: *On the properties of fields for the Lagrange problem in parametric form.*

For the problem of Lagrange in the calculus of variations, when stated in the parametric form in the usual manner, a field analogous to that of Mayer for the problem in non-parametric form is defined. Theorems analogous to those on the structure of a field for the problem in non-parametric form are then derived, and the construction of fields and some properties of transversal surfaces discussed.

9. Dr. Jesse Douglas (National Research Fellow): *The transversality relative to a surface* $\int F(x, y, z, y', z')dx = \text{minimum}$.

This paper has appeared in the November-December number of this Bulletin.

10. Dr. Jesse Douglas (National Research Fellow): *A characteristic property of minimal surfaces.*

This paper has appeared in the November-December number of this Bulletin.

11. Professor R. L. Moore: *Concerning paths that do not disconnect a given continuous curve.*

It is shown that if, in a plane S, M is a bounded continuous curve, A and B are points of $S-M$ which do not lie in the same complementary domain of M, and K is the point set obtained by adding together all simple closed curves which lie in M and separate A from B, then (1) K is closed, and (2) there exists a simple continuous arc from A to B which does not disconnect M, that is to say, there exists an arc AB such that $M-(AB)\cdot M$ is connected.

12. Professor W. A. Wilson: *On the separation of a plane by two irreducible continua.*

This paper is a generalization of A. Rosenthal's results. It is shown that the conclusion of his principal theorem is valid if the two continua C_1 and

C_2 have in common two point sets α and β, where $\alpha \cdot \beta = 0$, if each of the continua C_1 and C_2 is irreducible between any point a of α and any point b of β, and both C_1 and C_2 are decomposable or one of them is indecomposable and the other is not the union of two indecomposable continua. It is also shown that the frontier of each secondary region (Nebengebiet) is a part of some oscillatory set of C_1 or C_2 or is a part of the union of the oscillatory sets of C_1 and C_2 about a or b; consequently, such a frontier is either a continuum of condensation of C_1+C_2, an indecomposable continuum, or the union of two indecomposable continua.

13. Professor H. J. Ettlinger: *Note on continuity in several variables.*

This paper appears in full in the present issue of this Bulletin.

14. Professor L. M. Graves (National Research Fellow): *On the existence of the absolute minimum in space problems of the calculus of variations.*

Tonelli in his recent *Fondamenti di Calcolo delle Variazioni* has developed new and powerful methods for treating the problem of the absolute minimum. The present paper applies these methods, with some necessary alterations, to the ordinary or non-parametric integrals $I_C = \int f(x, y, y')dx$ in space of $k+1$ dimensions. The classes of curves in which the minimum is sought are those called "complete" by Tonelli, and are so general as to include cases of fixed or variable end points, and problems of Lagrange with finite side conditions. A case of the latter would arise from application of Hamilton's principle to problems in dynamics with holonomic constraints.

15. Mr. W. M. Whyburn: *The Green's function for systems of differential equations whose coefficients contain a parameter.*

This paper treats a system of n differential equations together with n linear boundary conditions connecting the values of the functions at two points of the interval of definition. The coefficients are continuous functions of a parameter and summable functions of the independent variable, and finally are uniformly bounded. A theorem due to Bôcher (this Bulletin, vol. 20 (1914), p. 1) on the reduction of the order of compatibility of a differential system is carried over, and this extended theorem is used to extend the author's previous results (Annals of Mathematics, (2), vol. 26 (1924), pp. 125–130) to more general systems. A matrix of $(n)^2$ Green's functions (Bounitzky, Journal de Mathématiques, (6), vol. 5 (1909), pp. 65–125) is set up, and the properties of these functions are discussed.

16. Dr. Louis Weisner (National Research Fellow): *A theorem concerning direct products.*

This paper appears in full in the present issue of this Bulletin.

17. Mr. L. H. Rice: *Compounds of Cayley products of determinants of higher class.*

Continuing the general subject treated in previous papers (Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 5, Nos. 1 and 4) which concerned both ordinary determinants and those of higher class, the writer shows that the mth compound of a Cayley product of two determinants of any classes and any signancies is the like Cayley product of the mth compounds of the factor determinants, and, as a corollary, that the same is true of the adjoint of a Cayley product. The paper will appear in volume 6 of the same Journal.

18. Professor T. R. Hollcroft: *Singularities of the Hessian.*

This paper appears in full in the present issue of this Bulletin.

19. Mr. H. C. Shaub: *Rational involutorial transformations in S_4 which leave invariant ∞^4 quadric varieties.*

By considering the (2, 1) correspondence between two spaces (x) and (x') of four dimensions for which a hyperplane in (x') has for image a quadric variety in (x), the author reduces the involutorial transformations which leave invariant ∞^2 quadratic varieties to four distinct types.

20. Professor Philip Franklin: *The classification of quadrics in euclidean three- and n-space, by means of covariants.*

Professor MacDuffee has recently given (American Mathematical Monthly, vol. 33, p. 243) the first complete classification of conics in the euclidean plane in terms of invariants and covariants. He uses the methods of the Lie theory. In this paper we give a complete classification of quadrics in n-space in terms of covariants. The system of covariants here used forms a Lie complete system, though our method of obtaining them is purely algebraic. The geometric significance of our results is also indicated.

21. Professor Philip Franklin: *The Simson lines of a triangle and the three-cusped hypocycloid.*

Steiner, in 1856, gave a discussion of the curve enveloped by the Simson lines of a triangle, and deduced many of its properties. He states (Werke, vol. 2, p. 642) that the curve is a three-cusped hypocycloid. While numerous analytical proofs of this have been given, the writer knows of no simple, direct synthetic proof in the literature. In this note such a proof is given. Certain related theorems are also obtained.

22. Professor G. D. Birkhoff: *On one-to-one transformations of surfaces.*

This paper deals with the structure of one-to-one analytic transformations T that do not possess an invariant area integral. The basis of the

treatment consists in the use of the author's extended form of the last geometric theorem of Poincaré, and of certain notions applicable to such transformations which are contained in a paper about to appear in the Göttinger Nachrichten.

23. Professor James Pierpont: *Classification of quadric surfaces in hyperbolic space.*

J. L. Coolidge (Transactions of this Society, vol. 4 (1903)) and T. J. Bromwich (the same Transactions, vol. 6 (1905)) have shown how to classify quadric surfaces in hyperbolic space. Coolidge's method rests on the discussion by Clebsch and Lindemann of the relative position of two quadric surfaces, while Bromwich's makes use of elementary divisors. Both methods are very neat, but require a considerable knowledge of geometry or algebra. A more elementary method has been employed by P. D. Schwartz to classify conics (see abstract in this Bulletin, vol.32 (1926), p. 315). An extension of these principles to space leads to a classification of quadric surfaces.

24. Dr. C. M. Huber: *On complete systems of irrational invariants of associated point sets.*

The principal object of this paper is the determination of complete systems of invariants for sets of n points P_n^1 on a line. For $n=2p+2$, it is shown that all invariant products of the differences of the roots of the binary $(2p+2)$-ic are expressible in terms of elementary products linear in each root. The invariants for a set Q_{2p+2}^p defined by P_{2p+2}^1 are expressed in terms of the linear invariants for P_{2p+2}^1. For n odd it is conjectured that a complete system is made up of cyclic invariants. This is proved for n equal to 5 and 7, and a method is given for testing the validity of the conjectured theorem for any particular value of n. A mapping problem in connection with the cyclic invariants for P_5^1 and P_7^1 is discussed.

25. Professor Solomon Lefschetz: *Intersections and complete transformations of complexes and manifolds.* Second paper.

In this paper the author extends the results of his paper in the January, 1926, number of the Transactions of this Society to manifolds with a boundary when the topological behavior at all points of the latter is the same. It is found necessary to restrict somewhat the transformations considered. The results are surprisingly parallel to those for the no-boundary case.

26. Dr. M. H. Stone: *Expansions associated with a regular differential system of the second order.*

Associated with a differential system $u''+(\rho^2+g)u=0$, $0\leqq x\leqq 1$, $\alpha_i u^{(k_i)}(0)+\beta_i u^{(k_i)}(1)+\cdots=0$, $i=1, 2$, where g is a Lebesgue integral,

and the boundary conditions are in normal form and regular, is a formal series expansion for any function integrable Lebesgue or Denjoy. The series for the same function in connection with two such systems are said to be equivalent on an interval (a, b) when the difference of partial sums of the two series is $o(1)$ uniformly on (a, b). It is shown here that a necessary and sufficient condition for equivalence on (0, 1) of the series for an arbitrary summable function is that the boundary conditions of the two systems satisfy one of the following three relations: $k_1=\bar{k}_1=k_2=\bar{k}_2=1$; $k_1=\bar{k}_1=k_2=\bar{k}_2=0$; $k_1=\bar{k}_1=1$, $k_2=\bar{k}_2=0$, $\alpha_1\bar{\beta}_1-\bar{\alpha}_1\beta_1=0$, $\alpha_2\bar{\beta}_2-\bar{\alpha}_2\beta_2=0$. This condition is also shown necessary and sufficient for the equivalence on an interval (a, b) completely interior to (0, 1) of the two series formed for an arbitrary function integrable in the sense of Denjoy. Finally, the first order Cesàro mean of the series of any function integrable in the sense of Denjoy is examined.

27. Dr. T. H. Gronwall and Professor V. K. LaMer: *The variation of the dielectric constant in the Debye-Hückel theory.*

In this paper a method is developed for determining the ion sizes and the variation of the dielectric constant with the concentration, from measurements of vapor pressure and freezing-point lowering in a solution of a strong electrolyte.

28. Dr. T. H. Gronwall: *A diophantine equation connected with the hydrogen spectrum.*

It is shown that in the Rydberg-Ritz formula for the wave length of a line in the hydrogen spectrum, the same wave length may correspond to two different sets of quantum numbers.

29. Professor Jules Drach: *Determination of Liouville's elements of length with algebraic integrals for the equation of geodesics.*

If $ds^2=4\lambda(u, v)\,du\,dv$ is an element of length in symmetric coordinates, the determination of the geodesics requires integration of the partial differential equation $pq=\lambda$, where $p=\partial z/\partial u$, $q=\partial z/\partial v$. In Liouville's case, where $\lambda=F(\alpha)-G(\beta)$, with $\alpha=u-v$, $\beta=u+v$, we know the quadratic integral $\phi=p^2+q^2+2pq(F+G)/(F-G)$; another integral, giving the general geodesic line, is ψ defined by $d\psi=d\alpha/(\phi+4F)^{1/2}+d\beta/(\phi+4G)^{1/2}$ and expressed in ϕ, α, β. The author wishes to determine all forms of F and G for which another integral f, rational in p and q, exists. In these cases a certain function of ψ is an algebraic function of ψ and f. For simplicity it is supposed that f is a polynomial in p, q.

R. G. D. Richardson,
Secretary.

ON THE METRIZATION PROBLEM AND RELATED PROBLEMS IN THE THEORY OF ABSTRACT SETS*

BY E. W. CHITTENDEN

1. *Topological Space.* In the theory of abstract sets we assume that we are given an arbitrary aggregate P and a relation between subsets of P which corresponds to the relation between a set and its derived set in the classical theory of sets of points.† That is, the mathematical concept abstract set in its current sense includes the notion limit point or point of accumulation. The introduction of limit points permits the definition of continuous 1-1 correspondence or homeomorphy. The study of such correspondences, particularly of invariants under homeomorphic transformations, constitutes the science of topology or analysis situs.‡ It seems proper therefore to speak of an abstract set as a topological space.§ Throughout this paper, the term *topological space* or *abstract set* refers to any system of the form (P, K) composed of an aggregate P and a relation of the form EKE' between the subsets E, E' of P which is subject to the condition, for every subset E of the aggregate P there is a unique set E' in the relation K to E. That is, the relation K defines a single-valued set-valued function on the class U of all subsets of the aggregate P.‖

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* Presented to the Society by invitation of the program committee, at the Summer Meeting, Columbus, Ohio, September 8, 1926.

† See M. Fréchet, *Esquisse d'une théorie des ensembles abstraits*, Sir Asutosh Mookerjee's Commemoration volumes, II, p. 360, The Baptist Mission Press, Calcutta, 1922; *Sur les ensembles abstraits*, Annales de l'École Normale, vol. 38 (1921), p. 341ff.

‡ See H. Tietze, *Beiträge zur allgemeinen Topologie*, I., Mathematische Annalen, vol. 88 (1923), p. 290.

§ This terminology is suggested by Fréchet. See Comptes Rendus, vol. 180 (1925), p. 419.

‖ These functions are studied in detail in an unpublished article by the writer.

2. *The Metrization Problem.* The problem is to state in terms of the concepts point, and point of accumulation the conditions that a topological space be metric.

A metric space* is any topological space in which the points of accumulation are defined or definable in terms of a function (p, q) called the distance between the points p and q and satisfying the following conditions.

(0) *The distance* (p, q) *is a definite real number for every pair of points* p, q.

(1) *Two points are coincident if and only if their distance is zero.*

(2) *For any three points* p, q, r,

$$(p,q) \leqq (p,r) + (q,r).$$

It follows readily from these conditions that the distance (p, q) is non-negative, and that it is symmetric in p and q, $(p, q) = (q, p)$.† In a metric space a point p is a point of accumulation of a set E provided its distance from a variable point of E which is distinct from p has the lower bound zero.

The following illustrations convey some notion of the scope of the concept metric space. If the aggregate P denotes the linear continuum of all real numbers and $(p, q) = |p-q|$, the resulting space is metric. Similarly euclidean space is also metric. The Hilbert‡ space of infinitely many dimensions in which the coordinates $x_1, x_2, x_3, \cdots, x_n, \cdots$ of each point are subject to the condition that the sum of their squares be a convergent series is a metric space in which distance is defined by the formula

$$(p,q) = [(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2 + \cdots]^{1/2},$$

* F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914, p. 211. The definition of distance is due to Fréchet, *Sur quelques points du calcul fonctionnel*, Rendiconti di Palermo, vol. 22 (1906).

† A. Lindenbaum, *L'espace métrique*, Fundamenta Mathematicae, vol. 8 (1926), pp. 209–222.

‡D. Hilbert, Göttinger Nachrichten, vol. 8 (1906). Hausdorff, loc. cit., p. 287.

in which the x_n, y_n are the coordinates of p and q, respectively.

If we take for our aggregate P the class of all functions $p=f(x)$, on an interval $a \leqq x \leqq b$, whose squares are summable, and define distance by the formula

$$(p,q) = \left[\int_a^b [f(x) - g(x)]^2 \, dx\right]^{1/2},$$

we obtain a metric space. It is necessary to make the convention that two points coincide if the corresponding functions differ only at a set of points of measure zero. The relation $p=Lp_n$ corresponds to convergence in the mean for the sequence of functions $p_n=f_n(x)$.

3. *Hausdorff Spaces.* A remarkable and important class of topological spaces has been defined by F. Hausdorff.* In a Hausdorff space the points of accumulation are defined in terms of a family of neighborhoods U conditioned by the following four postulates.

(A) *To every point p there corresponds at least one neighborhood U, and each neighborhood of p contains p.*

(B) *There is a neighborhood of a point p common to every two neighborhoods of p.*

(C) *If q is any element of a neighborhood U of a point p, then U contains all the points of a neighborhood of q.*

(D) *If p and q are distinct points, there exist neighborhoods of p and q, respectively, which have no common elements.*

The topological spaces of Hausdorff are evidently included among the classes (V) of Fréchet† in which the postulated family of neighborhoods is subject only to the condition (A). In a class (V) a point p is a point of accumulation of a set E in case every neighborhood of p contains a point of E distinct from p.

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* Loc. cit., p. 212.

† *Esquisse*, p. 346; Bulletin des Sciences Mathématiques, (2), vol. 42 (1918), pp. 1–19.

It is easy to show that every metric space is a Hausdorff space. Let $S(p, a)$ denote the generalized sphere composed of all the points q of a metric space which satisfy the inequality $(p, q) < a$. The family of all such spheres of center p and radius a has the four properties of Hausdorff and evidently defines the same points of accumulation as the distance (p, q).

The following set of properties form a necessary and sufficient condition that a topological space be a Hausdorff space.*

(I) $(A+B)' = A' + B'$.

(II) *A set containing but a finite number of points has no point of accumulation.*

(III) *The derived set of every set is closed.*

(IV) *If p and q are any two distinct points there exist open sets U and V which are disjoined and contain p and q respectively.*

To complete the solution of the metrization problem we need only add the conditions that a Hausdorff space be metric.

4. *Existence of Non-Constant Continuous Functions.* The metrization problem is included in another problem proposed by Fréchet in correspondence with Paul Urysohn and with me. It is evident that the distance (p, q) of two points p and q is a continuous function of its arguments, and is not constant in a space of two or more points. Thus the metrization problem is related to the more general problem, under what conditions does a topological space admit the existence of a non-constant continuous function.† The topological conditions for the existence of such functions in a Hausdorff space have been dis-

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* *Esquisse*, p. 367.

† The definition of continuous function for general topological space is given by Fréchet in the following form: *A point p of space is interior to a set I if it belongs to I and is not a point of accumulation of any subset of the of the complement of I, $P-I$. A function $f=f(p)$ is continuous at a point p if the oscillation of the function f on the sets I to which p is interior has the lower bound zero. Esquisse*, p. 363.

covered by Urysohn.* I have recently succeeded in formulating these conditions for topological spaces in general.

The following form of the question regarding the existence of non-constant continuous functions is of particular importance in the present discussion. Characterize those spaces of Hausdorff in which it is possible to define for every two disjoined closed sets A and B a continuous function $f(p)$ which is equal to zero on A, one on B, and satisfies the inequality

$$0 \leqq f(p) \leqq 1$$

everywhere else, that is, on the set $C = P - A - B$.

We shall show that it is both necessary and sufficient for the space to be *normal*. A space is normal provided every pair of closed disjoined sets A and B is separated by open sets, that is, that there exist open sets U, V which are disjoined and include A and B respectively.

Since the condition is evidently necessary we proceed to the proof of its sufficiency. Let P be a normal Hausdorff space and let A and B be any two disjoined subsets of P. From the hypothesis of normality there exist two open sets U_1, V_1 such that

$$A = U_1, \qquad B = V_1, \qquad U_1 V_1 = 0.$$

The set $P - U_1$ is closed and includes V_1. It follows that there exist disjoined open sets U_0, V_0 such that

$$A \leqq U_0, \qquad P - U_1 \leqq V_0.$$

Therefore†

$$A \leqq U_0 \leqq U_0^0 \leqq U_1, \qquad B \leqq P - U_1.$$

Since U_0^0 and $P - U_1$ are disjoined closed sets there is an open set $U_{1/2}$ for which

$$A \leqq U_0^0 \leqq U_{1/2} \leqq U_{1/2}^0 \leqq U_1.$$

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* *Über die Mächtigkeit der zusammenhängenden Mengen*, Mathematische Annalen, vol. 94 (1925), p. 290.

† $U^0 = U + U'$.

It follows readily that there exists for any positive integer n a series of the form

$$A \leqq U_0^0 \leqq U_{1/2^n} \leqq U^0_{1/2^n} \leqq \cdots \leqq U_{m/2^n} \leqq U^0_{m/2^n} \leqq \cdots \leqq U_1$$

where $B = P - U_1$.

If $x = Lr_n$, where $r_n = m_n/2^n$, and $0 \leqq r_n \leqq r_{n+1} \leqq x \leqq 1$, we set

$$U_x = \sum_{n=1}^{\infty} U_{r_n}.$$

By definition $U_x \leqq U_{x'}$ if $x < x'$. Furthermore $U_x^0 \leqq U_{x'}$. For if n and m are chosen so that $x < m/2^n < (m+1)/2^n < x'$, then

$$U_x^0 \leqq U^0_{m/2^n} \leqq U^0_{(m+1)/2^n} \leqq U_{x'}.$$

Let $L_0 = U_0$; $L_1 = P - U_1$; $L_x = U_x^0 - U_x$, $0 < x < 1$. Then the sets $L_x (0 \leqq x \leqq 1)$ are closed, and if $x \neq x'$, $L_x L_{x'} = 0$. The function $f(p)$ which is equal to x when p is a point of L_x has the required properties.

5. *Perfectly Separable Spaces.* Among the spaces which were considered by Hausdorff are those whose points of accumulation are definable in terms of an enumerable family of neighborhoods.* Such spaces are said to satisfy the second axiom of enumerability. Tychonoff and Vedenissof have called them separable spaces.† In a letter to me Fréchet calls attention to the fact that the word separable is already in use in a more general sense and suggests the term perfectly separable. The following important and remarkable theorem was discovered by Urysohn.‡

THEOREM. *A necessary and sufficient condition that a perfectly separable Hausdorff space be metric is that it be normal.*

It is easy to show that every metric space is normal.§ Let A and B be any two disjoined closed sets and let a_p denote the

* Loc. cit., p. 263.

† Bulletin des Sciences Mathématiques, vol. 50 (1926), p. 17.

‡ Mathematische Annalen, vol. 94 (1925), p. 309.

§ Tietze, loc. cit., p. 311.

lower bound of the distance (p, q) as q varies over B. Let U be the sum of the spheres $S(p, a_p/3)$ obtained by considering all the points of A. Let V denote the corresponding open set enclosing B. Then U and V are disjoined. For if r is a point common to U and V there exist points p and q such that $(p, r) < c/3$, $(q, r) < c/3$, where c is the greater of a_p, b_q. Then we have

$$(p, q) \leqq (p, r) + (r, q) < 2c/3,$$

contrary to the befininition of $a_p\, b_q$.

To show that the condition is sufficient let

$$U_1, \quad U_2, \quad U_3, \cdots, U_i, \cdots, U_j, \cdots$$

represent an enumerable set of neighborhoods determining the points of accumulation of a *normal* Hausdorff space P. For each of the enumerable family of pairs of neighborhoods U_i, U_j, such that $U_i^0 \leqq U_j$, there is a continuous function $f = f(p)$ which satisfies the conditions $f(p) = 0$ on U_i^0, $f(p) = 1$ on $P - U_j$, $0 \leqq f(p) \leqq 1$ on P. Let the set of all such functions be represented by the sequence

$$f_1, f_2, f_3, \cdots, f_n, \cdots.$$

Consider the function

$$(p, q) = \sum_{n=1}^{\infty} [f(p) - f(q)]/2^n.$$

This function is continuous because it is the sum of a uniformly convergent series of continuous functions. It is furthermore evident that $(p, p) = 0$, and that $(p, q) \leqq (p, r) + (q, r)$.

It is necessary to show that if $p \neq q$ then $(p, q) > 0$. Let U_j be a neighborhood of p which does not contain the point q. Since the sets $A = p$ and $B = P - U_j$ are closed and disjoined there is a neighborhood U_i such that $U_i^0 = U_j$. Let $f_n(p)$ be the continuous function corresponding to the two sets U_i, U_j. Since $f_n(p) = 0$ and $f_n(q) = 1$, we have $(p, q) \geqq 1/2^n$, which was to be proved.

It remains to show that the neighborhoods U_i and the distance (p, q) just defined determine the same space. This will

be the case if for each point p every neighborhood of p contains a sphere $S(p, a)$ and if every such sphere contains a neighborhood of p. Since the distance (p, q) is continuous in q it follows that a neighborhood of p, U_i, can be found on which the oscillation of the function (p, q) is less than $a/2$ for any value of a (>0). Hence the sphere $S(p, a)$ includes the neighborhood U_i.

Suppose that there is a point p and a neighborhood U_j of p such that for every positive value of a there is a point q_a of the sphere $S(p, a)$ which is exterior to U_j. As before there exists a neighborhood U_i of p such that $U_i \leqq U_j$. Let f_n be the function associated with the pair of neighborhoods U_i, U_j. Then $f_n(q) = 1$, and $(p, q) = 1/2^n$. This is impossible if $a < 1/2^n$.

6. *Axioms of Separation.* The property of normality is the third of a series of four axioms of separation which have been discussed in detail by Tietze.* Two point sets A and B are separated by open sets U, V if

$$A = U, \qquad B = V, \qquad UV = 0.$$

We consider the following four cases.

(1) *The sets A and B each contain one point only.*

(2) *The set A contains a single point, the set B is closed.*

(3) *A and B are any closed sets.*

(4) *A and B are disconnected. That is, neither set contains a point of accumulation of the other.*

The first of these axioms of separation is Axiom (D) of the set defining a Hausdorff space, and coincides with the fourth of the conditions that a topological space be a space of Hausdorff. We indicate the axiom which corresponds to case i by the symbol (IV_i). In the terminology of Paul Alexandroff and Urysohn† a space satisfying axiom (IV_2) is regular; (IV_3) is normal; (IV_4) is completely normal. Each of these axioms

* Loc cit., p. 300, etc.

† Mathematische Annalen, vol. 92 (1924), p. 263.

is stronger than its predecessor and independent of it. Metric spaces are completely normal. However there exist spaces which are completely normal but not metric.*

7. *Regular Spaces.* Is a regular perfectly separable Hausdorff space normal? This question was proposed by Urysohn† and answered in the affirmative by Tychonoff.‡ Let A and B be any disjoined closed sets and let

$$U_1, U_2, U_3, \cdots, U_m, \cdots$$

be an enumerable system of open sets defining the points of accumulation of a Hausdorff space P. For each point p of the set A there is a neighborhood U_{n_p} of p which with its derived set contains no point of B. The class of all the neighborhoods U_{n_p} determined by the points of A and the set B forms a sequence

$$V_1, V_2, V_3, \cdots, V_n, \cdots$$

of open sets whose sum includes the set A. In similar fashion we define a sequence of open sets

$$W_1, W_2, W_3, \cdots, W_n, \cdots$$

hose sum includes B, such that $W_n^0 A = 0$, $n = 1, 2, 3, \cdots$. Let $G_1 = U_1$, $H_1 = V_1 - G_1^0$, and in general,

$$G_n = U_n - \sum_{i=1}^{n-1} H_i^0, \qquad H_n = V_n - \sum_{i=1}^{n} G_i^0.$$

The sets G_n and H_n are evidently open. If we now set

$$G = \sum_{n=1}^{\infty} G_n, \qquad H = \sum_{n=1}^{\infty} H_n,$$

we can show that

$$A \leqq G, \qquad B \leqq H, \qquad GH = 0.$$

Since no point of A is contained in any set V_n, $G_n A = U_n A$, and therefore $A \leqq G$. Similarly $B \leqq H$. Suppose that there is a point p common to G and H. Then there must exist indices

* Tietze, loc. cit.

† Mathematische Annalen, vol. 94 (1925), p. 315.

‡ Mathematische Annalen, vol. 95 (1926), p. 139.

m and n for which p is common to the sets G_m, H_n. If $m \leqq n$, we have

$$G_m H_n = G_m\left(V_n - \sum_{i=1} G_i^0 \right) \leqq G_m(V_n - G_m^0) = 0,$$

a contradiction. A similar contradiction is obtained if $m > n$.

This result when combined with the theorem of the preceding section gives the fundamental theorem.

THEOREM. *A necessary and sufficient condition that a perfectly separable Hausdorff space be metric is that it be regular.*

8. *An Axiom of R. L. Moore.* The importance of the regular and perfectly separable, therefore metric, spaces in the analysis of continua is indicated by the fact that nine years before the publication of the discoveries of Urysohn, R. L. Moore assumed these properties in the first of a system of axioms for the foundations of plane analysis situs.* This axiom is furthermore of particular interest historically since it yields when slightly modified a necessary and sufficient condition that a topological space be metric and separable. The modified axiom of Moore may be stated as follows.

AXIOM (R. L. Moore). *We are given a space P in which point of accumulation is defined in terms of a family of classes of points called regions. Among the regions there exists a fundamental enumerable sequence*

$$R_1, R_2, R_3, \cdots, R_n, \cdots$$

with the following properties: (0) *for every region R there is an integer n such that R_n is a subset of R*; (1) *for every point p and integer n there is an integer n' greater than n such that $R_{n'}$ contains p*; (2) *if p and q are distinct points of a region R there is an integer m such that if n is greater than m and R_n contains p, then R_n is a subset of $R - q$.*†

* *On the foundations of plane analysis situs*, Transactions of this Society, vol. 17 (1916), pp. 131–164. The hypothesis of regularity was also made (apparently independently) by L. Vietoris, Monatshefte, vol. 31 (1921), p. 176.

† This differs from Axiom 1 (loc. cit.) with respect to the first sentence and condition (0) only.

It is easy to show that in a space satisfying this axiom the regions are open sets, and that the space is a regular and perfectly separable Hausdorff space, therefore metric and separable. The converse of this proposition is also true and is established by the following chain of propositions.

The axiom of Moore is a topological invariant. That is, if the axiom is satisfied in one of two homeomorphic spaces it is satisfied in the other. Furthermore if the axiom holds in a space P it holds in any relative subspace of P.

It is easy to show that every compact metric space admits the axiom. Since Urysohn* has shown that every separable metric space is homeomorphic with a subset of a compact domain in the Hilbert space it follows immediately that every separable metric space satisfies this axiom.†

THEOREM. *A necessary and sufficient condition that a topological space be metric and separable is that it satisfy the axiom of R. L. Moore.*‡

9. *Metrization of Compact Spaces.* It can now be shown that a compact and perfectly separable Hausdorff space is metrizable. It is sufficient to show that it is regular. We have to show that for each point p and closed set A there are open sets V, U which separate p and A. Enclose each point of a given closed set A in an open set which does not contain p and consider the system of open sets thus obtained together with the open set $P-A$. Since every compact perfectly separable space has the "any-to-finite"§ property of Borel, a finite subset of these open sets may be selected which covers the closed set A. Let

$$U_1, U_2, U_3, U_4, \cdots, U_n$$

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* Mathematische Annalen, vol. 92 (1924), p. 302.

† I have obtained a direct proof of this proposition.

‡ The fact that Axiom 1 is a sufficient condition for metrizability was inferred by R. L. Moore from the theorem of Tychonoff in §7 above.

§ The phrase was introduced by T. H. Hildebrandt in an article on *The Borel theorem and its generalizations*, this Bulletin, vol. 32 (1926), pp. 423–474.

be a finite family of open sets whose sum U includes A but not p. Then there is a neighborhood V of p which contains no point of the open set U. The proof of the following theorem of Urysohn* is now easily completed.

THEOREM. *A necessary and sufficient condition that an infinite compact Hausdorff space be metrizable is that it be perfectly separable.*

10. *The Metrization Conditions in Locally Compact Spaces.* The result just obtained has been applied by Alexandroff† to locally compact Hausdorff spaces. A space is locally compact provided there is for every point p a neighborhood V such that V^0 is compact. The following fundamental theorem may be stated.

THEOREM. *A necessary and sufficient condition that a locally compact Hausdorff space be metrizable is that the space be perfectly separable or else be the sum of a set (of arbitrary cardinal number) of disjoined domains which are perfectly separable subspaces of the given space.*

The proof that this condition is sufficient follows lines indicated previously. The essential part of the proof that the condition is necessary consists in showing that every locally compact metric space which is not perfectly separable is representable as a sum of disjoined perfectly separable spaces.

In a locally compact metric space every point p is the center of a sphere $S(p, a)$ which is compact and therefore perfectly separable. Let $G_0 = S(p_0, a_0)$ and assume that G_n is defined and perfectly separable. Then G_{n+1} is defined to be the sum of all the spheres $S(p, a)$ which contain a point of G_n. The set

$$G = G_0 + G_1 + \cdots + G_n + \cdots$$

is perfectly separable. If p_0 is replaced by any point of G the process just defined will lead to the same set G. Thus the

* Mathematische Annalen, vol. 92 (1924), pp. 275–293, and vol. 94 (1925), p. 313.

† Mathematische Annalen, vol. 92 (1924), pp. 294–301.

given space admits a decomposition into disjoined perfectly separable metric subspaces.

This theorem has the following interesting corollary.

COROLLARY. *A necessary and sufficient condition that a connected and locally compact space be metrizable is that it be perfectly separable.*

11. *Relations between Metrization and the Existence of Continuous Functions.* We have seen that the metrization problem is related to the more general problem of the existence of non-constant continuous functions and how the study of the latter problem led Urysohn to the solution of the metrization problem for perfectly separable spaces. It is therefore of interest to formulate the conditions that a space be metrizable in terms of continuous functions.

THEOREM. *A necessary and sufficient condition that a topological space which is equivalent to a class (V) of Fréchet be metrizable is that there exist a family of equally continuous functions with the following properties:*

(1) *each function of the family is defined and continuous throughout the space;*

(2) *for each point p there is a function of the family which vanishes at p and is bounded from zero on the complement of every neighborhood of p.*

To show that the condition is necessary, let P be a metric space and define the required family of functions to be the class of all functions $\phi(p) = (p, q)$, where (p, q) is the distance from p to q, the point q is held fixed and the point p allowed to vary. That this family of functions has the required properties is an immediate consequence of the conditions satisfied by the distance between two points.

It will now be shown that the condition is sufficient. The definition of distance is derived from the formula

$$(p,q) = \lim \sup \frac{|\phi(p) - \phi(q)|}{1 + |\phi(p) - \phi(q)|},$$

where all functions ϕ of the given equally continuous family are to be considered. It is evident that the function (p, q) just defined has the properties of distance. It remains to consider the relation between distance and point of accumulation.

Suppose that a point p is a point of accumulation of a set E. Because of the equal continuity of the functions ϕ there is for every positive number e a neighborhood V of p on which the oscillation of each function is less than e. Therefore the distance of the point p from the set E is zero.

Conversely, suppose that every sphere $S(p, a)$ of center p and radius a in the metric space just defined contains a point q of a set E. Let V be any neighborhood of the point p in the given space (V). By the second condition of the theorem there is a function ϕ and a number e such that $|\phi| > e$ on the set $P - V$. This implies that $(p, q) > e/2$ for all points q of $P - V$. Therefore the sphere $S(p, e/2)$ is contained in the neighborhood V. That is, V contains a point of E, and the point p is by definition a point of accumulation of E.

12. *The Equivalence of Distance and Uniformily Regular Écart.* Attempts have been made by Fréchet,* E. R. Hedrick,† A. D. Pitcher, and the writer‡ to obtain effective generalizations of the theory of metric spaces. That is, to impose hypotheses which yield substantially the same group of theorems about point sets and are less restrictive. It has however been established in each case that the conditions proposed imply that the resulting space is equivalent to a metric space.

In the theory proposed by Fréchet, distance is replaced by a function (p, q) with the properties (0), (1) of distance and the further property

(2′) *There is a function $f(e)$, approaching zero with e, such that for any three points p, q, r,*

$$(p,r) \leqq e, \qquad (q,r) \leqq e, \qquad \textit{imply} \quad (p,q) \leqq f(e).$$

* *Sur quelques points du calcul fonctionnel,* Rendiconti di Palermo, vol 30 (1906), p. 1–74.

† Transactions of this Society, vol. 12 (1911), pp. 285–294.

‡ Transactions of this Society, vol. 19 (1918), pp. 66–78.

The discovery by H. Hahn* that every space of this type which contains two or more distinct points admits continuous non-constant functions led Fréchet† to the conclusion that this class of spaces is metrizable. This conclusion was verified by Chittenden.‡

From this and other considerations Fréchet§ has in his later papers employed the term écart to refer to a non-negative single-valued real-valued function of two points. In this terminology the function (p, q) which was formerly called a voisinage becomes a uniformly regular écart.

THEOREM. *The concepts uniformly regular écart and distance are equivalent.*

Since distance is a uniformly regular écart for which $f(e) = 2e$ it is sufficient to show that a class of equally continuous functions satisfying the conditions of the theorem of §11 above may be defined in any space admitting a definition of its points of accumulation in terms of a uniformly regular écart.

If the given space P is singular the theorem is obvious. If it contains at least two points a number $a > 0$ can be chosen with the property: for every point p there is a point q such that $(p, q) > a$. This number a may be determined in the following manner. Let q', q'' be any two distinct points and let a be chosen so that $f(a) < (q', q'')$. Then one of the points q', q'' is effective as the required point q. For if $(p, q') \leqq a$, and $(p, q'') \leqq a$, then by the property (2′)

$$(q', q'') \leqq f(a),$$

contrary to the definition of a.

Suppose $a' < a$ chosen so that $f(a') < a$. Let p_0 be any fixed point and let‖

$$A = S(p_0, a'), \quad B = P - S(p_0, a), \quad C = P - A - B.$$

* Monatshefte, vol. 19 (1908), pp. 247–257.

† Rendiconti di Palermo, vol. 30 (1910), p. 22–23.

‡ Transactions of this Society, vol. 18 (1917), pp. 161–166.

§ See Bulletin des Sciences Mathématiques, (2), vol. 42 (1918), pp. 1–19.

‖ The validity of the following discussion is unimpaired in case the set C or any of its subdivisions is a null set.

The sets A^0, B^0 are disjoined. Suppose the contrary. Then there would exist points p, q, r of A, B, C, respectively, such that $(p, r) < a''$, $(q, r) < a''$, where $f(a'') < a'$. Then since $(p_0, p) < a'$, and $(p, q) \leqq f(a'') < a'$, we have $(p_0, q) \leqq a$, a contradiction.

The set C will be divided into disjoined sets C_0, C_1 according to the following rule. If $(r, A) = (r, B)$, where (r, A) denotes the distance from the point r to the set A, the point r is assigned to C_0, otherwise to C_1. The sets A^0, C_1^0 are disjoined, likewise the sets C_0^0, B^0. It is sufficient to give the proof for the first case. Let a number c be chosen so that $f(c) < a''$, and assume that there is a point r common to A^0, C_1^0. Then points p and q of A and C_1 respectively exist such that $(p, r) < c$, $(q, r) < c$. Therefore $(p, q) \leqq f(c) < a''$. From the definition of C_1 there is a point q' of B such that $(q, q') < (p, q) < a''$. Therefore $(p, q') \leqq f(a'') < a'$ contradicting the definition of the set B.

From the sets A, B, C we obtain by iteration of this method of subdivision a development of the space P of which the mth stage has the form

$$A,\ C_{00\cdots 0},\ C_{00\cdots 01},\ C_{i_1 i_2 \cdots i_m},\ \cdots,\ C_{11\cdots 1},\ B,$$

in which the indices i assume the values 0, 1 only. This sequence has the further property that there exists a number a_m (which is independent of p_0) such that the distance of any two non adjacent sets exceeds a_m. It is quite easily shown by mathematical induction that the numbers a_m may be so chosen that

$$a > a_1 > a_2 \cdots,\ La_m = 0,$$

and that

$$a > f(a_1),\ a_1 > f(a_2),\ \cdots.$$

For each point p of C there is a unique set $C_{i_1 i_2 \cdots i_m}$ of stage m of which it is an element. Consequently each point determines a unique sequence of indices,

$$i_1,\ i_2,\ i_3,\ \cdots,\ i_m,\ \cdots,$$

and therefore corresponds to the number in the binary scale determined by these indices.

We proceed to the definition of a function ϕ of primary importance. On the set A, $\phi(p)=0$, on B, $\phi(p)=1$, on C

$$\phi(p) = \frac{i_1}{1} + \frac{i_2}{2^2} + \frac{i_3}{3^3} + \cdots .$$

This function is continuous. In fact, if $(p, q) < a_m$ then p and q lie in adjacent classes of the mth stage of the development of the space P and therefore

$$|\phi(p) - \phi(q)| \leqq 1/2^{m-1}.$$

Since the function ϕ is defined in terms of a and p_0, and the numbers a_m are independent of p_0, it follows that the family of all such functions obtained by varying p_0 and keeping a fixed is equally continuous.

If we now make the definitions

$$A_n = S(p_0, a_n), \qquad B_n = P - S(p_0, a_{n-1}),$$

$n=1, 2, 3, \cdots$, $a_0=a$, and let ϕ_n denote the function defined relative to A_n, B_n by the foregoing process, we obtain the desired family of equally continuous functions by considering all possible functions of the form

$$\phi = \sum_{n=1}^{\infty} \phi_n/2^n.$$

Since $\phi_n=1$ on B_n for each value of n, it follows that $\phi=1/2^n>0$ on the set $P-S(p_0, a_{n-1})$. Since $La_n=0$ the second condition of the theorem of § 11 is satisfied.

13. *Coherent Spaces.* Another attempt to generalize effectively the theory of metric spaces was made by A. D. Pitcher and E. W. Chittenden.* They considered an écart (p, q) in which the second condition on distance is replaced by the condition:

(2″) if $L(p,p_n)=0$, and $L(p_n,q_n)=0$, then $L(p,q_n)=0$.

A space in which the points of accumulation are definable in terms of a symmetric écart satisfying the condition (2″) is

* Loc. cit.

said to be coherent. It has recently been shown by Niemytski* that if the écart satisfies the further condition: $(p, q)=0$ is equivalent to $p=q$, then the space is metric.

14. *Spaces Defined by Developments.* The metrization problem for spaces whose points of accumulation are defined in terms of a development Δ was studied by Pitcher and Chittenden.† A development Δ is an arbitrary system of subclasses P^{ml} ($m=1, 2, 3, \cdots, l=1, 2, \cdots, l_m$) of a fundamental class P in which the classes P^{ml} for a fixed index m form a stage Δ^m of the development.‡ The index l_m may have a finite or infinite range. If its range is finite for all values of m the development is said to be finite, otherwise infinite.

A point p is a point of accumulation of a set of points E relative to a development Δ provided there is a sequence of indices $m_1<m_2<m_3\cdots$ and a corresponding sequence of classes $P^{m_n l_n}$ each of which contains an element of the set $E-p$. The reader is referred to the original article for the details of this investigation. The metrization problem was solved for compact spaces. The paper contains a set of necessary and sufficient conditions that a compact Hausdorff space be metric.

15. *The General Metrization Problem.* The general problem, to determine the topological conditions for the metrization of a topological space, was first explicitly stated and solved by Alexandroff and Urysohn.§ It is however of interest to note that E. R. Hedrick‖ in continuing the search begun by Fréchet for a generalization of metric space discovered that a number of important theorems stated by Fréchet for classes

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* In an article to appear in the Transactions of this Society.

† Transactions of this Society, vol. 20 (1919), pp. 213–233.

‡ This definition is due to E. H. Moore, New Haven Mathematical Colloquium, Yale University Press, New Haven, 1910, pp. 1–150.

§ Comptes Rendus, vol. 177 (1923), p. 1274.

‖ Loc. cit.

(V) normales* could be proved in any class (L)† in which derived sets are closed, providing the given space admits a property called the *enclosable* property. While Fréchet‡ soon proved that the space thus defined by Hedrick was in fact a "classe (V) normale" (therefore metric), it is important to observe that a slight modification of the conditions imposed by Hedrick constitute a set of necessary and sufficient conditions for the metrization of an abstract set.

We shall give three solutions of the general metrization problem; the first contains a modified form of the enclosable property of Hedrick, the second is due to Alexandroff and Urysohn, and the third is based upon the notion of coherence introduced by Pitcher and Chittenden. These three sets of conditions are alike in requiring the existence of a type of development Δ of fundamental importance which it is proposed to call *regular*.

Each stage of a *regular* development of a topological space is a family Δ^m of open sets V^m which covers the space P. The development proceeds by consecutive stages, that is, each set V^{m+1} of the $(m+1)$st stage is a subset of a set of the mth stage. Furthermore, if

$$V^1, V^2, V^3, \cdots, V^m, \cdots$$

is any infinite sequence of open sets one from each stage of the development and if there is a point p which is common to the sets of the sequence, then that point is *determined* by the sequence. That is, if V is any neighborhood of the point p, then for some value of the integer m, we have $V^m = V$. The sets V^m of the mth stage of the development are said to be of *rank* m.

The following additional definitions will be needed. Two points p, q are *developed* of stage or rank m provided there is a set of rank m which contains them both. In a regular develop-

* A "classe (V) normale" is a class that is separable, perfect, admits a definition of voisinage and a generalization of the theorem of Cauchy.

† A class (L) is essentially a space in which point of accumulation is defined in terms of the relation limit of a sequence.

‡ Transactions of this Society, vol. 14 (1913), pp. 320–324.

ment two points which are developed of rank m are developed of any lower rank.

Two sequences p_m, q_m are connected by a regular development provided the points p_m, q_m are developed of stage $m(m=1, 2, 3, \cdots)$.

A regular development Δ will be said to be *coherent* provided the connection of sequences is transitive. That is, a sequence $\{p_m\}$ is connected with a sequence $\{q_m\}$ whenever there is a sequence $\{r_m\}$ such that $\{p_m\}$ is connected with V_m and $\{r_m\}$ is connected with $\{q_m\}$.

The $(m+1)$st stage of a development is said to be *inscribed* in the mth if every pair of sets of rank $m+1$ which have a common point is contained in a set of rank m.

THEOREM. *Let P be a Hausdorff space admitting a regular development Δ in open sets V^m. Each of the following three conditions is a necessary and sufficient condition that P be equivalent to a metric space.*

I. (Hedrick) *For any positive integer m there is an integer n such that for any point p there is a set V^m of rank m which includes all sets of rank n which contain p.*

II. (Alexandroff and Urysohn) *For each value of the integer m the $(m+1)$st stage of the development Δ is inscribed in the mth.*

III. *The development Δ is coherent.*

Since it is evident that the three conditions of the theorem are necessary we may proceed at once to the proofs of their sufficiency.

Let us consider first the condition of Hedrick. It will be convenient to assume that the class P is a set of zero rank and allow the index m to have the values $m=0, 1, 2, 3, \cdots$. Then from condition II there exists for each integer n an integer $m=g(n)$, the greatest value of m for which n is the integer determined by condition II. The function $g(n)$ is unbounded. Suppose the contrary. Then there would exist an integer m' such that for every value of n, $g(n)<m'$. But there is an inte-

ger n' determined by m', contrary to the definition of $g(n)$ as the greatest such integer.

The écart (p, q) of two points is defined as follows. We set $(p, p)=0$. If m is the largest integer for which p and q are developed of rank m then $(p, q)=1/2^m$. Evidently the écart is symmetric. And if $p \neq q$ then $(p, q)>0$. For if the points p and q are developed of every rank m, then there is a sequence of sets V^m, one from each stage of the development to which both are common, contrary to the definition of a regular development, and the fourth condition of Hausdorff.

It remains to be shown that the écart thus defined is uniformly regular and that it defines the space P.

Let n be any integer and let p, q, r be three points such that $(p, r)<1/2^n$, $(q, r)<1/2^n$. If $m=g(n)$ we have at once, from condition II, that p and q are developed of stage m and therefore $(p, q)<1/2^m$. If $1/2^{n-1}<e\leqq 1/2^n$, and we set $f(e)=1/2^m$ we obtain a function $f(e)$ satisfying the required condition (2').

It remains to show that the given space and the derived space are equivalent. It is evident that if every set V^m which contains p contains a point of a set E then the écart (p, E) is null. Conversely, if $(p, E)=0$ there is for every integer n a point q_n of E such that $(p, q_n)<1/2^n$. That is, p and q_n are common to some neighborhood V^n. It follows that q_n is an element of $V^{q(m)}(p)$. Since $Lg(n)=\infty$, it follows from the regularity of the development Δ that every neighborhood of p contains a point of E.

It will now be shown that condition II implies condition I. We assume a regular development Δ satisfying condition II. A regular development Δ_1 satisfying condition I for $n=m+1$ may be defined in terms of Δ as follows. Stage $\Delta_1{}^m$ of Δ_1 is the family of all open sets $U^m(p)$ obtained by forming the sum for each point p of all the sets V^m of rank m of Δ which contain p. To show that if a point p is common to a sequence of sets

$$U^1, U^2, U^3 \cdots, U^m, \cdots$$

then p is determined by that sequence, let V be any neighborhood of p, and suppose that for each value of m there is a point

q_m of U^m which does not belong to V. Since q_m is in the set U^m there must be a set V^m of rank m of which contains both p and q_m. But the sequence of sets V^m determines p. Therefore for some value of m, V^m (and therefore q_m) is contained in V, contrary to the hypothesis on q_m.

In the proof of the sufficiency of condition I the écart (p, q) of two points was defined, and without reference to that condition. This definition will be applied in the proof of the sufficiency of condition III. Because of the result obtained by Niemytski stated in § 13, we have only to show that this écart satisfies the condition (2''), since the equivalence of the spaces has already been established.

It is at once evident from the definition of écart that if two sequences are connected by the development then $L(p_n, q_n) = 0$ and conversely. If then, $L(p, p_n) = 0$, and $L(p_n, q_n) = 0$, the sequence composed of the repeated element p must because of condition III be connected with the sequence q_n. Therefore $L(p, q) = 0$, as required by condition (2'').

16. *The Borel Theorem.* A set E will be said to possess the "any-to-finite" form of the property of Borel in case every infinite proper covering F of E contains a finite proper covering. A family F of sets G is a proper covering of a set E if every element of E is interior to some set E. In a metric space a necessary and sufficient condition that a set E admit this property is that E be compact and closed. Thus the problem, to characterize those spaces in which the "any-to-finite" form of the property of Borel is equivalent to compact (or self-compact) and closed, is very closely related to the metrization problem. As Hildebrandt* has presented an excellent and full account of recent work on this problem, it is unnecessary to discuss it further here.

THE UNIVERSITY OF IOWA

* Loc. cit.

A THEOREM ON FACTORIZATION*

BY D. N. LEHMER

In a note in this Bulletin† I observed that if $R=pq$ is the product of two odd factors whose difference is less than twice the fourth root of R then the factors of R are obtainable directly from the expansion of $R^{1/2}$ in a continued fraction. This theorem comes from the fact that in view of a theorem due to Lagrange, $(p-q)^2/4$ will appear as a denominator of a complete quotient in that expansion, and that therefore the diophantine equation $x^2-Ry^2=(p-q)^2/4$ will have the integral solution $x=\frac{1}{2}(r+q)$, $y=1$.

The object of the present note is to point out that the method is of much wider application than the above statement would indicate. For consider the identity

$$\left(\frac{mp+nq}{2}\right)^2-\left(\frac{mp-nq}{2}\right)^2=mnpq.$$

From this it appears that if mn is a square and if m and n are both odd or both even, we will have an integral solution of the equation

$$x^2-Ry^2=\frac{1}{4}(mp-nq)^2,$$

namely

$$x=\frac{1}{2}(mp+nq), \qquad y=(mn)^{1/2}.$$

By Lagrange's theorem, therefore, if $mp-nq<2R^{1/4}$ one of the denominators in the expansion of $R^{1/2}$ will certainly be $(mp-nq)^2/4$ and since the numerator of the preceding convergent will be $(mp+nq)/2$ these two numbers will serve to furnish the factors p and q of R. We have then the following theorem.

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* Presented to the Society, San Francisco Section, October 30, 1926.

† Vol. 13 (1906-7), p. 501. Translated in Sphinx-Oedipe, 1911. Given also in Kraitchik's *Recherches sur la Théorie des Nombres*, p. 73.

THEOREM. *If $R=pq$ is the product of two odd factors, and if two numbers m and n, both even or both odd, are obtainable such that their product is a square and also such that $mp-nq<2R^{1/4}$ then the continued fraction for $R^{1/2}$ will furnish without trial the factors p and q of R.*

It should be noted that if the difference $mp-nq$ is less than the fourth root of R the restriction that m and n be both even or both odd may be disregarded, for in that case $2m$ and $2n$ are suitable multipliers. Also it is worth noting that the square denominator will appear in the complete quotient when the denominator of the preceding convergent is $(mn)^{1/2}$. This means that in the original theorem the desired square is under the third complete quotient.

An example will indicate the method of attacking a number by this method. Let $A=1564,08789$. The square root expansion gives the following series of denominators for the complete quotients: 1, 8753, 15013, 3740, 529, $\cdots$, the partial quotients being 12506, 2, 1, 6, 47, $\cdots$. The convergent preceding the complete quotient with the square denominator 529 is found to be 250127/20. We have then

$$(mp+nq)/2 = 250127, \qquad (mp-nq)/2 = 23.$$

Whence

$$mp = 250150, \quad nq = 250104.$$

Since, now, $pq=A$ and $mn=20^2=400$ it is easily found that $p=31263$, $q=5003$, $m=8$, $n=50$. The success of the method was due to the fact that the difference $mp-nq=46$, which is less that $2A^{1/4}=222$. In this example also we see that $36p-225q=207$, which is also less than 222; but since here the values of m and n differ in parity, these values will not appear in the expansion. Similarly the difference $mp-nq=23$ for $m=4$, $n=25$, and m and n being different in parity these values will also not appear in the expansion. But since 23 is less than $A^{1/4}$, we will have $2m$ and $2n$ for suitable values, and these are indeed the ones that do appear.

THE UNIVERSITY OF CALIFORNIA

ON CONTINUITY IN SEVERAL VARIABLES*

BY H. J. ETTLINGER

The following theorem on the continuity of a function of several variables is contained implicitly in a theorem on the existence of solutions of differential equations by Carathéodory.† It is of a general nature and independent of the context in which it is found. It is, therefore, worth while isolating and signalizing it.

THEOREM. *Hypotheses*: (1) $f(x, [y])$ *is defined in the domain* $R: a<x<b$, $y_0-k<[y]<y_0+k$, *where* $[y]\equiv(y_1, \cdots, y_n)$.

(2) *For each* $[y]$ *in* R, $f_x'(x, [y])$ *exists almost everywhere on* $a<x<b$, *and is summable on* $a<x<b$.

(3) $|f_x'(x, [y])|\leqq M(x)$, *where* $M(x)$ *is summable on* $a<x<b$.

(4) *For every* x *on* $a<x<b$, $f(x, [y])$ *is continuous in* $[y]$ *at* $y_1=y_0$.

Conclusion: $f(x, [y])$ *is continuous in* $(x, [y])$ *at* $(x, [y_0])$ *where* x *is on* $a<x<b$.

PROOF. Let us put $F_1\equiv f(x+h,[y_0+k_i])$, where $|k_i|\leqq k$; $F_2\equiv f(x,[y_0+k_i])$; $F_3\equiv f(x,[y_0])$. From hypothesis (2), we have

$$F_1 - F_2 = \int_x^{x+h} f_t'(t, [y_0 + k_i])dt,$$

which becomes by hypothesis (3),

$$|F_1 - F_2| \leqq \int_x^{x+h} M(t)dt. \tag{1}$$

Also, from hypothesis (4), it follows that given a positive ϵ, there exists a positive number k'_{xy_0}, such that

$$|F_2 - F_3| < \epsilon, \quad \text{for} \quad |k_i| < k'_{xy_0}. \tag{2}$$

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* Presented to the Society, October 30, 1926.

† *Vorlesungen ueber Reelle Funktionen*, Leipzig, 1918, p. 678, Satz 5.

On the other hand, since $F_1 - F_3 \equiv F_1 - F_2 + F_2 - F_3$. it follows that

$$|F_1 - F_3| \leqq |F_1 - F_2| + |F_2 - F_3|.$$

By applying (1) and (2) to this inequality, we obtain the result $f(x, [y])$ is continuous in $(x, [y])$ at $(x, [y_0])$, where x is any value on $a<x<b$.

Let $F(x, y)$ be defined in the interior of the square S: $a<x$, $y<b$, and have the following properties, (1) for a fixed y, it is summable in x on $a<x<b$, (2) $|\ F(x, y)\ | \leqq K(x)$, for all values of y on $a<y<b$, where $K(x)$ is summable on $a<x<b$, (3) for every fixed x except a null set, $F(x, y)$ is continuous in y. Consider

$$f(x, y) = \int_a^x F(t, y)dt.$$

From a well known theorem concerning an indefinite integral we have that $f(x, y)$ is continuous in x for y fixed. From a theorem first stated by Carathéodory,* $f(x, y)$ is continuous in y for fixed x. *From the theorem stated above*, $f(x, y)$ is continuous in (x, y) for all points in R.

The conclusion of the preceding paragraph is of importance in showing that if a system of first order differential equations which satisfy the conditions of Carathéodory's† existence theorem be solved by the method of successive approximations, not only will the solutions be continuous functions of the independent variable and the parameter when both vary arbitrarily (as given by Carathéodory's theorem) but also the approximating functions will have continuity of this same kind.‡

The University of Texas

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* Loc. cit., p. 639.

† Loc. cit., p. 678.

‡ For a pair of linear first order differential equations, see Sturdivant, *Properties of second order linear systems with Lebesgue integrable coefficients*, offered to the Transactions of this Society.

ON A PROBLEM IN CLOSURE*

BY VIRGIL SNYDER

The following note concerns finite groups of birational transformations which leave an algebraic curve of genus 1 invariant.

Let the curve be expressed as a C_4 in S_3, intersection of two general quadric surfaces. This curve is invariant under the linear group G_8, of order eight, generated by the harmonic homologies defined by the self-conjugate tetrahedron associated with C_4. The points of C_4 are thus arranged in sets of 8, forming a linear I_8^1 of genus 0. If the curve be projected upon a plane from an arbitrary point, a plane quartic C_4 results with nodes at K_1, K_2. The four central homologies become four perspective quadratic involutions T_i with centers O_i not on C_4; the three axial involutions become three non-perspective quadratic inversions, with fundamental points O_{ik} for $T_iT_k = T_kT_i = T_{ki} = T_{lm}$ at the diagonal points of the quadrangle $O_1O_2O_3O_4$. The nodes K_1, K_2 are the other fundamental points for all seven operations.†

From the theorem of Bertini it follows that in any plane nodal quartic one and only one conic can be found meeting

* Presented to the Society, September 8, 1926.

† A brief synthetic outline of the properties of G_8 was first given by C. Segre, *Su una trasformazione irrazionale dello spazio . . .* , Giornale di Matematiche, vol. 21 (1883), pp. 355–378. This was amplified in connection with a larger problem by D. Montesano, *Su alcuni gruppi chiusi di trasformazioni involutorie nel piano e nello spazio*, Atti Istituto Veneto, (6), vol. 6 (1888), pp. 1425–1444. It is also contained in the papers by K. Meister, *Ueber die Systeme, welche durch Kegelschnitte mit einem gemeinsamen Polardreieck, bez. durch Flächen zweiten Grades mit einem gemeinsamen Polartetraeder gebildet werden*, Zeitschrift der Mathematik und Physik, vol. 31 (1886), pp. 321–347; vol. 34 (1889), pp. 6–24; 73–91 and by H. E. Timerding, *Ueber die quadratische Transformation durch welche die Ebenen des Raumes in ein System von Flächen zweiter Ordnung mit gemeinsamen Poltetraeder übergeführt werden*, Annali di Matematica, (3), vol. 1 (1898), pp. 95–117.

any line through the node in two points which form with the residual intersections of the line and the quartic a harmonic set. Using this conic as conic of invariant points and the node as center in a perspective quadratic inversion, the quartic is transformed into itself. In case there are two nodes there is a conic of Bertini associated with each, and the conic associated with one node passes through the other. Let these operations be denoted by S_1, S_2. In a recent memoir* the question is raised whether the group generated by S_1, S_2 and G_8 can be finite, but the author does not answer it. The answer may be difficult if the method is restricted to pure geometry. By assistance of the parametric representation in terms of elliptic functions it can be answered in every case.

Let

$$F = x_1^2 - x_2^2 + x_3^2 - x_4^2 = 0, \qquad \sum a_i x_i^2 = 0$$

define the curve. The tetrahedron of reference is the self-conjugate tetrahedron of the pencil of quadrics, and the equations of the harmonic homology H_1 with center at vertex (1, 0, 0, 0), invariant plane $x_1 = 0$, are $\rho x_1' = -x_1$, $\rho x_i' = x_i$, $i = 2, 3, 4$; similarly for H_2, H_3, H_4.

By projecting $F = 0$ upon $x_4 = 0$ from (1, 1, 1, 1), the operation H_1 becomes the perspective inversion

$$(T_1) \qquad \begin{cases} x_1' = (x_2 - x_3)(x_1 - x_2 - x_3), \\ x_2' = x_2(x_1 - x_2 + x_3), \\ x_3' = x_3(x_1 + x_2 - x_3). \end{cases}$$

The center is $O_1 = (1, 0, 0)$, and the other fundamental points are

$$K_1 = (0, 1, 1), \qquad K_2 = (1, 1, 0).$$

The conic of invariant points is

$$(x_1 - x_2)^2 + 2x_1x_3 - x_3^2 = 0.$$

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* E. Ciani, *Le quartiche piane invertibili*, Giornale di Matematiche, vol. 57 (1919), 47 pages.

Similar expressions exist for T_i, $i=2, 3, 4$, and for T_{ik}, $O_4=(1, 1, 1)$. The triangle formed by any three of the points $O_i\ O_k\ O_l$ is self-conjugate as to the conic of invariant points associated with the fourth.*

Among the space quartics belonging to the same self-conjugate tetrahedron are ∞^1 through the center of projection. This pencil is projected into a pencil of plane cubics through K_1, K_2, the four vertices O_1, O_2, O_3, O_4, and the three diagonal points O_{ik}. Hence by suppressing any side of the quadrangle, the four remaining basis points and K_1, K_2 lie on a conic.

The equation can be reduced to the form

$$k_1x_1(x_2-x_3)(x_2+x_3-x_1)+k_2x_2(x_3-x_1)(x_1-x_2+x_3)=0.$$

The points K_1, K_2 form a pair of conjugate points in the Geiser net determined by the other seven basis points. The line joining them meets any C_3 of the pencil in just one point, which with the seven basis points makes a group of I_8 on that curve. Any position on C_3 can be chosen for one of them, and then the other is uniquely fixed.

The points O_i are all cotangential, and any one of them can be assumed at will. The fundamental points O_{ik} are collinear with O_iO_k and with O_lO_m. Associated with any point P on C_3 are the points of tangency of the four tangents from P. The other three from the first tangential of P have their points of contact at O_{ik}. Expressed in parametric form, assuming the form to which every elliptic cubic can be reduced, that three points are collinear when the sum of their parameters is congruent to zero, then if P has the parameter u, we have

$$O_1 = -\frac{u}{2}, \qquad O_2 = -\frac{u}{2}+\omega, \qquad O_3 = -\frac{u}{2}+\omega',$$

$$O_4 = -\frac{u}{2}+\omega+\omega', \qquad O_{12} = u+\omega, \qquad O_{13} = u+\omega',$$

$$O_{14} = u+\omega+\omega'.$$

* The equations of the operations of the group and those of an ∞^2 family of binodal quartics each invariant under it were determined in my seminar by Miss Bertha I. Hart, of Western Maryland College.

Let K_1 have the parameter v; then K_2 has the parameter $-u-v$. A conic through K_1 and through the four points remaining of the seven basis points when three on any line have been suppressed will also pass through K_2.

The involution in which K_1, K_2 are a pair of conjugate points is that having the polar conic at P for conic of invariant points, and P for vertex.

If K_1, K_2 be regarded as fixed, then a cubic of the pencil is fixed by its residual intersection with K_1, K_2. Since P_1 K_1 can be chosen at will and satisfy all the conditions of the problem it follows that the group generated by K_1, O_1 is generally of infinite order. If finite, K_1, G_8 is also finite.

Since P can be chosen at will on C_3 and K_1 can be chosen at will, there are ∞^2 groups of I_8 in the plane. As groups of operations there appear in sets of pencils, those of any set (O_i on any curve of the pencil) having the three operations T_{ik} in common, so far as the transformations of points on the curve are concerned.

These results can be interpreted in terms of C_4 in space directly. By an appropriate linear transformation we may write*

$$x_1=\sigma_1(2u),\quad x_2=\sigma_2(2u),\quad x_3=\sigma_3(2u),\quad x_4=\sigma(2u),$$

the σ_i being those defined in the Schwarz-Weierstrass *Formeln und Lehrsätze*, pp. 21–2.

Four points with parameters $u_1, \cdots, u_4$ are collinear if $\Sigma u_i=0$. The operations of G_8 are defined as follows:

$$\begin{aligned} u' &= -u+\omega+\omega' && \text{is } H_1,\\ u' &= -u+\omega' && \text{is } H_2,\\ u' &= -u+\omega && \text{is } H_3,\\ u' &= -u && \text{is } H_4, \end{aligned}$$

from which the H_{ik} at once follow.

A quadric of the pencil through C passes through any given point P. The quadric contains a generator of each system,

* E. Lange, *Die sechzehn Wendeberührungspunkte der Raumcurve vierter Ordnung, erster Species*, Schlömilchs Zeitschrift, vol. 28 (1883), pp. 1–23.

through P, each meeting C_4 twice. If this quadric be projected stereographically from P, the plane quadric will have a double point at K_1 on one generator and at K_2 on the other. Consider the line PK_1. It contains P_1, P_2 on C_4 and any plane through PK_1 meets C_4 in two points P_3, P_4 which project into two points collinear with K_1. These points are interchanged by S_1, hence in space we may choose u_1 and u_2 at will, then pass a plane through the points determined by them and any third point u_3. The space operation consists in interchanging the point u_3 with the fourth point u_4 as the plane turns about u_1 u_2, as u_3 describes C_4.

The operation has the form

$$u' = -u + c,$$

where c is any given constant. If this be associated with H_i the product is not periodic unless c is of the form $r\omega + r'\omega'$ wherein r, r' are both rational. But if this product is rational, then the group generated by S and G_3 is finite. The operation S_2 is defined by

$$u' = -u - c,$$

hence the necessary and sufficient condition that S_1, S_2, G_8 generate a finite group is that S_1, S_2 generate a finite group.

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A THEOREM CONCERNING DIRECT PRODUCTS*

BY LOUIS WEISNER†

The theorem in question may be stated as follows. *A group of order mn, where m and n are relatively prime, in which every element whose order divides m is commutative with every element whose order divides n, is the direct product of two groups of orders m and n.*

Burnside‡ has proved the theorem for the special case in which either m or n is a power of a prime. Hence, to prove the theorem, we need only show that it is true for groups of order mn if it is true for groups of order $<mn$. To avoid trivial cases we assume that m and $n>1$. We denote by m_1, m_2, and n_1, n_2, divisors >1 of m and n respectively.

If the group G contains an element of order m, let p be a prime factor of m and p^α the highest power of p that divides m. Every element of G whose order divides p^α is commutative with every element whose order is prime to p. It follows from the special case referred to, that G contains an invariant subgroup of order mn/p^α, which contains an invariant subgroup of order n, as every element whose order divides m/p^α is commutative with every element whose order divides n.

We suppose now that G contains no element of order m. The normaliser H of an element s of order m_1 includes all the Sylow subgroups of G whose orders divide n, and is therefore of order $m_2 n$, where $m_2 \geqq m_1$. If $m_2 < m$, H contains an invariant subgroup of order n. If $m_2 = m$, s is invariant under G. An element t of G corresponding to an element t' of $G/(s)$ of order n_1 is of order $n_1\mu$, where μ divides m_1. Hence G contains§ two elements t_1 and s_1 of orders n_1 and μ respectively, such that

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‡ W. Burnside, *Theory of Groups*, 2d edition, p. 327.

§ Loc. cit., p. 16.

$t=t_1s_1$. Since $(t_1s_1)^{n_1}=s_1^{n_1}$ is in (s), and n_1 is prime to the order of s_1, s_1 is a power of s. Thus t_1, as well as t, corresponds to t' in the isomorphism of G with $G/(s)$; but t_1 and t' are of the same order n_1. Every element of $G/(s)$ whose order is a divisor of m/m_1 corresponds to an element of G whose order is a divisor of m. It follows that t', and hence every element of $G/(s)$ whose order divides n, is commutative with every element whose order divides m/m_1. Hence $G/(s)$, being of order $<mn$, contains an invariant subgroup of order n. The corresponding subgroup of G, being of order $m_1n<mn$, also contains an invariant subgroup of order n.

Thus in all cases G contains a subgroup of order n. Similarly G contains a subgroup of order m. G is evidently the direct product of these two subgroups.

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A CUBIC CURVE CONNECTED WITH TWO TRIANGLES

BY H. BATEMAN

1. *Introduction.* If ABC, XYZ are two triangles, a cubic curve Γ_3 may be associated with them as follows.* Let (PQ, RS) denote the point of intersection of the lines PQ, RS; then Γ_3 is the locus of a point O such that (OA, YZ), (OB, ZX), (OC, XY) are collinear and also the locus of a point O for which (OX, BC), (OY, CA), (OZ, AB) are collinear. In fact when one set of three points is collinear the other set of three is also collinear. Take ABC as triangle of reference and let the points X, Y, Z have coordinates (x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3) respectively, then if (α, β, γ) are current coordinates

* H. Grassmann, *Die lineale Ausdehnungslehre*, 1844, p. 226. The corresponding quartic surface connected with two tetrahedra is mentioned by H. Fritz, Pr. Ludw. Gymn. Darmstadt [reference taken from Jahrbuch der Fortschritte der Mathematik, vol. 21 (1889), p. 725] and by C. M. Jessop, *Quartic Surfaces*, Cambridge, 1916, p. 189.

the condition for the collinearity of the second sets of three points may be expressed in the form

$$(\gamma x_1 - \alpha x_3)(\alpha y_2 - \beta y_1)(\beta z_3 - \gamma z_2)$$
$$= (\alpha x_2 - \beta x_1)(\beta y_3 - \gamma y_2)(\gamma z_1 - \alpha z_3).$$

This equation represents a cubic curve passing through the points* A, B, C, X, Y, Z, (BC, YZ), (CA, ZX), (AB, XY), (CY, BZ), (AZ, CX), (BX, AY). There is evidently only one cubic curve passing through all these points, and since a cubic through all these points is obtained for the locus when the three points (OA, YZ), (OB, ZX), (OC, XY) are collinear, Γ_3 must be the locus in both cases.

This is easily verified analytically. If X_1, X_2, etc. denote the co-factors of the constituents x_1, x_2, etc. in the determinant

$$\Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix},$$

the equation of the locus in the first case may be expressed in the form

$$\begin{vmatrix} \beta X_2 + \gamma X_3 & -\beta X_1 & -\gamma X_1 \\ -\alpha Y_2 & \alpha Y_1 + \gamma Y_3 & -\gamma Y_2 \\ -\alpha Z_3 & -\beta Z_3 & \alpha Z_1 + \beta Z_2 \end{vmatrix} = 0,$$

and the equivalence of the two equations is easily verified with the aid of a number of identities, one of which is

$$2X_1Y_2Z_3 - Y_1X_3Z_2 - Z_1X_2Y_3$$
$$= \Delta(2x_1y_2z_3 - y_1x_3z_2 - z_1x_2y_3)$$
$$= \Delta(x_1X_1 + y_2Y_2 + z_3Z_3 - \Delta).$$

2. *Salmon's Projective Invariant.* By a well known theorem due to Salmon the cross ratio of the four tangents to a cubic from the point on the curve is constant and represents

* The properties of this configuration of 12 points and some degenerate cases of the cubic curve are discussed by H. M. Taylor, Proceedings of the London Society, (1), vol. 28 (1897), pp. 545–555.

an invariant of the curve. In the present case this invariant will be an invariant of the two triangles* and it seems worth while to calculate it. A direct calculation is laborious when use is made of the equation of the curve in one of the forms already given; the work may be simplified, however, by noticing that the four lines BC, YZ, BZ, CY form a complete quadrilateral whose six vertices lie on the curve. Denoting the equations of these lines by $\alpha=0$, $\theta=0$, $\phi=0$, $\psi=0$ respectively, the equation of the curve may be written in the form

$$\frac{A}{\alpha}+\frac{B}{\theta}+\frac{C}{\phi}+\frac{D}{\psi}=0,$$

where the identical relation between $\alpha, \theta, \phi, \psi$ is

$$\alpha+\theta+\phi+\psi=0,$$

and

$$A = y_2z_1^2X_2^2Z_3, \qquad B = y_1^2z_3X_3^2Y_2,$$
$$C = y_1^2z_1^2X_1\Delta, \qquad D = x_1X_2^2X_3^2.$$

If λ is one of the cross ratios of the four tangents, λ is connected with the coefficients A, B, C, D by the relation†

$$(\lambda+1)^2(\lambda-2)^2(\lambda-\tfrac{1}{2})^2I^3 = 27(\lambda^2-\lambda+1)^3J^2,$$

where I and J, the usual invariants of a biquadratic equation, are given by the equations

$$12I = \Theta^2 - 48ABCD, \quad 216J = \Theta[72ABCD - \Theta^2],$$
$$I^3 - 27J^2 = A^2B^2C^2D^2[\Theta^2 - 64ABCD],$$

where

$$\Theta^2 = A^2+B^2+C^2+D^2-2BC-2CA-2AB-2AD-2BD-2CD.$$

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* A complete set of invariants of two triangles is given in a paper by D. D. Leib, Transactions of this Society, vol. 10 (1909), p. 361. In our case there is a correspondence between the vertices of the triangles and the invariant theory is a little different.

† The coefficient m in the canonical equation $x^3+y^3+z^3+6mxyz=0$ is connected with I and J by the relation

$$\frac{64m^3(m^3-1)^3}{I^3} = \frac{(8m^6+20m^3-1)^2}{27J^2}.$$

See H. Hilton, *Plane Algebraic Curves*, Oxford, 1920, p. 235.

Substituting the expressions for A, B, C, D, we find that

$$\Theta = y_1^2 z_1^2 X_2^2 X_3^2 [(x_3 y_1 z_2 - x_2 y_3 z_1)^2 - 4\Delta x_1 y_2 z_3 - 4X_1 Y_2 Z_3],$$
$$ABCD = y_1^4 z_1^4 X_2^4 X_3^4 [x_1 X_1 y_2 Y_2 z_3 Z_3 \Delta].$$

The nature of the cubic is thus seen to depend on the value of the single invariant

$$K = \frac{[(x_3 y_1 z_2 - x_2 y_3 z_1)^2 - 4\Delta x_1 y_2 z_3 - 4X_1 Y_2 Z_3]^2}{x_1 y_2 z_3 X_1 Y_2 Z_3 \Delta}.$$

The cubic has a double point when this invariant is either infinite or 64. When $\Theta = 0$ we have $J = 0$ and the four tangents form a harmonic pencil. If A, B, C, Y, and Z are given, the locus of X when $\Theta = 0$ is a conic. This, however, is not the complete locus when the tangents form a harmonic pencil because J vanishes also when $\Theta^2 = 72ABCD$ or $K = 72$. It should be noticed that the expression K remains the same when we replace each constituent in the determinant by its co-factor. We have in fact an identity of type

$$X_3 Y_1 Z_2 - X_2 Y_3 Z_1 = \Delta(z_3 Z_3 - y_2 Y_2 + x_3 X_3 - x_2 X_2)$$
$$= \Delta(x_3 y_1 z_2 - x_2 y_3 z_1),$$

with the aid of which the preceding remark is easily verified.

Quantities that possess this property may be called *invariants* of the determinant. To find such quantities we may commence by finding pseudo-invariants, that is, quantities which retain the same form except for a power of the determinant. The quantities $x_1 X_1$, $x_2 X_2$, $x_3 X_3$, $y_1 Y_1$, $y_2 Y_2$, $y_3 Y_3$, $z_1 Z_1$, $z_2 Z_2$, $z_3 Z_3$ all possess this last property when considered as functions of the constituents x_1, x_2, etc. and a similar remark may be made in the case of a determinant of higher order. For the general case of a determinant of order n I do not know definitely how many pseudo-invariants of degree n in the constituents are linearly independent. A consideration of the cases $n = 2$, 3, and 4 leads to the conclusion that this number N may be given by the formula

$$N = (n - 1)^2 + 1.$$

When $n=2$, it is evident that $N=2$. When $n=3$ it appears that $N=5$, for each of the nine quantities x_1X_1, x_2X_2, x_3X_3, y_1Y_1, y_2Y_2, y_3Y_3, z_1Z_1, z_2Z_2, z_3Z_3 can be expressed as a linear function of the five linearly independent quantities x_1X_1, x_2X_2, y_1Y_1, y_2Y_2, Δ.

In the case of a determinant of the fourth order

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ t_1 & t_2 & t_3 & t_4 \end{vmatrix},$$

the problem is more complex because in addition to the pseudo-invariants x_sX_s etc. there are pseudo-invariants of type $(y_2z_3-y_3z_2)(x_1t_4-x_4t_1)$.

Adopting the notation

$$\binom{yz}{14} = (y_2z_3 - y_3z_2)(x_1t_4 - x_4t_1) = \binom{xt}{23},$$

we observe that the relations of type

$$y_1Y_1 = x_2X_2 + \binom{xt}{12} + \binom{zx}{12} - \binom{xy}{31} - \binom{xy}{14},$$

$$y_2Y_2 = x_1X_1 + \binom{zx}{12} + \binom{xt}{12} + \binom{xy}{24} + \binom{xy}{23},$$

$$y_3Y_3 = x_4X_4 + \binom{xt}{34} + \binom{zx}{34} - \binom{xy}{31} - \binom{xy}{23},$$

$$y_4Y_4 = x_3X_3 + \binom{xt}{34} + \binom{zx}{34} - \binom{xy}{14} - \binom{xy}{24}$$

indicate that all the pseudo-invariants of type y_sY_s, z_sZ_s, t_sT_s can be expressed in terms of the pseudo-invariants x_1X_1, x_2X_2, x_3X_3, x_4X_4 and the 18 quantities of type

$$\binom{yz}{14}.$$

It would be wrong, however, to suppose that there are 22 linearly independent pseudo-invariants of degree four, for there are the six identities

$$x_1X_1 + x_4X_4 = \binom{xy}{23} + \binom{zx}{23} + \binom{xt}{23},$$

$$x_2X_2 + x_4X_4 = \binom{xy}{31} + \binom{zx}{31} + \binom{xt}{31},$$

$$x_3X_3 + x_4X_4 = \binom{xy}{12} + \binom{zx}{12} + \binom{xt}{12},$$

$$x_2X_2 + x_3X_3 = \binom{xy}{14} + \binom{zx}{14} + \binom{xt}{14},$$

$$x_3X_3 + x_1X_1 = \binom{xy}{24} + \binom{zx}{24} + \binom{xt}{24},$$

$$x_1X_1 + x_2X_2 = \binom{xy}{34} + \binom{zx}{34} + \binom{xt}{34},$$

three relations of type

$$\Delta = \binom{xy}{34} + \binom{xy}{14} + \binom{xy}{24} + \binom{xy}{23} + \binom{xy}{31} + \binom{xy}{12},$$

three relations of type

$$\Delta = \binom{yz}{12} + \binom{zx}{12} + \binom{xy}{12} + \binom{xt}{12} + \binom{yt}{12} + \binom{zt}{12},$$

and the relation $x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4 = \Delta$. The number of linearly independent pseudo-invariants thus seems to be ten.

California Institute of Technology

ON THE INTEGRATION IN FINITE TERMS OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER*

BY J. F. RITT

1. *Introduction.* Liouville, in 1840, investigated the equation

$$\text{(A)} \qquad \frac{d^2w}{dz^2} = \chi(z)w,$$

with $\chi(z)$ an elementary function of z, to determine under what circumstances the solutions of the equation are elementary.†

By an "elementary function", Liouville understood, in this connection, any function of z obtained in a finite number of steps by performing algebraic operations, taking logarithms and exponentials, and performing integrations. An example of such a function would be

$$\int \left[\log \arcsin z + \int z^2 dz \right]^{1/2} dz + \tan \log_z \left[1 + (z)^{1/2} \right].$$

It is a consequence of Liouville's work that the solutions of Bessel's equation

$$z^2w'' + zw' + (z^2 - \nu^2)w = 0$$

are elementary functions only when 2ν is an odd integer.‡ But it would be improper to conclude from Liouville's results alone, as some authors seem to do, that Bessel's equation cannot be solved by the simpler formal methods of the theory of differential equations. For instance, the equation

$$\frac{du}{dz} = \frac{u}{u - 1}$$

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† Journal de Mathématiques, vol. 5 (1840). See also Watson, *Theory of Bessel Functions*, Cambridge, 1922, p. 111. No acquaintance with Liouville's work is necessary for the understanding of the present paper.

‡ Watson, loc. cit.

integrates directly into

$$u - \log u = z + c,$$

but the functions u thus obtained cannot be expressed in terms of z by performing algebraic operations and taking exponentials and logarithms. Can one be sure in advance that Bessel's equation is not susceptible to similar treatment?

Investigations on the possibility of integrating differential equations of the first order by elementary operations performed upon both the dependent variable and the independent variable were carried out by Lorenz, Steen, Hansen and Petersen,* and by Mordukhai-Boltovskoi.† Lagutinski has studied systems of equations of the first order.‡

In the present paper, we prove a general theorem (§3) which shows that the solutions of Bessel's equation do not satisfy elementary equations,§ except for the given values of ν.

Our theorem states that if any solution of equation (A) satisfies an elementary equation in w and z, then the general solution of (A) is an elementary function of z.

The procedure in the present paper is purely formal. The many questions of a function-theoretic nature which will arise will be found treated in our papers on elementary functions published in the Transactions of this Society.‖

2. *The l-Functions.* We shall call any algebraic function of w and z an *l-function of order zero*, and the variables w and z *monomials of order zero.*

We introduce now two classes of *monomials of order one.* The first class will consist of the exponentials of non-constant algebraic functions of w and z. The second will consist of all

* Nyt Tidskrift for Matematik, 1874–1876.

† Communications de la Société Mathématique de Kharkow, vol. 10 (1909), p. 34.

‡ Kharkow Technological Institute, 1909. I have not been able to secure Lagutinski's paper, but the abstract in the Jahrbuch indicates that the present paper does not overlap on it.

§ This term, whose meaning is fairly clear from what precedes, is made precise below.

‖ Vol. 25 (1923), p. 211, and vol. 27 (1925), p. 68.

functions which are not algebraic, but whose partial derivatives are both algebraic. Evidently the logarithm of any non-constant algebraic function is a monomial of the second class.

By an *l-function of order one*, we shall mean any non-algebraic function u of w and z which is obtained by performing algebraic operations upon monomials of orders zero and one; that is, a function which satisfies an equation

$$(1) \qquad \alpha_0(w,z)u^n + \alpha_1(w,z)u^{n-1} + \cdots + \alpha_n(w,z) = 0,$$

in which every $\alpha(w, z)$ is a rational integral combination of monomials of orders zero and one.

We now define, by induction, l-functions of any order n. A function which is not among the functions of order 0, 1, $\cdots$, $n-1$, will be called a *monomial of order n* if it is of one of the following two types:

(a) an exponential of a function of order $n-1$;

(b) a function both of whose partial derivatives are among the functions of order 0, 1, $\cdots$, $n-1$.

Evidently a logarithm of a function of order $n-1$ is a monomial of order n if it is not a function of order less than n.

Any function u will be called an *l-function of order n* if it is not an l-function of order less than n, and if it satisfies an equation like (1) in which each $\alpha(w, z)$ is a rational integral combination of monomials of orders 0, 1, $\cdots$, n.*

The result of performing algebraic operations upon l-functions of the first n orders is always an l-function of one of those orders.

The partial derivatives of any l-function u of order n are both l-functions of order n or less. If, in the expression for u, the maximum of the orders of the monomials which involve w is r, then the derivatives have expressions for which the same maximum is at most r, and in which no monomials of order r involving w appear which do not also appear in u. These facts are a consequence of the rules for differentiation.

* It is not important here to know whether functions of every order actually exist. See Liouville, Journal de Mathématiques, vol. 3 (1838), p. 523.

3. *Linear Differential Equations.* We prove the following theorem, which shows that the solutions of Bessel's equation do not satisfy an elementary equation, except for the stated values of ν.

THEOREM. *If w, any solution of the differential equation $w''=\chi(z)w$, in which $\chi(z)$ is an l-function, satisfies an equation $F(w, z)=0$, where $F(w, z)$ is an l-function, not identically zero, then w is an l-function of z.*

Differentiating the equation $F(w, z)=0$ with respect to z, we find

$$F_w w' + F_z = 0. \tag{2}$$

Here F_w and F_z are l-functions. We have thus, in the first member of (2), an expression algebraic in w', and in l-functions of w and z, which vanishes when w is the given solution of $w''=\chi(z)w$, *without vanishing identically.**

With every such expression, $f(w', w, z)$, we associate two integers; first r, the maximum of the orders of those monomials in w and z, contained in the expression, which involve w; secondly s, the number of such monomials of order r which involve w. Many different expressions will represent the same function; the integers will vary with the expression.

There is a class of expressions $f(w', w, z)$ for which r is a minimum, and in this class there are expressions for which s is as small as it can be if r is a minimum. We assume that we have in hand an $f(w', w, z)$ with r and s minima, and write

$$f(w', w, z) = 0. \tag{3}$$

We are going to prove that $r=0$. Let the contrary, namely that $r>0$, be assumed, and let θ be one of the monomials of order r contained in f which involve w. Since f involves θ algebraically, we find from (3), upon solving for θ,

$$\theta - g(w', w, z) = 0, \tag{4}$$

* Of course, $F(w, z)$ is also such an expression, but (2) gives a better example.

where g is an algebraic function of w' and of l-functions of w and z, with $s-1$ monomials of order r which involve w.

If θ is an exponential, e^v, where v is of order $r-1$, we write (4) in the form

$$v - \log g(w', w, z) = 0. \tag{5}$$

If θ is an integral, that is, a monomial of the second class, we let (4) stand. We let the first member of the equation

$$G(w', w, z) = 0 \tag{6}$$

represent the first member of (4), or that of (5), according as θ is an integral or an exponential.

On differentiating (6) we find, remembering that $w'' = \chi(z)w$,

$$\chi(z)wG_{w'} + G_w w' + G_z = 0. \tag{7}$$

Now if the first member of (7) did not vanish identically with respect to w', w and z, we would have an equation of the form (3) involving at most $s-1$ monomials of order r which contain w. For, in (4) and (5), the partial derivatives of θ and v are of order less than r, and the derivatives of g and $\log g$ contain at most $s-1$ monomials of order r involving w.* Hence the first member of (7) vanishes identically.

Let μ be any constant. If w is the given solution of the equation $w'' = \chi(z)w$, the derivative with respect to z of

$$G(\mu w', \mu w, z) \tag{8}$$

will evidently be the first member of (7) with w' and w replaced by $\mu w'$ and μw.† As (7) holds for arbitrary values of the variables, the derivative just mentioned must vanish for every z. Hence the function of (8) depends only on μ, and not on z. We write

$$G(\mu w', \mu w, z) = \beta(\mu). \tag{9}$$

* Final remarks of §2, which obviously apply also to $G_{w'}$.

† The device now being used is analogous to, though not identical with, one which underlies Liouville's work on the elementary functions. We use below a method given by us in the Comptes Rendus for Aug. 21, 1926, which permits a great simplification of the work of Liouville and his followers.

We differentiate (9) partially with respect to μ, and then put $\mu=1$. Writing $\beta'(1)=a$, we have

$$w'G_{w'} + wG_w = a, \tag{10}$$

and, as in the case of (7), we see that (10) is an identity in w', w and z.

A particular solution of (10) is $G=a \log w$, and w'/w is a solution of the equation obtained on replacing the second member of (10) by zero. Hence

$$G = a \log w + H\left(\frac{w'}{w}, z\right). \tag{11}$$

As G is algebraic in w', H must be an algebraic function of w'/w and of l-functions of z. By (6) and (11), we have, for w as in the hypothesis,

$$a \log w + H\left(\frac{w'}{w}, z\right) = 0. \tag{12}$$

If $a=0$, we have a contradiction of the assumption that $r>0$.

Suppose that $a\neq 0$. Writing $u=w'/w$ and differentiating (12), we find

$$au + [\chi(z) - u^2]H_u + H_z = 0. \tag{13}$$

This equation must be an identity in u and z, else it would be an equation (3) with $r=0$. But we shall show that no H which involves u algebraically can satisfy (13), thus making untenable, finally, the assumption that $r>0$.

If H is algebraic in u, any of its branches has, for the neighborhood of $u=\infty$, a development in descending powers of u,

$$H = \alpha_1(z)u^{p_1} + \alpha_2(z)u^{p_2} + \cdots + \alpha_n(z)u^{p_n} + \cdots, \tag{14}$$

where the decreasing exponents p_n are rational, and all have the same denominator, which may be unity. We may, and shall, assume that no $\alpha(z)$ is zero for every z.

If $p_1\neq 0$, the development of $(\chi-u^2)H_u$ begins with a term in u^{p_1+1}, whereas, in H_z, the first exponent does not exceed p_1. Hence the first term of $(\chi-u^2)\,H_u$ must balance with au

in (13), an impossibility if $p_1 \neq 0$. If $p_1 = 0$, $(\chi - u^2)H_u$ begins with u^{p_2+1}, and H_z begins with a zero or negative power. Now, as $a \neq 0$, we have to balance au in (13). This is impossible, because $p_2 + 1 < 1$. Thus no H algebraic in u satisfies (13).

We have proved that $r = 0$ for the first member of (3). If, now, (3) does not involve w', then, since it is algebraic in w, we can solve for w, finding w to be an l-function of z. If (3) does involve w', we solve for w', finding

$$w' = H(w,z), \tag{15}$$

where H is an algebraic function of w, and of l-functions of z.

On differentiating (15), remembering that $w'' = \chi(z)w$, we find

$$\chi(z)w = H_w H + H_z. \tag{16}$$

If (16) is not an identity in w and z, w is determined as an l-function of z. Let, then, (16) be an identity, and consider, for the neighborhood of $w = \infty$, a development of H in descending powers,

$$H = \alpha_1(z)w^{p_1} + \alpha_2(z)w^{p_2} + \cdots + \alpha_n(z)w^{p_n} + \cdots .$$

As the coefficients α are found by algebraic operations from the equation which H satisfies together with w and l-functions of z, the α's are all l-functions.

When we substitute the development of H into (16), we find that $p_1 = 1$ and that

$$\alpha_1'(z) + [\alpha_1(z)]^2 = \chi(z).$$

Thus, $\alpha_1(z)$ is a solution of the Riccati equation $u' + u^2 = \chi(z)$, obtained from $w'' = \chi(z)w$ by putting $u = w'/w$. Hence the exponential of the integral of $\alpha_1(z)$, which is an l-function not identically zero, satisfies the equation $w'' = \chi(z)w$.

But it is well known, and easy to see, that when a single solution of a linear homogeneous equation of the second order is known, the general solution can be found from it by a quadrature. Hence every solution of (A) is an l-function, and the theorem is proved.

COLUMBIA UNIVERSITY

ON SMALL DEFORMATIONS OF CURVES

BY C. E. WEATHERBURN

1. *Introduction.* This paper is concerned with small deformations of a single tortuous curve, of a family of curves on a given surface, and of a congruence of curves in space. In all cases, the displacement $\mathbf{s}$ is supposed to be a small quantity of the first order, quantities of higher order being negligible.*

2. *Single Twisted Curve.* Consider first a given curve in space. The position vector $\boldsymbol{r}$ of a point on the curve may be regarded as a function of the arc-length s of the curve, measured from a fixed point on it. Let $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$ be the unit tangent, principal normal and binormal. These are connected with the curvature κ and the torsion τ as in the Serret-Frenet formulas. Imagine a small deformation of the curve, such that the point of the curve originally at $\boldsymbol{r}$ suffers a small displacement $\mathbf{s}$, its new position vector $\boldsymbol{r}_1$ being then

$$\boldsymbol{r}_1 = \boldsymbol{r} + \mathbf{s}. \tag{1}$$

Let a suffix unity be used to distinguish quantities belonging to the deformed curve, and let primes denote differentiations with respect to the arc-length s. Then the element $d\boldsymbol{r}_1$ of the deformed curve, corresponding to the element $d\boldsymbol{r}$ of the original, is given by $d\boldsymbol{r}_1 = d\boldsymbol{r} + d\mathbf{s}$, and its length ds_1 by

$$(ds_1)^2 = (d\boldsymbol{r}_1)^2 = (d\boldsymbol{r})^2 + 2d\boldsymbol{r} \cdot d\mathbf{s} = ds^2(1 + 2\boldsymbol{t} \cdot \mathbf{s}').$$

Consequently $ds_1 = ds(1 + \boldsymbol{t} \cdot \mathbf{s}')$.

The quantity $\boldsymbol{t} \cdot \mathbf{s}'$ represents the increase of length per unit length of the curve, or the extension of the curve at

*See also a paper by Perna, Giornale di Matematiche, vol. 36 (1898), pp. 286–299; and another by Salkowski, Mathematische Annalen, vol. 66 (1908), pp. 517-557.

the point considered. Let it be denoted by ϵ. Then

$$ds_1 = ds(1 + \epsilon) \; ; \quad ds = ds_1(1 - \epsilon). \tag{2}$$

The unit tangent $\boldsymbol{t}_1$ to the deformed curve is given by

$$\boldsymbol{t}_1 = \frac{d\boldsymbol{r}_1}{ds_1} = (1 - \epsilon)(\boldsymbol{r}' + \mathrm{s}') = (1 - \epsilon)\boldsymbol{t} + \mathrm{s}'. \tag{3}$$

Consequently, if κ_1 is the curvature and $\boldsymbol{n}_1$ the unit principal normal for the new curve,

$$\kappa_1\boldsymbol{n}_1 = \frac{d\boldsymbol{t}_1}{ds_1} = (1 - \epsilon)\frac{d\boldsymbol{t}_1}{ds} = (1 - 2\epsilon)\kappa\boldsymbol{n} - \epsilon'\boldsymbol{t} + \mathrm{s}''.$$

On "squaring" both members, and neglecting small quantities of the second order, we have

$$\kappa_1^2 = (1 - 4\epsilon)\kappa^2 + 2\kappa\boldsymbol{n} \cdot \mathrm{s}'',$$

and therefore

$$\kappa_1 = (1 - 2\epsilon)\kappa + \boldsymbol{n} \cdot \mathrm{s}''. \tag{4}$$

Inserting this value in the above product $\kappa_1\boldsymbol{n}_1$ we find

$$\left\{\begin{aligned} \boldsymbol{n}_1 &= \boldsymbol{n} - \frac{1}{\kappa}((\boldsymbol{n} \cdot \mathrm{s}'')\boldsymbol{n} + \epsilon'\boldsymbol{t} - \mathrm{s}'') \\ &= \boldsymbol{n} - (\boldsymbol{n} \cdot \mathrm{s}')\boldsymbol{t} + \frac{1}{\kappa}(\boldsymbol{b} \cdot \mathrm{s}'')\boldsymbol{b}. \end{aligned}\right. \tag{5}$$

The unit binormal $\boldsymbol{b}$ to the deformed curve is then given by

$$\left\{\begin{aligned} \boldsymbol{b}_1 &= \boldsymbol{t}_1 \times \boldsymbol{n}_1 = (1 - \epsilon)\boldsymbol{b} + \mathrm{s}' \times \boldsymbol{n} - \frac{1}{\kappa}(\boldsymbol{b} \cdot \mathrm{s}'')\boldsymbol{n} \\ &= \boldsymbol{b} - (\boldsymbol{b} \cdot \mathrm{s}')\boldsymbol{t} - \frac{1}{\kappa}(\boldsymbol{b} \cdot \mathrm{s}'')\boldsymbol{n}. \end{aligned}\right. \tag{6}$$

The torsion τ_1 may then be found by differentiating the unit binormal, using the Serret-Frenet formulas. Thus on differentiating (6), and using the preceding results, we find on reduction

$$\tau_1 = (1 - \epsilon)\tau + \kappa\boldsymbol{b} \cdot \mathrm{s}' + \frac{d}{ds}\left(\frac{1}{\kappa}\boldsymbol{b} \cdot \mathrm{s}''\right). \tag{7}$$

We have thus determined the geometric characteristics of the deformed curve in terms of those of the original curve and the small displacement $\mathbf{s}$. The rectangular trihedron, consisting of the unit tangent, the principal normal, and the binormal, undergoes a small rotation which is represented by the vector

$$\boldsymbol{R} = \frac{1}{\kappa}(\boldsymbol{b} \cdot \mathbf{s}'')\boldsymbol{t} - (\boldsymbol{b} \cdot \mathbf{s}')\boldsymbol{n} + (\boldsymbol{n} \cdot \mathbf{s}')\boldsymbol{b} \tag{8}$$

or

$$\boldsymbol{R} = \frac{1}{\kappa}(\boldsymbol{b} \cdot \mathbf{s}'')\boldsymbol{t} + \boldsymbol{t} \times \mathbf{s}'. \tag{9}$$

The coefficients of t, n, b in (8) represent the small rotations of the trihedron about the tangent, the principal normal, and the binormal, respectively.

An *inextensional deformation* of the curve is one for which ϵ vanishes identically.* If $\mathbf{s}$ is expressed in the form

$$\mathbf{s} = Pt + Qn + Rb,$$

the vanishing of $t \cdot \mathbf{s}'$ gives $P' = \kappa Q$ as the necessary and sufficient condition for inextensional deformation.

3. *Family of Curves on a Surface.* Consider next a family of curves on a given surface. Suppose that the curves of the family suffer a small deformation such that they remain on the same surface. Then the displacement $\mathbf{s}$ at any point is tangential to the surface, and is a function of two parameters that specify the point considered. On any one curve $\mathbf{s}$ is a function of the arc-length s, and the formulas found above hold for the deformed curve.

Other results may be very neatly expressed in terms of the two-parametric differential invariants for the surface, introduced and examined by the author in a recent paper *On differential invariants in geometry of surfaces.*† If ϕ is the value of any function associated with the deformed curve at the point $\boldsymbol{r}+\mathbf{s}$, the new value of this function at

*This case has been considered in some detail by Sannia, Rendiconti di Palermo, vol. 21 (1906), pp. 229–256.

† Quarterly Journal, vol. 50 (1925), pp. 230–269.

the point $\boldsymbol{r}$ originally occupied by this point of the curve is $\phi - \mathbf{s} \cdot \nabla\phi$, where ∇ is the two-parametric differential operator for the surface. Thus after the deformation the unit tangent $\bar{t}$ to the new curve through the point $\boldsymbol{r}$ is, by (3),

$$(10) \quad \begin{cases} \bar{t} = t_1 - s \cdot \nabla t_1 = (1 - \epsilon)t + t \cdot \nabla s - s \cdot \nabla t \\ \quad = (1 - \epsilon)t + \operatorname{curl}(s \times t) - s \operatorname{div} t + t \operatorname{div} s. \end{cases}$$

We have shown elsewhere* that the line of striction of a family of curves with unit tangent $\boldsymbol{t}$ is given by $\operatorname{div} t = 0$. Hence the line of striction of the deformed family has for its equation $\operatorname{div} \bar{t} = 0$, which may be expressed in the form

$$(11) \qquad (1 - \epsilon) \operatorname{div} t - t \cdot \nabla\epsilon - s \cdot \nabla \operatorname{div} t + t \cdot \nabla \operatorname{div} s = 0,$$

since the divergence of the curl of the normal vector $s \times t$ vanishes identically.†

If the original family of curves is one of parallels, $\operatorname{div} t$ vanishes identically.‡ Hence *a family of parallels will remain parallels after the deformation provided*

$$(12) \qquad t \cdot \nabla(\epsilon - \operatorname{div} s) = 0.$$

The geodesic curvature of a curve of the original family§ is $n \cdot \operatorname{curl} t$, and that of one of the deformed curves is

$$n \cdot \operatorname{curl}\left[(1 - \epsilon)t + t \cdot \nabla s - s \cdot \nabla t\right].$$

If then the family of curves is a family of geodesics, it will remain so after the deformation provided

$$n \cdot [t \times \nabla\epsilon + \operatorname{curl}(t \cdot \nabla s - s \cdot \nabla t)] = 0.$$

Similarly, the original family will be *lines of curvature*‖ if $t \cdot \operatorname{curl} t = 0$; and they will remain lines of curvature after the change provided $\bar{t} \cdot \operatorname{curl} \bar{t} = 0$.

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* *Some new theorems in geometry of a surface*, § 2, Mathematical Gazette, vol. 13 (1926), pp. 1–6.

† Quarterly Journal, loc. cit., § 7.

‡ See a paper by the author *On families of curves and surfaces*, § 7, recently communicated to the Messenger of Mathematics.

§ Quarterly Journal, loc. cit., § 8.

‖ Mathematical Gazette, loc. cit., § 4.

4. *Congruence of Curves in Space.* Consider finally a small deformation of a congruence of curves in three dimensions. On any one curve the displacement $\mathbf{s}$ is a function of a single parameter; but on the congruence it is a point-function in space. Let ∇ now represent the three-parametric differential operator for three dimensions. Then the extension ϵ is $\epsilon = \boldsymbol{t} \cdot \mathbf{s}' = \boldsymbol{t} \cdot (\nabla \mathbf{s}) \cdot \boldsymbol{t}$, and, by the same argument as above, the unit tangent $\bar{\boldsymbol{t}}$ to the curve of the congruence which passes through the point $\boldsymbol{r}$ after the deformation is $\bar{\boldsymbol{t}} = (1 - \epsilon)\boldsymbol{t} + \boldsymbol{t} \cdot \nabla \mathbf{s} - \mathbf{s} \cdot \nabla \boldsymbol{t}$, which may also be expressed in the same form as (10).

The surface of striction (or orthocentric surface) of a congruence with unit tangent $\boldsymbol{t}$ is given by* $\operatorname{div} \boldsymbol{t} = 0$. Hence the surface of striction of the deformed congruence has for its equation $\operatorname{div} \bar{\boldsymbol{t}} = 0$, which may be expanded in the form $(1 - \epsilon) \operatorname{div} \boldsymbol{t} - \boldsymbol{t} \cdot \nabla \epsilon - \mathbf{s} \cdot \nabla \operatorname{div} \boldsymbol{t} + \boldsymbol{t} \cdot \nabla \operatorname{div} \mathbf{s} = 0$, since the divergence of the curl now vanishes identically. The limit surface† of the congruence before deformation is

$$\operatorname{div} (\boldsymbol{t} \operatorname{div} \boldsymbol{t} - \boldsymbol{t} \cdot \nabla \boldsymbol{t}) = 0, \dagger \tag{13}$$

and that of the deformed congruence is given by a similar equation in which $\bar{\boldsymbol{t}}$ takes the place of $\boldsymbol{t}$. The deformed curves will constitute a normal congruence provided we have

$$\bar{\boldsymbol{t}} \cdot \operatorname{curl} \bar{\boldsymbol{t}} = 0,$$

and they will constitute an isometric congruence if, in addition, $\bar{\boldsymbol{t}}$ satisfies the equation§

$$\operatorname{curl} (\bar{\boldsymbol{t}} \operatorname{div} \bar{\boldsymbol{t}} - \bar{\boldsymbol{t}} \cdot \nabla \bar{\boldsymbol{t}}) = 0 \tag{14}$$

which may be expanded in terms of $\boldsymbol{t}$ and $\mathbf{s}$.

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* See a paper by the author *On congruences of curves*, § 6, recently communicated to the Tôhoku Mathematical Journal.

† Ibid., § 4.

‡ This equation for the limit surface of a curvilinear congruence was first given by the author in a paper *On isometric systems of curves and surfaces*, recently communicated to the American Journal.

§ Ibid., § 6.

INTEGERS REPRESENTED BY POSITIVE TERNARY QUADRATIC FORMS*

BY L. E. DICKSON

1. *Introduction.* Dirichlet† proved by the method of §2 the following two theorems:

THEOREM I. *$A=x^2+y^2+z^2$ represents exclusively all positive integers not of the form $4^k(8n+7)$.*

THEOREM II. *$B=x^2+y^2+3z^2$ represents every positive integer not divisible by 3.*

Without giving any details, he stated that like considerations applied to the representation of multiples of 3 by B. But the latter problem is much more difficult and no treatment has since been published; it is solved below by two methods.

Ramanujan‡ readily found all sets of positive integers a, b, c, d such that every positive integer can be expressed in the form $ax^2+by^2+cz^2+du^2$. He made use of the forms of numbers representable by

$$\begin{aligned} A, B, C &= x^2+y^2+2z^2, \\ D &= x^2+2y^2+2z^2, \\ E &= x^2+2y^2+3z^2, \\ F &= x^2+2y^2+4z^2, \\ G &= x^2+2y^2+5z^2. \end{aligned}$$

He gave no proofs for these forms and doubtless obtained his results empirically. We shall give a complete theory for these forms. These cases indicate clearly methods of procedure for any similar form.

For a new theorem on forms in n variables, see §9.

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* Presented to the Society, December 31, 1926.

† Journal für Mathematik, vol. 40 (1850), pp. 228–32; French translation Journal de Mathématiques, (2), vol. 4 (1859), pp. 233–40; Werke, vol. II, pp. 89–96.

‡ Proceedings of the Cambridge Philosophical Society, vol. 19 (1916–19), pp. 11–15. He overlooked the fact that $x^2+2y^2+5z^2+5u^2$ fails to represent 15.

2. *The Form B.* Let B represent a multiple of 3. Since -1 is a quadratic non-residue of 3, x and y must be multiples of 3. Thus $B=3\beta$, $\beta=3X^2+3Y^2+z^2$. Since $\beta\equiv 0$ or $1 \pmod 3$, β represents no integer $3n+2$. If β is divisible by 3, z is divisible by 3 and B is the product of a like form by 9. We shall prove that β represents every positive integer $3n+1$. These results and Theorem II give

THEOREM III. *$x^2+y^2+3z^2$ represents exclusively all positive integers not of the form $9^k(9n+6)$.*

We shall change the notation from β to f and employ the fact that the only reduced positive ternary forms of Hessian 9 are*

$$f = x^2 + 3y^2 + 3z^2, \qquad g = x^2 + y^2 + 9z^2,$$
$$h = x^2 + 2y^2 + 5z^2 - 2yz, \qquad l = 2x^2 + 2y^2 + 3z^2 - 2xy.$$

No one of g, h, l represents an integer $8m+7$. For g this follows from Theorem I, since $g\equiv A \pmod 8$. Suppose $l\equiv 7 \pmod 8$. Then z is odd, $2s\equiv 4 \pmod 8$, where $s=x^2+y^2-xy$. Thus s is even and $(1+x)(1+y)\equiv 1 \pmod 2$, x and y are even, and $s\equiv 0 \pmod 4$, a contradiction. Finally, let $h\equiv 7 \pmod 8$. If y is even, $h\equiv x^2+z^2 \pmod 4$. Hence y is odd and

$$3\equiv h\equiv x^2+(z-1)^2+1 \pmod 4,$$

so that x and $z-1$ are odd. Write $z=2Z$. Then $h\equiv 3+4Z(Z-1)\equiv 3 \pmod 8$.

Consider the ternary form lacking the term xy:

$$(1) \qquad \phi = ax^2 + by^2 + cz^2 + 2ryz + 2sxz.$$

Its Hessian H is $a\Delta-bs^2$, where $\Delta=bc-r^2$. Take $H=9$, $s=1$, $\Delta=24t$, $t=6k+1$. Then $b=3\beta$, $\beta=8at-3$. If a is not divisible by 3, $\beta=48ak+8a-3$ is a linear function of k with relatively prime coefficients and hence represents an infinitude of primes.

* Eisenstein, Journal für Mathematik, vol. 41 (1851), p. 169. By the Hessian H of ϕ we mean the determinant whose elements are the halves of the second partial derivatives of ϕ with respect to x, y, z. Eisenstein called $-H$ the determinant of ϕ. The facts borrowed in this paper from Eisenstein's table have been verified independently by the writer.

Take $a=3n+1$. Then $\beta\equiv -1 \pmod 6$,

$$(2)\qquad \left(\frac{-3}{\beta}\right) = -1, \quad \left(\frac{2}{\beta}\right) = -1,$$
$$\left(\frac{t}{\beta}\right) = \left(\frac{\beta}{t}\right) = \left(\frac{-3}{t}\right) = 1, \quad \left(\frac{-\Delta}{\beta}\right) = 1.$$

Hence $w^2\equiv -\Delta \pmod \beta$ is solvable. We can choose a multiple r of 3 such that $r\equiv w \pmod \beta$. Then $(\Delta+r^2)/b$ is an integer c. Since ϕ represents $b\equiv 7 \pmod 8$, it is equivalent to no one of g, h, l and hence is equivalent to f. Thus f represents every $a=3n+1$.

THEOREM IV. *$x^3+3y^2+3z^2$ represents exclusively all positive integers not of the form $9^k(3n+2)$.*

This theory for B made use of forms of the larger Hessian 9. We shall next show how to deduce a theory making use only of forms having the same Hessian 3 as B.

3. *A New Theory for B.* We proved that f is equivalent to a form (1) having $a=3n+1$, $b=3\beta$, $r=3\rho$, $s=1$, where $\beta=8at-3$ is a prime. In $9=H=a(bc-r^2)-3\beta$, replace β by its value. Thence $c=(8t+3\rho^2)/\beta\equiv 1 \pmod 3$, $c=1+3\gamma$. In (1) replace x by $X-z$. We get

$$\psi = aX^2 - 6nXz + 3(n+\gamma)z^2 + 3\beta y^2 + 6\rho yz.$$

Write $3z=Z$, $3y=Y$. Then

$$3\psi = 3aX^2 - 6nXZ + (n+\gamma)Z^2 + \beta Y^2 + 2\rho YZ$$

is equivalent to $3f=3x^2+Y^2+Z^2$ and hence to B. In 3ψ, replace a by α, β by b, ρ by r. We conclude that (1) is equivalent to B if

$$(3)\qquad a = 3\alpha, \quad \alpha = 3n+1, \quad b = 8\alpha t - 3,$$
$$t = 6k+1, \quad s = -3n = 1-\alpha.$$

We shall now give a direct proof that there exists a form (1) of Hessian 3 which satisfies conditions (3) and is equivalent to B. In $H=3$, replace a and s by their values in (3). We get

$$b + 3 + 3\alpha r^2 - \alpha bP = 0, \qquad P = 3c + 2 - \alpha.$$

Replace the first term b by its value in (3), and cancel α. We get

$$8t + 3r^2 - bP = 0, \quad -24t \equiv (3r)^2 \pmod{b}. \tag{4}$$

This congruence is solvable by (2) with β replaced by b. By (4), $8t(1-P) \equiv 0$, $P \equiv 1 \pmod 3$. Hence the value of c determined by P is an integer. The only two reduced forms of Hessian 3 are B and $\chi = x^2 + 2\sigma$, where $\sigma = y^2 + yz + z^2$. Suppose $\chi \equiv 5 \pmod 8$. Then x is odd and $\sigma \equiv 2 \pmod 4$. Thus

$$(1 + y)(1 + z) \equiv 1, \quad y \equiv z \equiv 0 \pmod 2, \quad \sigma \equiv 0 \pmod 4.$$

This contradiction shows that b is not represented by χ. Since (1) represents b, it is not equivalent to χ and hence is equivalent to B. Thus B as well as ϕ represents* $a = 3\alpha$. This completes the new proof of Theorem III by using only forms of Hessian 3. The numbers represented by χ are given by Theorem XI.

4. *The Form* $C = x^2 + y^2 + 2z^2$. By Theorem I, A represents every positive $4k+2$. Then just two of x, y, z are odd, say x and y, while $z = 2Z$. Then $x = X + Y$, $y = X - Y$ determine integers X and Y. Hence

$$X^2 + Y^2 + 2Z^2 = 2k + 1,$$

so that C represents all positive odd integers.† If

$$m \neq 4^k(8n+7), m = X^2 + Y^2 + z^2,$$

by Theorem I. Hence

$$2m = (X + Y)^2 + (X - Y)^2 + 2z^2.$$

Conversely, if C is even, it is of the latter form.

THEOREM V. *C represents exclusively all positive integers not of the form* $4^k(16n+14)$.

5. *The Form* $D = x^2 + 2y^2 + 2z^2$. If m is odd and $\neq 8n+7$, $m = x^2 + Y^2 + Z^2$ by Theorem I. Then $x + Y + Z$ is odd. Permuting, we may take x odd, and write $Y + Z = 2y$, $Y - Z = 2z$.

* If we apply the method of §2 when $H = 3$ and hence take $s = 1$, $b = 3\beta$, where β is a prime $\equiv -1 \pmod 8$, we find that it fails for all choices of Δ.

†Lebesque, Journal de Mathématiques, (2), vol. 2 (1857), p. 149, gave a long proof by the method of §2.

Then $m=D$. Next, let $m=2r$ be any even integer not of the form $4^k(8n+7)$. Then $r\neq 4^t(16n+14)$. By Theorem V, r is represented by C. Then $m=2r$ is represented by D with x even.

THEOREM VI. *D represents exclusively* all positive integers not of the form $4^k(8n+7)$.*

6. *The Form $F=x^2+2y^2+4z^2$.* Every odd integer is represented by C with $x+y$ odd, whence one of x and y is even. Any integer $\neq 4^k(8n+7)$ is represented by D, and $2D=(2y)^2+2x^2+4z^2$.

THEOREM VII. *F represents exclusively† all positive integers not of the form $4^k(16n+14)$.*

The simple methods used in proving Theorems V–VII apply also to $x^2+2^ry+2^sz$ when r and s are both $\leqq 3$, and when $r=1$ or 3, $s=4$.

7. *The Form $G=x^2+2y^2+5z^2$.* The only reduced forms of Hessian 10 are G, $J=x^2+y^2+10z^2$, $K=2x^2+2y^2+3z^2+2xz$, and $L=2x^2+2y^2+4z^2+2yz+2xz+2xy$. Neither J nor K represents a number of the form $2(8n+3)$. For, if K is even, $z=2Z$, $K=2M$, $M=X^2+y^2+5Z^2$, where $X=x+Z$. Since M is congruent to a sum of three squares modulo 4, it is congruent to 3 if and only if each square is odd, and then $M\equiv 7 \pmod 8$. If J is even, $x=y+2t$, $J=2N$, $N=(y+t)^2+t^2+5z^2\not\equiv 3 \pmod 8$.

We now apply the method of §2 to prove that G represents every positive integer prime to 10. Take $\Delta=16k$, $k=10l\pm 3$. Then $b=2\beta$, where $\beta=8ka-5$ represents infinitely many primes. Now

$$\left(\frac{-\beta}{k}\right)=\left(\frac{5}{k}\right)=\left(\frac{k}{5}\right)=-1,$$

$$\left(\frac{-\Delta}{\beta}\right)=-\left(\frac{k}{\beta}\right)=-\left(\frac{-\beta}{k}\right)=1.$$

Since (1) represents b, which is of the form $2(8n+3)$, it is not equivalent to J or K. Since it represents the odd a, it is not

* $D=x^2+(y+z)^2+(y-z)^2\neq 4^k(8n+7)$ by Theorem I.

† If F is even, x is even and F is the double of a form D.

equivalent to L. Hence (1) is equivalent to G, which therefore represents a.

If G represents a multiple of 5, it is the product of 5 by $g=5X^2+10Y^2+z^2$, whence G represents no $5(5n\pm2)$. Also, g is divisible by 5 only when z is. Thus G is divisible by 25 only when it is a product of 25 by a form like G.

To prove* that G represents every 5α if $\alpha=5n\pm1$ is odd, employ (1) with $a=5\alpha$, $b=2\beta$, $\beta=8\alpha t-5$, $r=2\rho$, $s=1\mp\alpha$. The Hessian of (1) is 10 if

$$\beta + 5 + 10\alpha\rho^2 - \alpha\beta P = 0, \qquad P = 5c \pm 2 - \alpha.$$

Take t prime to 10 and replace the first β by its value. Thus $8t+10\rho^2-\beta P=0$, $P\equiv\pm1\pmod 5$. Hence P yields an integral value for c. Also,

$$\left(\frac{t}{\beta}\right) = \left(\frac{-\beta}{t}\right) = \left(\frac{5}{t}\right),$$

$$\left(\frac{5}{\beta}\right) = \left(\frac{\beta}{5}\right) = \left(\frac{\pm 8t}{5}\right) = -\left(\frac{t}{5}\right), \tag{5}$$

$$\left(\frac{-80t}{\beta}\right) = -\left(\frac{5t}{\beta}\right) = 1.$$

Next, if G is even, $x=z+2w$ and $G=2T$, where

$$T = y^2 + 2w^2 + 2wz + 3z^2, \qquad S = x^2 + y^2 + 5z^2$$

are the only reduced forms of Hessian 5. Every positive integer a prime to 5 is represented by T. Take $\Delta=8k, k=10m\pm1$. Then $b=a\Delta-5$ represents an infinitude of primes, and

$$\left(\frac{-2}{b}\right) = 1, \quad \left(\frac{-\Delta}{b}\right) = \left(\frac{k}{b}\right) = \left(\frac{-b}{k}\right) = \left(\frac{5}{k}\right) = \left(\frac{k}{5}\right) = 1.$$

Now $b\equiv3\pmod 8$ is not represented by S, as proved for M. Hence (1) is equivalent to T and not S. Thus T represents a.

* Or we may use the method of §2. Of the ten properly primitive reduced forms of Hessian 50, all except g fail to represent numbers $\equiv14\pmod{16}$. To prove that g represents $\alpha=5n\pm1$ when odd, take $\Delta=80t$, whence $b=10\beta$, $\beta=8\alpha t-5$; apply (5). From this proof was reconstructed the shorter one in the text.

We saw that $2T=G$ represents no $5(5m\pm2)$. Thus T represents no $5(5n\pm1)$. To prove that T represents every 5α, where $\alpha=5n\pm2$, employ (1) with $a=5\alpha$, $s=1\pm2\alpha$. Its Hessian is 5 if

$$b + 5 + 5\alpha r^2 - \alpha bP = 0, \qquad P = 5c - 4\alpha \pm 4.$$

Replace the first term b by $8t\alpha-5$, where t is prime to 10. Thus $8t+5r^2-bP=0$. Hence $8t(1\mp2P)\equiv0$, $P\equiv\pm3(\text{mod } 5)$, and the resulting value of c is an integer. Also

$$\left(\frac{5}{b}\right)=\left(\frac{b}{5}\right)=\left(\frac{\pm 16t}{5}\right)=\left(\frac{t}{5}\right),$$

$$\left(\frac{t}{b}\right)=\left(\frac{-b}{t}\right)=\left(\frac{5}{t}\right), \quad \left(\frac{-40t}{b}\right)=1.$$

We have now proved the two theorems:

THEOREM VIII. *T represents exclusively all $\neq 25^k(25n\pm5)$.*

THEOREM IX. *G represents exclusively all $\neq 25^k(25n\pm10)$.*

8. *The Form $E=x^2+2y^2+3z^2$.* We shall outline a proof of the following theorem.

THEOREM X. *E represents exclusively all $\neq 4^k(16n+10)$.*

The only reduced forms of Hessian 6 are E and

$$Q = x^2 + y^2 + 6z^2, \qquad R = 2x^2 + 2y^2 + 2z^2 + 2xy.$$

To prove that E represents every positive odd integer a, take $\Delta=9k$, $k=8t+3$. Then $b=3\beta$, where β represents primes. Also $(-\Delta/\beta)=1$. The resulting form (1) represents the odd a and hence is not equivalent to R. Since it represents $b=3(3n+1)$, it is not equivalent to Q. For, if Q is divisible by 3, both x and y are.

If E is even, then $x=z+2t$ and $E=2U$,

$$U=y^2+2z^2+2zt+2t^2$$

and conversely. In place of U we employ the like form χ of §3. To show that χ represents $a=2\alpha$ when α is odd, take $\Delta=9k$, $k=4t+1$. Then $b=a\Delta-3\equiv6(\text{mod } 9)$ is not represented by the remaining reduced form B of Hessian 3 (Theorem III). Also $b=3\beta$, $(-\Delta/\beta)=+1$.

Finally, χ represents every positive odd integer $a \not\equiv 5 \pmod 8$. Write $\alpha = \frac{1}{2}(3a-1)$ and take $\Delta = 9k$, $k = 2h+1$. Then $b = 6q$, $q = 3ah+\alpha$. If $a = 8A+1$, take $h = 4t$, t odd. Then $q = 12Ak + 12t + 1$,

$$\left(\frac{-\Delta}{q}\right) = \left(\frac{k}{q}\right) = \left(\frac{q}{k}\right) = \left(\frac{12t+1-k}{k}\right) = \left(\frac{4t}{k}\right) = \left(\frac{k}{t}\right) = 1.$$

If $a = 8A+3$, take $h = 4t+1$. If $a = 8A+7$, take $h = 4t+1$. In each case $(-\Delta/q) = 1$. In all three cases, q represents an infinitude of primes.

THEOREM XI. *$x^2+2y^2+2yz+2z^2$ represents exclusively all positive integers not of the form $4^k(8n+5)$.*

9. *Forms in n Variables.* By a simple modification of Ramanujan's determination of quaternary forms which represent all positive integers, we readily prove*

THEOREM XII. *If, for $n \geqq 5$, $f = a_1x_1^2 + \cdots + a_nx_n^2$ represents all positive integers, while no sum of fewer than n terms of f represents all positive integers, then $n = 5$ and*

$$f = x^2 + 2y^2 + 5z^2 + 5u^2 + ev^2, \quad (e = 5, 11, 12, 13, 14, 15),$$

and these six forms f actually have the property stated.

After this paper was in type, I saw that J. G. A. Arndt gave† the Dirichlet type of proof which appears in §2 above, but not the improved new proof of §3. For the form G of §7, he treats only numbers not divisible by 5.

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*Note to appear in Proceedings of the National Academy.

†*Ueber die Darstellung ganzer Zahlen als Summen von sieben Kuben*, Dissertation, Göttingen, 1925.

A DIOPHANTINE AUTOMORPHISM*

BY E. T. BELL

1. *Introduction.* We define a *diophantine automorphism* of a form $f \equiv f(x, y, \cdots, w)$ to be a transformation of f into f^n, where $n > 1$ is an integer, by means of a substitution of the type

$$x = X(x, y, \cdots, w), \qquad y = Y(x, y, \cdots, w), \cdots,$$
$$w = W(x, y, \cdots, w),$$

so that

$$f(X, Y, \cdots, W) = f^n(x, y, \cdots, w),$$

where $X, Y, \cdots, W$ are rational integral functions, with rational integral coefficients, of $x, y, \cdots, w$ and the coefficients of f.

Obviously powers of norms of algebraic integers, powers of sums of 4 or 8 squares, the generalizations of the latter which are composable forms and, generally, any power greater than the first of any composable form, have the automorphic property just defined. Excluding powers of composable forms we shall call any other form having a diophantine automorphism *proper*.

Examples of proper forms are extremely rare; their interest for diophantine analysis is evident. Apparently there are known but two instances, the discriminant of a binary cubic, this observation being due to Eisenstein† (see §9), and Cayley's‡ generalization of this to a certain octenary quartic with numerical coefficients $(1, -2, 4)$.

Being given a proper automorphism for a form with numerical coefficients, we devise a method of generalization

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* Presented to the Society, San Francisco Section, October 30, 1926.

† F. M. G. Eisenstein, Crelle's Journal, vol. 27 (1844), p. 105.

‡ A. Cayley, Collected Mathematical Papers, vol. 1, pp. 89–91; p. 113; p. 353; p. 532.

which yields a proper form with literal coefficients, whose automorphism degenerates to that of the given form when each of the literal coefficients is replaced by unity. The method leads to systems of linear diophantine equations and inequalities whose solution gives the required generalization. Both Eisenstein's and Cayley's automorphisms can be generalized in this way. The ordinary diophantine problem for Cayley's is so complicated, however, (leading as one detail to a system of 76 equations and inequalities in 8 variables), that we shall reproduce here only the work for Eisenstein's. The method extends and systematizes that used in a former paper* to generalize the 8 square identity, and it is applicable to all problems discussed there.

2. *Matric Exponents and Multipliers of a Transformation.* Let a, b, c be constant integers >0, and $u_j, v_j, r_j (j=1, \cdots, g)$ variable integers $\geqq 0$. Let $\phi_j (j=1, \cdots, g)$ be independent parameters. Write, symbolically,

$$\prod_{j=1}^{g} \phi_j^{au_j} \equiv \phi^{au},$$

and call au the *matric exponent* of ϕ. Let $\phi^{au}, \phi^{bv}, \phi^{cr}$ be any three such symbolic powers between which there is the relation

$$\phi^{au}\phi^{bv} = \phi^{cr},$$

the indicated multiplication on the left being as in common algebra, so that

$$au_j + bv_j = cr_j, \qquad (j = 1, \cdots, g).$$

Then we shall write these g equations as the single equation

$$au + bv = cr$$

between matric exponents, and similarly for matric equations in any number of variables $u, v, r, s, \cdots, x$.

The laws of combination of such equations are abstractly identical with those for ordinary linear systems, being merely a translation of the laws of algebraic exponents. It is unneces-

* E. T. Bell, Annals of Mathematics, (2), vol. 27 (1925), p. 99.

sary to elaborate them formally as they will be clear from the application made later.

Let $x, y, \cdots, z$ be independent variables, and $a, b, \cdots, c$, $u, v, \cdots, r, \phi$ as before. Then $\phi^{au}, \phi^{bv}, \cdots, \phi^{cr}$ will be called the *multipliers* of the transformation which replaces $x, y, \cdots, z$ by $\phi^{au}x, \phi^{bv}y, \cdots, \phi^{cr}z$ respectively.

In generalizing automorphisms we deal with a great many multipliers, matric exponents and variables simultaneously, and it is a saving of labor, also a suggestive impulse to the algebra to use a device which will show at a glance the specific variable to which a given exponent or multiplier appertains.

We shall write $\phi^x, \phi^y, \cdots, \phi^z$ for the multipliers appertaining to the variables $x, y, \cdots, z$. The exponents of these multipliers are $x, y, \cdots, z$. Finally, the multipliers will be denoted by their exponents. Thus $x, y, \cdots, z$ will denote either variables, the respective multipliers of these variables, or the matric exponents of the latter. No confusion can result from this triple interpretation of $x, y, \cdots, z$, as in each case it will be stated what the letters are. This supple device enables us to follow the multiplicative transformations of a set of variables without elaborate notations and with a minimum of algebra. It also reduces the particular kind of diophantine problem occurring in the subject from degree $n>1$ to degree 1.

3. *Eisenstein's Automorphism.* As we start from Eisenstein's identity we transcribe it here. Write

$$f(x,y,z,w) \equiv x^2w^2 - 3y^2z^2 + 4xz^3 + 4wy^3 - 6xyzw,$$

$$X \equiv 3xyz - x^2w - 2y^3$$

$$Y \equiv 2xz^2 - xyw - y^2z,$$

$$Z \equiv xzw - 2y^2w + yz^2,$$

$$W \equiv xw^2 - 3yzw + 2z^3,$$

so that

$$X = -\frac{1}{2}\frac{\partial f}{\partial w}, \quad Y = \frac{1}{6}\frac{\partial f}{\partial z}, \quad Z = -\frac{1}{6}\frac{\partial f}{\partial y}, \quad W = \frac{1}{2}\frac{\partial f}{\partial x}.$$

The automorphism is then

$$f(X,Y,Z,W) = f^3(x,y,z,w).$$

This striking result is today evident in many ways, for example by observing the discriminant of the cubicovariant of the binary cubic of which $f(x, y, z, w)$ is the discriminant. Nevertheless Eisenstein's remarks on it still stand, as no proof has indicated a method of attack on the general problem of finding all composable forms or upon the equally (or more) difficult one of diophantine automorphism which his theorem illustrates, and of which it was the first example. He says:

"Es scheint, man müsse dieser Gattung von identischen Gleichungen eine um so grössere Wichtigkeit zuschreiben, als man keine allgemeine Methode zu ihrer Auffindung besitzt, und weil dieselben namentlich in der Zahlenlehre oft die Grundlage zu ganzen Theorieen bilden; in welcher Hinsicht sich auf die Theorieen der quadratischen und ternären Formen, so wie auf diejenige der Zerlegung einer Zahl in die Summe von vier Quadraten und andere verweisen lasst."

It was proved by Dickson* that f is proper in the sense of §1.

4. *Summary of Extension.* We put here for convenience, in ordinary notation, the conclusion which follows from the subsequent algebra in §8. Let ϕ_j $(j=1, \cdots, g)$ be parameters, l_j, m_j, n_j, r_j arbitrary integers $\geqq 0$, such that

$$2m_j = l_j + n_j + r_j, \qquad (j = 1, \cdots, g),$$

and write

$$\lambda \equiv \prod_{j=1}^{g} \phi_j^{l_j}, \qquad \mu \equiv \prod_{j=1}^{g} \phi_j^{m_j}, \qquad \nu \equiv \prod_{j=1}^{g} \phi_j^{n_j},$$

$$\rho \equiv \prod_{j=1}^{g} \phi_j^{r_j}.$$

Then

$$\begin{aligned}&\left[\mu^2(\lambda xw - yz)^2 - 4\lambda(\mu xz - \rho y^2)(\mu yw - \nu z^2)\right]^3 \\ &= \mu^2(\lambda XW - YZ)^2 - 4\lambda(\mu XZ - \rho Y^2)(\mu YW - \nu Z^2),\end{aligned}$$

* L. E. Dickson, Comptes Rendus du Congrès International des Mathématiciens, Strassbourg, 1921, (pub. Toulouse): *Homogeneous polynomials with a multiplication theorem*, §9. The property of the Hessian noted by Haskell, American Mathematical Monthly, vol. 10 (1903) p. 2 and cited by Dickson, loc. cit., p. 9, was first proved by Cayley.

where

$$X \equiv 3\mu xyz - \lambda\mu x^2w - 2\rho y^3,$$
$$Y \equiv 2\lambda\nu xy^2 - \lambda\mu xyw - \mu y^2z,$$
$$Z \equiv \lambda\mu xzw - \lambda\rho y^2w + \mu yz^2,$$
$$W \equiv \lambda\mu xw^2 - 3\mu yzw + 2\nu z^3.$$

For $0=l_j=m_j=n_j=r_j$ $(j=1, \cdots, g)$ this becomes Eisenstein's identity in §3. An alternative form of the above is

$$\lambda^2[\nu\rho(\lambda xw - yz)^2 - 4(\mu xz - \rho y^2)(\mu yw - \nu z^2)]^3$$
$$= \nu\rho(\lambda XW - YZ)^2 - 4(\mu XZ - \rho Y^2)(\mu YW - \nu Z^2),$$

where λ, μ, ν, ρ are any parameters subject only to the condition

$$\mu^2 = \lambda\nu\rho.$$

For these parameters we have in fact

$$\mu^2(\lambda xw - yz^2) - 4\lambda(\mu xz - \rho y^2)(\mu yw - \nu z^2)$$
$$= \lambda[\nu\rho(\lambda xw - yz)^2 - 4(\mu xz - \rho y^2)(\mu yw - \nu z^2)],$$

and obviously the original λ, μ, ν, ρ (in terms of the ϕ_j) satisfy $\mu^2=\lambda\nu\rho$.

Although it is not of the type discussed in this paper, we may note in passing the following curious identity:

$$(px^2w^2 - 3qy^2z^2 + 4rxz^3 + 4swy^3 - 6txyzw)^3$$
$$= \frac{1}{p}X^2W^2 - \frac{3}{q}Y^2Z^2 + \frac{4}{r}XZ^3 + \frac{4}{s}WY^3 - \frac{6}{t}XYZW,$$

where

$$X \equiv 3txyz - px^2w - 2sy^3,$$
$$Y \equiv 2rxz^2 - txyz - qy^2z,$$
$$Z \equiv txzw - 2sy^2w + qyz^2,$$
$$W \equiv pxw^2 - 3tyzw + 2rz^3,$$

and p, q, r, s, t are any parameters such that

$$(pq - t^2)(rs - qt) = 0.$$

This is a consequence of either set of formulas in §§ 3, 4.

5. *Transformation of f.* Transform f in §3 by

$$\Sigma \equiv \begin{pmatrix} x, & y, & z, & w \\ x\phi^x, & y\phi^y, & z\phi^z, & w\phi^w \end{pmatrix},$$

the exponents of ϕ being matric, into

$$F \equiv Px^2w^2 - 3Qy^2z^2 + 4Rxz^3 + 4Swy^3 - 6Txyzw$$

and let the respective matric exponents of P, Q, R, S, T be p, q, r, s, t. Then

$$\begin{aligned} p &= 2x + 2w, \\ q &= 2y + 2z, \\ r &= x + 3z, \\ s &= w + 3y \\ t &= x + y + z + w, \end{aligned} \tag{1}$$

a set of diophantine matric equations.

Under Σ the matric exponents of the respective terms, in the order in which they occur, in X, Y, Z, W of §3 are

$$\begin{array}{lll} x + y + z\ , & 2x + w, & 3y, \\ x + 2z, & x + y + w, & 2y + z, \\ x + z + w\ , & 2y + w, & y + 2z, \\ x + 2w, & y + z + w\ , & 3z, \end{array} \tag{2}$$

and the multipliers having these respective exponents are

$$\begin{array}{lll} xyz\ , & x^2w\ , & y^3\ , \\ xz^2\ , & xyw\ , & y^2z\ , \\ xzw\ , & y^2w\ , & yz^2\ , \\ xw^2\ , & yzw\ , & z^3\ . \end{array} \tag{3}$$

6. *Conditions for Diophantine Automorphism.* As in the theory of numbers, if a, b, c, $\cdots$ are integral elements of any ray, and the quotient of b by a is an element of the ray, say c, we write $a \mid b$ for "a divides b." Hence $a \mid b$ implies that there exists an integral element c such that $b = ac$. Note that we are concerned throughout with arithmetic division and not merely algebraic. As in algebra a/b is the result of dividing a by b.

In order that F of §5 shall have a diophantine automorphism it is sufficient by (3) that the following 12 relations of divisibility for multipliers shall subsist,

$$
\begin{aligned}
&x \mid (xyz,\ x^2w,\ y^3),\\
&y \mid (xz^2,\ xyw,\ y^2z),\\
&z \mid (xzw,\ y^2w,\ yz^2),\\
&w \mid (xw^2,\ yzw,\ z^3).
\end{aligned} \tag{4}
$$

Passing from (4) to matric exponents we see that the following must be integers $\geqq 0$:

$$
\begin{array}{lll}
y+z, & x+w, & 3y-x,\\
x+2z-y, & x+w, & y+z,\\
x+w, & 2y+w-z, & y+z,\\
x+w, & y+z, & 3z-w.
\end{array} \tag{5}
$$

Hence it is sufficient that the following inequalities shall hold in integers (matric exponents) $\geqq 0$,

$$3y \geqq x, \quad x+2z \geqq y, \quad w+2y \geqq z, \quad 3z \geqq w. \tag{6}$$

The problem is therefore reduced to satisfying (1), (6) in integers $x, y, z, w, p, q, r, s, t \geqq 0$, the general solution being required.

7. *Solution of* (1), (6). From (1) we have

$$p+q=2t, \qquad r+s=q+t, \tag{7}$$

and hence from (1)

$$
\begin{aligned}
6y &= 2x+2s-p,\\
3z &= r-x,\\
2w &= p-2x,\\
3q &= 2r+2s-p,\\
3t &= r+s+p,
\end{aligned} \tag{8}
$$

which must be solved in integers $\geqq 0$ subject to (6).

Treat x, r, p, s as parameters. Then, for integrality we must have

$$\begin{aligned} 2x + 2s &\equiv p, && \mod 6, \\ r &\equiv x, && \mod 3, \\ p &\equiv 0, && \mod 2, \\ 2r + 2s &\equiv p, && \mod 3, \\ r + s + p &\equiv 0, && \mod 3. \end{aligned} \tag{9}$$

From the first of (9) we get $p \equiv 0, \mod 2$; hence the third is redundant. Now if $p = 2k$, then $p \equiv 4p, \mod 6$, and hence from the first, $2x+2s \equiv 4p, \mod 6$; thus, by the second, $r+s \equiv 2p, \mod 3$, and this implies the fifth. The set (9) is therefore equivalent to

$$2x + 2s \equiv p, \quad \mod 6, \qquad r \equiv x, \quad \mod 3\ ; \tag{10}$$

and from (6), (8) we get

$$2r \geqq p, \quad 2s \geqq p, \quad 4r + p \geqq 2s, \quad 4s + p \geqq 2r \tag{11}$$

and we have also

$$q = (2r + 2s - p)/3, \qquad t = (r + s + p)/3. \tag{12}$$

The array (5) now becomes

$$\begin{array}{lll} (2r - p + 2s)/6, & p/2, & (-p + 2s)/2, \\ (4r + p - 2s)/6, & p/2, & (2r - p + 2s)/6, \\ p/2, & (-2r + p + 4s)/6, & (2r - p + 2s)/6, \\ p/2, & (2r - p + 2s)/6, & (-p + 2r)/2. \end{array} \tag{13}$$

If (10), (11) are satisfied, it follows from the way in which (13) was obtained that its 12 members are integers $\geqq 0$. To exhibit this solution in terms of ordinary algebra we make some reductions.

8. *Transition to Ordinary Algebra.* By §5, beginning, we have

$$(P, Q, R, S, T) = (\phi^p, \phi^q, \phi^r, \phi^s, \phi^t)\ ;$$

hence the equivalent in terms of multipliers of (13) is

$$(14)\qquad \begin{matrix} (R^2S^2/P)^{1/6}, & P^{1/2}, & (S^2/P)^{1/2}, \\ (PR^4/S^2)^{1/6}, & P^{1/2}, & (R^2S^2/P)^{1/6}, \\ P^{1/2}, & (PS^4/R^2)^{1/6}, & (R^2S^2/P)^{1/6}, \\ P^{1/2}, & (R^2S^2/P)^{1/6}, & (R^2/P)^{1/2}, \end{matrix}$$

and from (12) we have

$$(15)\qquad Q = (R^2S^2/P)^{1/3}, \qquad T = (PRS)^{1/3},$$

and P, R, S are not similarly reducible, since we took x, p, r, s as independent (matric) parameters.

As a check, we should have, by (7),

$$(16)\qquad (PQ - T^2)(RS - QT) = 0,$$

a relation used in what follows, and this is verified by (15).

Applying (16) to (14), we get the array in the form

$$(17)\qquad \begin{matrix} Q^{1/2}, & P^{1/2}, & S/P^{1/2}, \\ R/Q^{1/2}, & P^{1/2}, & Q^{1/2}, \\ P^{1/2}, & S/Q^{1/2}, & Q^{1/2}, \\ P^{1/2}, & Q^{1/2}, & R/P^{1/2}, \end{matrix}$$

whose twelve members must be rational integral functions of parameters together with Q, T as given by (15).

For rationality and integrality (17) demands

$$(18)\qquad Q = Q_1^2, \quad P = P_1^2, \quad S = S_1P_1 = S_2Q_1, \quad R = R_1Q_1 = R_2P_1,$$

the letters with suffixes being new parameters. From (18) we get

$$(19)\qquad \begin{matrix} Q_1^6\,P_1^2 = R^2S^2, & T^3 = RSP_1^2, \\ P_1/Q_1 = S_2/S_1 = R_1/R_2 = K, \end{matrix}$$

where K is a new parameter. Hence

$$(20)\qquad P = K^2Q_1^2, \quad Q = Q_1^2, \quad R = KQ_1R_2, \quad S = KQ_1S_1, \quad T = KQ_1^2,$$

expressing P, Q, R, S, T in terms of 4 parameters K, Q_1, R_2, S_1 between which is the single relation

$$Q_1^2 = KR_2S_1. \tag{21}$$

Finally then, making the change in notation

$$\lambda \equiv K, \qquad \mu \equiv Q_1, \qquad \nu \equiv R_2, \qquad \rho \equiv S_1,$$

we have the results summarized in §§ 3, 4.

The use of the symbolic method is a rapid means for proving the existence or non-existence of a diophantine automorphism; the passage to ordinary algebra is then a simple translation of the matric and multiplier equations of the symbolic solution. Notice that the symbolic method avoids all irrationalities; its equations and all of its processes are carried out in terms of positive integers.

9. *References.* Many of Cayley's papers contain references to Eisenstein's theorem and to his own generalization of it, including the geometry of the related developable surface in homogeneous coordinates. Cayley's form is of interest from another point of view; it is that (an octenary quartic) which occurs incidentally in section 235 of the *Disquisitiones Arithmeticae* in connection with the composition of binary quadratic forms.

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THE CAUCHY-HEAVISIDE EXPANSION FORMULA AND THE BOLTZMANN-HOPKINSON PRINCIPLE OF SUPERPOSITION*

BY F. D. MURNAGHAN

1. *Introduction.* It is the object of the present note to point out that the well known expansion theorem of Heaviside is an immediate corollary of the classical method given by Cauchy for finding a particular solution of a linear, non-homogeneous, ordinary differential equation when the general solution of the corresponding homogeneous equation is known. Various proofs of Heaviside's theorem have been given by Bromwich and Wagner, using contour integrals, by Carson and others and it seems hardly possible that the direct connection with Cauchy's method can have escaped attention. This method is interesting, furthermore, since it furnishes a rigorous proof of the useful Boltzmann-Hopkinson principle of superposition.† The explicit form of the expansion theorem in the case when the characteristic equation has multiple roots is also given.

2. *Cauchy's Method.* Although Cauchy's method is classical and to be found in the older books, it does not seem to find a place in the more modern texts and it will, therefore, be convenient to give an indication of its derivation. Let us suppose

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* Presented to the Society, under a different title, September 9, 1926.

† The following citations may serve to give a proper orientation: Bromwich, T. J., *Normal coordinates in dynamical systems*, Proceedings of the London Society, vol. 15 (1916), pp. 401–448. Wagner, K. W., *Über eine Formel von Heaviside zur Berechnung von Einschaltvorgängen*, Archiv für Elektrotechnik, vol. 4 (1916), pp. 159–193. Carson, J. R., *On a general expansion theorem for the transient oscillations of a connected system*, Physical Review, vol. 10 (1917), pp. 217–225. Carson, J. R., *Theory of the transient oscillations of electrical networks and transmission lines*, Transactions A. I. E. E., vol. 38 (1919), pp. 345–427.

that we wish to find a particular solution of the linear differential equation of the nth order

$$(1) \qquad \phi(D)u = f(t) \; ; \qquad \phi(D) \equiv a_0 D^n + \cdots + a_n,$$

where u is the dependent variable, t the independent variable and $D \equiv d/dt$. The coefficients a are either constants or functions of t. If $(v_1, \cdots, v_n)$ are distinct solutions of the corresponding homogeneous equation

$$(1') \qquad \phi(D)v = 0,$$

its general solution is

$$(2) \qquad v = C_1 v_1 + \cdots + C_n v_n,$$

where the C's are arbitrary constants. We choose these so that v and its derivatives with respect to t up to the $(n-2)$d vanish, when $t=\tau$ the $(n-1)$st derivative taking the value $f(\tau)/a_0(\tau)$; τ being a fixed but arbitrary value of t. To find the proper values for the C's, we have merely to solve a system of n linear equations whose determinant is the Wronskian $|v_1 v_2' \cdots v_n^{(n-1)}|$ and this does not vanish on account of the distinctness of the v's. When the values of the C's as thus determined are entered in (2), v becomes a function of t and τ, and to stress this we may write it as $v(t, \tau)$. Then Cauchy's contention is that the integral

$$(3) \qquad u(t) = \int_0^t v(t,\tau)d\tau$$

is a particular solution of (1). We have, in fact, from (3)

$$Du = \int_0^t Dv(t,\tau)d\tau + v(t,t) = \int_0^t Dv(t,\tau)d\tau,$$

since $v(\tau, \tau)$ vanishes by definition for every value of τ. Similarly

$$D^2u = \int_0^t D^2v(t,\tau)d\tau, \cdots, D^{n-1}u = \int_0^t D^{n-1}v(t,\tau)d\tau,$$

while

$$D^n u = \int_0^t D^n v(t,\tau)d\tau + \left\{D^{n-1}v(t,\tau)\right\}_{\tau=t} = \int_0^t D^n v(t,\tau)d\tau + \frac{f(t)}{a_0(t)},$$

so that

$$\phi(D)u = \int_0^t \phi(D)v(t,\tau)d\tau + f(t) = f(t).$$

The particular solution u of (1) obtained in this way is characterized by the fact that it vanishes with its derivatives up to the $(n-1)$st when $t=0$; the nth derivative having, consequently, the value $f(0)/a_0(0)$ when $t=0$. It is apparent from the definition of the C's occurring in (2) that they may be obtained from the values they would have if $f(\tau)$ were $\equiv 1$ by multiplication by $f(\tau)$. Hence if we denote by $\psi(t,\tau)$ the particular function $v(t,\tau)$ corresponding to the particular problem where $f(t)\equiv 1$ we have $v(t,\tau) \equiv f(t)\psi(t,\tau)$, and (3) may be written in the form

$$(3\text{ bis}) \qquad u(t) = \int_0^t f(\tau)\psi(t,\tau)d\tau.$$

Denoting by $\xi(t,0)$ the particular solution corresponding to $f(t)\equiv 1$ we have

$$\xi(t,0) \equiv \int_0^t \psi(t,\tau)d\tau,$$

which suggests the notation

$$(4) \qquad \xi(t,\tau) \equiv \int_\tau^t \psi(t,s)ds,$$

with the relation

$$\frac{\partial \xi(t,\tau)}{\partial \tau} = -\psi(t,\tau).$$

An integration of (3 bis) by parts then yields

$$(5) \qquad \begin{cases} u(t) = [-f(t)\xi(t,t) + f(0)\xi(t,0)] + \int_0^t \xi(t,\tau)\dfrac{df}{d\tau}d\tau \\ \qquad = f(0)\xi(t,0) + \int_0^t \xi(t,\tau)\dfrac{df}{d\tau}d\tau, \quad \text{since } \xi(t,t) = 0. \end{cases}$$

This is the mathematical statement of the Boltzmann-Hopkinson principle of superposition according to which we are able

to build up from a particular solution $\xi(t, \tau)$ of the differential equation (1) when $f(t) \equiv 1$, a particular solution of this equation when $f(t)$ is an arbitrary function of t. The particular solution obtained in this way is characterized by the fact that it vanishes, together with its derivatives with respect to t up to the $(n-1)$st, when $t=0$.

3. *The Particular Case when the Coefficients of $\phi(D)$ are Constant.* In the first case we shall consider the general case where the zeros of the polynomial $\phi(D)$ are all distinct and we shall write

$$\phi(D) \equiv a_0(D - r_1)(D - r_2) \cdots (D - r_n).$$

We may choose as our n distinct solutions of (1′) the functions

$$v_1 = e^{r_1(t-\tau)}, \cdots, v_n = e^{r_n(t-\tau)},$$

and our equations to determine the constants C of (2) are

$$(6) \qquad \begin{cases} C_1 + \cdots + C_n = 0, \\ r_1C_1 + \cdots + r_nC_n = 0, \\ \cdots\cdots\cdots\cdots\cdots \\ r_1^{n-1}C_1 + \cdots + r_n^{n-1}C_n = \dfrac{f(\tau)}{a_0}. \end{cases}$$

These equations show that $(C_1, \cdots, C_n)$ are the coefficients of the analysis of $f(\tau)/\phi(D)$ into its simple fractions. In fact if we write

$$\frac{f(\tau)}{\phi(D)} \equiv \frac{C_1}{D - r_1} + \cdots + \frac{C_n}{D - r_n},$$

and expand each of the fractions on the right-hand side near $D=\infty$, we find on observing that $D=\infty$ is a zero of order n of the left-hand side, the first term in its expansion being $f(\tau)/(a_0D^n)$, exactly the equations (6) to determine the C's. Furthermore on observing that $D=r_p$ is a simple pole with residue $f(\tau)/\phi'(r_p)$ of the left-hand side we obtain the relations

$$(7) \qquad C_p = \frac{f(\tau)}{\phi'(r_p)}, \qquad (p = 1, \cdots, n).$$

Hence the function $v(t, \tau)$ is

$$v(t,\tau) \equiv f(\tau) \sum_{p=1}^{n} \frac{e^{r_p(t-\tau)}}{\phi'(r_p)},$$

so that

$$\psi(t,\tau) \equiv \sum_{p=1}^{n} \frac{e^{r_p(t-\tau)}}{\phi'(r_p)},$$

and

$$\xi(t,\tau) \equiv \sum_{p=1}^{n} \frac{e^{r_p(t-\tau)}}{r_p\phi'(r_p)} - \sum_{1}^{n} \frac{1}{r_p\phi'(r_p)}.$$

The relation

$$\frac{1}{\phi(D)} \equiv \sum_{p=1}^{n} \frac{1}{\phi'(r_p)(D - r_p)}$$

obtained above shows that

$$\frac{1}{\phi(0)} = -\sum_{p=1}^{n} \frac{1}{r_p\phi'(r_p)},$$

so that we may write

$$(8) \qquad \xi(t,\tau) = \frac{1}{\phi(0)} + \sum_{p=1}^{n} \frac{e^{r_p(t-\tau)}}{r_p\phi'(r_p)},$$

and, in particular,

$$(9) \qquad \xi(t) \equiv \xi(t,0) = \frac{1}{\phi(0)} + \sum_{p=1}^{n} \frac{e^{r_p t}}{r_p\phi'(r_p)}.$$

It is this particular form of Cauchy's result which yields immediately the Heaviside expansion theorem. Before proceeding to show this it will be well, however, to see what modification is introduced when two or more of the zeros of $\phi(D)$ coincide. For simplicity of presentation let us suppose that $r_2 = r_1$, the other roots being all different from r_1 and from one another. Then we take

$$v_1 = e^{r_1(t-\tau)}, \quad v_2 = (t-\tau)e^{r_1(t-\tau)}, \quad v_3 = e^{r_3(t-\tau)}, \quad \cdots, \quad v_n = e^{r_n(t-\tau)},$$

and the equations to determine the constants C are

$$\begin{aligned}
C_1 + 0 \cdot C_2 + C_3 + \cdots + C_n &= 0,\\
r_1C_1 + 1C_2 + r_3C_3 + \cdots + r_nC_n &= 0,\\
r_1^2C_1 + 2r_1C_2 + r_3^2C_3 + \cdots + r_n^2C_n &= 0,\\
\cdots\cdots\cdots

the coefficient of C_2 in each equation being the derivative with respect to r_1 of the coefficient of C_1. These equations show, just as before, that $(C_1, \cdots, C_n)$ are the coefficients in the analysis of $f(\tau)/\phi(D)$ into its simple fractions; i.e.,

$$\frac{f(\tau)}{\phi(D)} \equiv \frac{C_1}{D - r_1} + \frac{C_2}{(D - r_1)^2} + \frac{C_3}{D - r_3} + \cdots + \frac{C_n}{D - r_n}.$$

Developing this identity in the neighborhood of the double zero $D = r_1$, and setting

$$\frac{(D - r_1)^2}{\phi(D)} \equiv F(D),$$

we have

$$C_1 = f(\tau)F'(r_1), \qquad C_2 = f(\tau)F(r_1).$$

The part of the function $\xi(t)$ corresponding to the double zero $r = r_1$ may be conveniently expressed as follows. Writing $C_p = f(\tau)A_p$, $(p = 1, \cdots, n)$, so that

$$(10) \qquad \frac{1}{\phi(D)} = \frac{A_1}{D - r_1} + \frac{A_2}{(D - r_1)^2} + \cdots + \frac{A_n}{D - r_n},$$

we have, for the part of $\xi(t)$ in question, to evaluate the integral

$$\int_0^t \{A_1 + A_2(t - \tau)\} e^{r_1(t-\tau)} d\tau.$$

If we write $t - \tau = s$, this becomes

$$\int_0^t (A_1 + A_2 s) e^{r_1 s} ds = \frac{A_2}{r_1} t e^{r_1 t} + \left(\frac{A_1}{r_1} - \frac{A_2}{r_1^2}\right) e^{r_1 t} - \left(\frac{A_1}{r_1} - \frac{A_2}{r_1^2}\right);$$

as $A_1 = F'(r_1)$ and $A_2 = F(r_1)$, the variable part of this may be written in the form

$$\frac{F(r_1)}{r_1} t e^{r_1 t} + \left\{\frac{d}{dr_1}\left(\frac{F(r_1)}{r_1}\right)\right\} e^{r_1 t}.$$

In an exactly similar manner it can be shown that the variable part of $\xi(t)$ corresponding to a multiple root of order m is

$$\frac{1}{(m-1)!}\left[\left(\frac{F(r_1)}{r_1}\right)t^{m-1}e^{r_1 t}+(m-1)\left\{\frac{d}{dr_1}\left(\frac{F(r_1)}{r_1}\right)\right\}t^{m-2}e^{r_1 t}\right.$$
$$\left.+\cdots+\left\{\frac{d^{m-1}}{dr_1{}^{m-1}}\left(\frac{F(r_1)}{r_1}\right)\right\}e^{r_1 t}\right],$$

where $F(D)\equiv(D-r_1)^m/\phi(D)$; the expression in parenthesis being written with binomial coefficients. Thus in the case of a triple root the constants C are given by the equation

$$\frac{f(\tau)}{\phi(D)}\equiv\frac{C_1}{D-r_1}+\frac{C_2}{(D-r_1)^2}+\frac{2!C_3}{(D-r_1)^3}+\cdots.$$

It follows from (10), on putting $D=0$, that the constant part of $\xi(t)$ is $1/\phi(0)$. These results furnish the general expression for the particular solution of $\phi(D)u=1$ which vanishes with its derivatives up to the $(n-1)$st when $t=0$.

4. *The Heaviside Problem.* In his study of electric oscillations Heaviside had to consider a system of ordinary linear differential equations of the type

$$\sum_{\alpha=1}^{k} a_{r\alpha}x_\alpha = E_r(t), \qquad (r=1,\cdots,k). \tag{11}$$

Here the a's are polynomials in the operator D whose coefficients may involve the independent variable t (although Heaviside was particularly interested in the case where these coefficients are constants) and expressions for the k dependent variables are sought for, the functions $E(t)$ on the right-hand sides of the equations (11) being given. All the dependent variables but one, x_s say, may be eliminated by operating on the equations (11) with the cofactors of the sth column of the determinant $|a_{rs}|\equiv\phi(D)$. Denoting these cofactors by A_{1s}, $\cdots$, A_{ks} we find

$$\phi(D)x_s=\sum_{r=1}^{k}A_{rs}E_r(t).$$

Let us now confine our attention to the case where $E_p(t)=1$, all other E's being zero, and where, in addition, the coefficients

of the polynomials a_{rs} are constants. In this event, the operators $\phi(D)$ and A_{ps} are commutative and we first set up that particular solution of

$$\phi(D)y_p = 1$$

which vanishes, together with its derivatives up to that of the $(n-1)$st order, when $t=0$, n being the degree of the polynomial $\phi(D)$. Then $x_{sp}=A_{ps}y_p$ is a particular solution of the equation $\phi(D)x_s=A_{ps}$ and it is characterized by the fact that its derivatives up to an order one less than the degree in D of the elements a_{rs} vanish when $t=0$. From the exponential form already given for $y_p(t,0)$ in the case where the zeros of $\phi(D)$ are simple, it follows that

$$(12) \qquad x_{sp}(t,0)=A_{ps}y_p(t,0) = \sum_{\alpha=1}^{n} \frac{A_{ps}(r_\alpha)e^{r_\alpha(t)}}{r_\alpha\phi'(r_\alpha)} + \frac{A_{ps}(0)}{\phi(0)},$$

which is Heaviside's expansion formula. If now we consider the equation $\phi(D)u_p = E_p(t)$, a particular solution is

$$u_p(t) = E_p(0)y_p(t,0) + \int_0^t \frac{dE_p}{d\tau} y_p(t,\tau)d\tau,$$

this particular solution being characterized by the fact that it, together with all its derivatives up to the $(n-1)$st, vanish when $t=0$. The function

$$\bar{x}_{sp} = A_{ps}u_p(t) = E_p(0)x_{sp}(t,0) + \int_0^t \frac{dE_p}{d\tau} x_{sp}(t,\tau)d\tau$$

is, accordingly, a particular solution of the equation

$$\phi(D)\bar{x}_{sp} = A_{ps}E_p(t).$$

Adding with respect to p from 1 to k, we find that the set of functions

$$(13) \qquad x_s = \sum_{p=1}^{k} \left\{ E_p(0)x_{sp}(t,0) + \int_0^t \frac{dE_p}{d\tau} x_{sp}(t,\tau)d\tau \right\}$$

furnish that particular solution of the equations(11) which is

characterized by the fact that all the x's and their derivatives up to an order one less than the degree of the a_{rs} in D vanish when $t=0$.

When the zeros of $\phi(D)$ are not distinct we avail ourselves of the relations

$$D(te^{rt}) = rte^{rt} + e^{rt}, \quad D(t^m e^{rt}) = rt^m e^{rt} + mt^{m-1}e^{rt},$$

$$D^2(t^m e^{rt}) = r^2 t^m e^{rt} + 2rmt^{m-1}e^{rt} + m(m-1)t^{m-2}e^{rt},$$

$$D^3(t^m e^{rt}) = r^3 t^m e^{rt} + 3r^2 mt^{m-1}e^{rt} + 3rm(m-1)t^{m-2}e^{rt} + m(m-1)(m-2)t^{m-3}e^{rt},$$

which show that

$$A_{ps}(t^m e^{rt}) = A_{ps}(r)\, t^m e^{rt} + A'_{ps}(r) mt^{m-1}e^{rt} + A''_{ps}(r)\frac{m(m-1)}{1\cdot 2}t^{m-2}e^{rt} + \cdots.$$

When, then, we are operating with A_{ps} on a term such as

$$\frac{1}{(m-1)!}\left\{\frac{F(r_1)}{r_1}t^{m-1}e^{r_1 t} + (m-1)\frac{d}{dr_1}\frac{F(r_1)}{r_1}t^{m-2}e^{r_1 t} + \cdots\right\}$$

which corresponds to a multiple root r_1 of order m, we obtain

$$\frac{1}{(m-1)!}\left\{\frac{A_{ps}(r_1)F(r_1)}{r_1}t^{m-1} + (m-1)\frac{d}{dr_1}\left(\frac{A_{ps}(r_1)F(r_1)}{r_1}\right)t^{m-2} + \frac{(m-1)(m-2)}{1\cdot 2}\frac{d^2}{dr_1^2}\left(\frac{A_{ps}(r_1)F(r_1)}{r_1}\right)t^{m-3} + \cdots\right\}e^{r_1 t};$$

and these terms must be used in the expansion formula (12).

JOHNS HOPKINS UNIVERSITY

SINGULARITIES OF THE HESSIAN*

BY T. R. HOLLCROFT

1. *Introduction.* It has been proved† that when a curve f has no point singularities its Hessian H has no point singularities. Then point singularities occur on H when and only when f has point singularities. Moreover, singular points of H can occur only at those points which are singular points of f.

The number of intersections of f and H at any singularity of f is

$$6\delta_1 + 8\kappa_1 + \iota_1$$

where δ_1, κ_1, ι_1 are the numbers of nodes, cusps, and inflections respectively contained in the singularity of f. A given singularity of f needs but to be resolved and the number of intersections of f and H are thus found without reference to H. In order for this number of intersections to occur, there must be a singularity of H at this point, but except for cusps and simple multiple points with distinct tangents these singularities of H have not been investigated.

The purpose of this paper is to explain geometrically how the intersections of f and H at a given singularity of f occur. The principal problem involved is to find the singularity of H corresponding to a given singularity of f.

2. *Simple Multiple Points.* It has long been known that at a simple r-fold point of f with r distinct tangents, H has a $(3r-4)$-fold point, r of whose tangents coincide, one each, with r tangents of the r-fold point on f; also that at a cusp of f, H has a triple point two of whose tangents coincide with the cuspidal tangent.

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† A. B. Basset, *On the Hessian, the Steinerian, and the Cayleyan*, Quarterly Journal, vol. 47 (1916), p. 227.

Any number of cusps up to and including $r-1$ may occur in a simple r-fold point. The occurrence of κ_1 cusps in such a multiple point causes κ_1+1 tangents at that point to become consecutive. Also each cusp implies two additional intersections of f and H at this point.

Consider that f has an r-fold point at the origin O, the tangents at which are given by the equation $u_r=0$. Of the $3r-4$ tangents to H at O, r have the equation $u_r=0$ and the remaining $2(r-2)$ have the equation

$$T \equiv \frac{\partial^2 u_r}{\partial x^2} \cdot \frac{\partial^2 u_r}{\partial y^2} - \left(\frac{\partial^2 u_r}{\partial x\, \partial y}\right)^2 = 0.*$$

If the singularity of f at O contains $c-1$ cusps, it has c consecutive tangents. The tangents to f at O are now given by

$$u_r \equiv (ax + by)^c u_{r-c} = 0.$$

For this form of u_r, the equation T becomes

$$T \equiv (ax + by)^{2(c-1)} U_{2(r-c-1)} = 0,$$

where $U_{2(r-c-1)}$ is a function of $ax+by$, u_{r-c} and their first and second partial derivatives. Then each cusp in the r-fold point O of f causes two of the $2(r-2)$ additional tangents to H at O to become consecutive with the two consecutive tangents common to H and f.

If, in the above, $c=r$, the expression for T is identically satisfied so that it no longer is the equation of the additional tangents to H at O. This shows that the above conclusions are justified when and only when the number of cusps in the r-fold point of f does not exceed $r-2$. When it contains $r-2$ cusps, there are but two distinct tangents to f at P and these two are the only tangents to H at P. The $(3r-4)$-fold point on H now contains $3(r-2)$ cusps. If a general $(3r-4)$-fold point, it could contain one more cusp, causing all its tangents to become consecutive, but this special Hessian multiple point can not have this additional cusp, that is, a simple

* Hilton, *Plane Algebraic Curves*, 1920, Ex. 11, p. 103.

multiple point on a Hessian can not consist of a single superlinear branch.

The r-fold point P on f, however, may contain $r-1$ cusps and consist of a single superlinear branch. At P, H now has a $3(r-1)$-fold point with $2(r-1)$ of its tangents consecutive with the single tangent to f at P and the remaining $r-1$ tangents distinct from this and from each other. The well known triple point of H at a cusp of f is a special case of this for $r=2$.

Consider the r-fold point of f first without cusps and let nodes be changed into cusps one by one up to and including the maximum. Each cusp up to and including the $(r-2)$th causes three nodes of the $(3r-4)$-fold point of H to be converted into three cusps. The genus of H is not affected by this exchange so long as the number of cusps of f at P does not exceed $r-2$. When the $(r-1)$th node is changed into a cusp, nothing different from usual happens on f, but something very different occurs on H. The multiplicity of P on H is increased by unity causing the addition of $3r-4$ nodes and a corresponding decrease of $3r-4$ in the genus of H and, in addition, $r-2$ cusps in the multiple point of H at P are changed back into nodes causing $r-2$ branches of H at P that had been coincident to become distinct.

If $r=n-1$, H is of order $3(n-2)$ with a $(3n-7)$-fold point at P. In this case and in this case only can the Hessian be rational. If $r=n-1$, H is a proper curve when and only when the number of cusps of f at P does not exceed $n-3$. If the $(n-1)$-fold point of f contains $n-2$ cusps, H degenerates into $3(n-2)$ lines through P of which $2(n-2)$ coincide with the single tangent to f at P. A well known special case of this is the Hessian of a cuspidal cubic.

3. *Compound Singularities.* When the singularity of f at P is compound, the singularity of H at P is also compound.

Any number s of consecutive r-fold points at P on f with r distinct branches determines at P, on H, s consecutive $3(r-1)$-fold points with $3(r-1)$ distinct branches and with the same tangent as that to f at P. There are, then, at P, r branches of

f, each of which has contact of order s with each of the $3(r-1)$ branches of H that pass through P.

Any series of consecutive multiple points of orders $r_1, r_2, r_3, \cdots$ where $r_1 > r_2 \geqq r_3 \geqq \cdots$ and $r_1 > r_2 + 1$ gives rise to a series of consecutive multiple points on H of the respective orders $3r_1-4$, $3(r_2-1)$, $3(r_3-1), \cdots$ and through the same respective penultimate points and therefore with the same tangent. If, however, $r_2 \leqq r_1 \leqq r_2+1$, the series on H are of the respective orders $3(r_1-1)$, $3(r_2-1)$, $3(r_3-1), \cdots$.

In case $r_1 = r_2$, the $3(r_1-1)$ branches of H all have the same tangent at P as the r_1 branches of f. In case $r_1 = r_2+1$, at P on f there are r_2 branches with the same tangent and one branch with a distinct tangent. At P on H there are $3r_1-4$ branches that have the tangent common to the r_2 branches of f and the remaining branch of H has the same tangent as the remaining branch of f.

Since any series of consecutive multiple points of f gives rise to a series of consecutive multiple points of H through the same penultimate points respectively, the order of nearness* of any two consecutive points of H is always equal to the order of nearness of the two corresponding consecutive multiple points of f.

Any series of consecutive multiple points of orders $r_1, r_2, r_3, \cdots$ where $r_1 \geqq r_2 \geqq r_3 \geqq \cdots$ may contain any number of cusps up to and including $r_1 - 1$. For each node changed into a cusp in the part of the r_1-fold point involved in the consecution or in any of the other consecutive points, a bitangent of the singularity is changed into a stationary tangent. Then if the number of cusps k is such that

$$k \leqq r_2 - 1$$

the singularity contains k inflections, but if

$$r_1 - 1 \geqq k \geqq r_2 - 1$$

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* See F. Enriques, *Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche*, vol. 2, pp. 404–408.

the singularity contains r_2-1 inflections provided that the cusps are considered as being introduced so far as possible into that part of the singularity involved in the consecution. Then the introduction of $k \leqq r_2-1$ cusps causes the addition of $3k$ intersections of f and H at this point, $2k$ for the cusps that replace k nodes and k for the k inflections that replace k bitangents. The introduction of k cusps into the portion of the r_1-fold point not involved in the consecution causes the addition of only $2k$ intersections as for simple multiple points.

If the consecutive multiple points of f at P contain any number k of cusps all of which are involved in the consecution, the consecutive multiple points of H at P contain $3k-1$ cusps. All the branches of both f and H at P (except the simple branches of the r_1-fold point of f and the $(3r_1-4)$-fold point of H in case $r_1>r_2$) have the same tangent whether the points contain cusps or not. The change of k nodes into k cusps on f causes $k+1$ of the branches to coincide, while on H the $3k-1$ cusps cause the coincidence of $3k$ branches. H and f must now have $3k$ additional intersections at P. None of these are accounted for by the mere coincidence of the branches, but all are accounted for by the fact that the superlinear branches of f and H have closer contact than had each of the component branches when distinct. If there are s consecutive r-fold points of f at P, each branch of f has s-point contact with each branch of H at P. When the multiple points of f contain k cusps, however, each of the $k+1$ partial branches of the superlinear branch of f at P has $[(s+1)/(k+1)]$-point contact with each of the $3k$ partial branches of the superlinear branch of H at P.

A flecnode on f gives rise to a flecnode on H with the same stationary and simple tangents respectively. Also a biflecnode on f gives rise to a biflecnode on H with the same two stationary tangents. In general, if any of the tangents at an r-fold point of f are stationary tangents, these are also stationary tangents to H at this point.

4. *Line Singularities.* Simple line singularities of f cause on H either no singularities or simple line singularities. If the

line singularity of f be composed of only bitangents, that is, if all its contacts are distinct, H avoids both this line and its contacts and has no singularity caused by such a multiple line of f. Moreover, a multiple line of class ρ of f may contain any number up to $(\rho+1)/2$ inflections, no more than one at each contact except that one point of contact may contain two inflections, and H will have no line singularity. It is only when more than two inflections occur at one contact or more than one inflection at two or more contacts of a multiple line p of f that H has p as a multiple line.

In general, if f has a multiple line p of class ρ and q is the number of contacts of p with f that contain two or more inflections and i_j is the number of inflections contained in any one of these q contacts, then p is a multiple tangent of H of class

$$\rho_1 \equiv \sum_{j=1}^{q} i_j - q.$$

Since the number of inflections of a multiple line of class ρ with q distinct contacts can not exceed $\rho-q$, there is an upper limit to the class ρ_1 of the line p of H

$$\rho_1 \leqq \rho - 2q.$$

The equality sign holds only when no contacts of p with f contain less than two inflections. It is also apparent that the fewer the number of distinct contacts, the larger the class of the multiple line of H may be. Thus the maximum class of a multiple line of H that results from a multiple line of class ρ of f is $\rho-2$. This occurs when and only when p contains $\rho-1$ stationary tangents to f, in which case $q=1$. In this case, the multiple line p contains $\rho-3$ stationary tangents to H and all its contacts are at one point.

5. *Singularities that a Hessian Cannot Have.* In the preceding sections, the multiple points and lines that a Hessian can have were discussed. There are, however, certain singular points that a Hessian can not have under any circumstances. These include all simple or compound double points except

distinct nodes, multiple points of order $3a+1$ for a any integer and simple multiple points with all tangents consecutive.

There are no such restrictions on multiple lines of the Hessian.

6. *The Hessian as a Jacobian.* Since the Hessian of f is the Jacobian of the net of first polars of f, the singularity P of H caused by a given singularity P of f is the singularity of the Jacobian of the net of first polars of f corresponding to this basis point P on the first polars. Since an r-fold point P of f causes a $(3r-4)$-fold point P of H and an $(r-1)$-fold point P on each first polar, then H as the Jacobian has a $(3i-1)$-fold point at an i-fold basis point of the net. Since this result is general, it might be inferred that the singularities of the Jacobian corresponding to any kind of basis point could be now found by finding the kind of basis point on the first polars determined by a given singularity of f. However, the result is general only for simple multiple points with distinct branches. If the curves of the net have contact of any given order with each other at P, the nature of the singularity P on the Jacobian can not be predicted unless more is known about the net. For example, simple contact in a first polar net may result from a cusp on f or a tacnode on f. In these cases, the singularity on the Jacobian is respectively a triple point two of whose tangents coincide with the cuspidal tangent or two consecutive triple points whose tangent coincides with the tacnodal tangent. But a simple contact in a net of otherwise general curves causes on the Jacobian a triple point with three distinct tangents one of which coincides with the common tangent of the curves of the net.

For $n>3$, first polar nets are not the most general nets of order $n-1$ and for all values of n, if f has singularities, first polar nets are very highly specialized. For this reason (except in the case of a simple multiple point with distinct tangents) general results for the singularity of the Jacobian at a given singularity of the net can not be obtained by regarding the Hessian as the Jacobian of the net of first polars.

WELLS COLLEGE

THREE THEOREMS ON CLOSURE OF BIORTHOGONAL SYSTEMS OF FUNCTIONS*

BY R. E. LANGER

1. *Introduction.* Professor Birkhoff has given a theorem† which asserts the closure of a given normal orthogonal set of functions provided it is sufficiently near to another such set which is closed. In its applications this theorem is frequently used in conjunction with asymptotic forms for the functions involved, and the information at hand applies only to a portion of the entire set. For this reason an extension of the theorem to sets which involve a certain lack of closure is useful. Such an extension together with a generalization of the theorem to biorthogonal systems of functions is given below in Theorem I.

In Theorems II and III the discussion is restricted to the closure of such biorthogonal systems as are composed of the characteristic functions of an integral equation.

A set of continuous functions $\{u_n(x)\}$ will be said to involve a k-fold lack of closure on a given interval if there exist precisely k continuous functions not identically zero which are linearly independent and are orthogonal on the interval to every function of the set.

THEOREM I. *Let* $\{u_n(x), v_n(x)\}$, *that is,*

$$\left\{\begin{array}{l} u_0(x),\ u_1(x),\ u_2(x),\ \cdots, \\ v_0(x),\ v_1(x),\ v_2(x),\ \cdots, \end{array}\right. \tag{1}$$

be a normalized biorthogonal system of continuous functio on the interval $a \leqq x \leqq b$, *and let* $\{\bar{u}_n(x), \bar{v}_n(x)\}$, *i. e.,*

$$\left\{\begin{array}{l} \bar{u}_0(x),\ \bar{u}_1(x),\ \bar{u}_2(x),\ \cdots, \\ \bar{v}_0(x),\ \bar{v}_1(x),\ \bar{v}_2(x),\ \cdots, \end{array}\right. \tag{2}$$

* Presented to the Society, December 28, 1926.

† Proceedings of the National Academy, vol. 3 (1917), pp. 656–659.

be a second normalized biorthogonal system which is such that (a) *the series*

$$\sum_{n=0}^{\infty}[u_n(x) - \bar{u}_n(x)]v_n(y), \qquad a \leqq x, y \leqq b, \tag{3}$$

converges to a function $H(x, y)$ less than $1/(b-a)$ in absolute value; and

(b) *the convergence is such that the series upon being multiplied by any continuous function may be integrated term by term as to x to yield a uniformly convergent series.*

*Then if the set $\{u_n(x)\}$ involves a k-fold lack of closure on the interval (a, b) the lack of closure of the set $\{\bar{u}_n(x)\}$ is at most k-fold.**

By hypothesis there exist k linearly independent continuous functions $\phi_i(x)$, $i=1, 2, \cdots, k$, which are orthogonal to all functions of the set $\{u_n(x)\}$, and every continuous function $\Phi(x)$, which is orthogonal to all functions $u_n(x)$, is expressible in the form

$$\Phi(x) = \sum_{i=1}^{k} c_i\phi_i(x).$$

It is clear that the entire set

$$\phi_1(x), \phi_2(x), \cdots, \phi_k(x), u_0(x), u_1(x), \cdots \tag{4}$$

is closed.

Suppose now that there exist $(k+1)$ continuous functions $f_i(x)$, $i=1, 2, \cdots, k+1$, which are orthogonal to all the functions $\bar{u}_n(x)$. Then set

$$F(x) = \sum_{i=1}^{k+1}\alpha_i f_i(x), \tag{5}$$

the $(k+1)$ constants α_i being chosen not all zero and so as to satisfy the k linear relations

* A slightly different generalization of Birkhoff's theorem to biorthogonal systems in the case $k=0$ (closure) is given by the author, Transactions of this Society, vol. 28, p. 585.

$$\int_a^b \Psi(y)\phi_i(y)dy = 0, \quad (i = 1,2, \cdots, k), \tag{6}$$

where

$$\Psi(y) = F(y) - \int_a^b F(x)H(x,y)dx. \tag{7}$$

Because of hypothesis (b), the function $\Psi(y)$ is continuous. Substituting for $H(x, y)$ the series (3), and observing from (5) that $F(x)$ is orthogonal to all functions $\bar{u}_n(x)$, we obtain from (7) the relation

$$\int_a^b \Psi(y)u_n(y)dy = \int_a^b F(y)u_n(y)dy - \sum_{j=0}^{\infty} \int_a^b F(x)u_j(x)dx \int_a^b v_j(y)u_n(y)dy.$$

Since the system (1) is normalized biorthogonal, however, the right member of this equation vanishes for all n. From this fact and relation (6), it is seen that $\Psi(x)$ is orthogonal to every function of the closed set (4). It follows that $\Psi(y) \equiv 0$, that is,

$$F(y) = \int_a^b F(x)H(x,y)dx.$$

If, now, the maximum numerical value of $F(y)$ on (a, b) is F, and if this value is taken on for $y = y_0$, we have

$$\left| \int_a^b F(x)H(x,y_0)dx \right| = F.$$

But since $|H(x, y)| < 1/(b-a)$, this is impossible unless $F = 0$, and therefore $F(x) \equiv 0$. Hence the $(k+1)$ functions $f_i(x)$ must be linearly dependent and the set $\{\bar{u}_n(x)\}$ involves at most a k-fold lack of closure.

2. *A Theorem in Integral Equations.* The theory of the integral equation with a symmetric kernel contains a theorem to the effect that if the system of characteristic functions

$\{u_n(x)\}$ is not closed then every function which is orthogonal to $u_n(x)$ for all n must be also orthogonal to the kernel.* The following theorem is in a sense analogous for the theory of the integral equation with a general kernel. We consider the integral equation

$$u(x) = \lambda \int_a^b K(x,\xi)u(\xi)d\xi \tag{8}$$

in which the kernel is bounded and has its discontinuities regularly distributed.†

THEOREM II. *If the set of characteristic functions* $\{u_n(x)\}$ *of equation* (8) *is not closed but involves only a finite lack of closure, then there always exists at least one continuous function* $F(x) \not\equiv 0$ *which is orthogonal to the kernel, that is,*

$$\int_a^b K(x,\xi)F(x)dx = 0.$$

By hypothesis there exist k linearly independent continuous functions $f_i(x)$, $i = 1, 2, \cdots, k$, such that

$$\int_a^b f_i(\xi)u_n(\xi)d\xi = 0 \tag{9}$$

for all values of n; and every function orthogonal to all functions $u_n(x)$ is expressible linearly in terms of the functions $f_i(x)$. Consider the function

$$\theta_i(x) = \int_a^b K(\xi,x)f_i(\xi)d\xi. \tag{10}$$

Multiplying it by $u_n(x)$ and integrating, we find because of (8) and (9) that $\theta_i(x)$ is orthogonal to $u_n(x)$ for all values of n. Hence

$$\int_a^b K(\xi,x)f_i(\xi)d\xi = \sum_{j=1}^{k} c_{ij}f_j(x). \tag{11}$$

* T. Lalesco, *Théorie des Équations Intégrales*, Paris, 1912, p. 69.

† The reader will have no difficulty in observing where restrictions of this kind can be lightened if desired.

Let the function $F(x)$ be defined now by the relation

$$F(x) = \sum_{i=1}^{k} \alpha_i f_i(x). \tag{12}$$

Then, from (11), we have

$$\int_a^b K(\xi, x) F(\xi) d\xi = \sum_{j=1}^{k} f_j(x) \sum_{i=1}^{k} c_{ij} \alpha_i. \tag{13}$$

Now the equation

$$\begin{vmatrix} c_{11} - \rho & c_{21} & \cdots & c_{k1} \\ c_{12} & c_{22} - \rho & \cdots & c_{k2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ c_{1k} & c_{2k} & \cdots & c_{kk} - \rho \end{vmatrix} = 0$$

admits of at least one root $\rho = \rho_1$; hence we may choose the constants α_i in (12) not all zero, and so that

$$\sum_{i=1}^{k} c_{ij} \alpha_i = \rho_1 \alpha_j.$$

Substituting this in (13), we find that

$$\int_a^b K(\xi, x) F(\xi) d\xi = \rho_1 F(x). \tag{14}$$

Suppose now that $\rho_1 \neq 0$. Then $F(x)$ is a characteristic function of the associated equation for $\lambda = 1/\rho_1$, and it follows that

$$F(x) = \sum \beta_l v_l(x), \tag{15}$$

where the coefficients on the right are constants and the sum includes all the characteristic functions $v_l(x)$ of the associated equation, which occur in the biorthogonal normal system $\{u_n(x), v_n(x)\}$, and which correspond to the value $\lambda = 1/\rho_1$. From (12), however, $F(x)$ is orthogonal to all

functions of the set $\{u_n(x)\}$, and $F(x) \not\equiv 0$ since the functions $f_i(x)$ are linearly independent. This contradicts (15), since the system $\{u_n(x), v_n(x)\}$ is normalized biorthogonal. It follows that $\rho_1 = 0$, which proves the theorem.

3. *A more General Form of Theorem* II. For simplicity in the formulation of Theorem III we shall call a function $u(x)$ which satisfies the relation

$$\int_a^b K(x,\xi)u(\xi)d\xi = 0$$

a *characteristic function* of equation (8) for the value $\lambda = \infty$. If the equation (8) admits of m linearly independent characteristic functions $u_{-i}(x)$, $i = 1, 2, \cdots, m$, for $\lambda = \infty$, and the associated equation also admits of m characteristic functions $v_{-i}(x)$, $i = 1, 2, \cdots, m$, for $\lambda = \infty$, the system $\{u_{-i}(x), v_{-i}(x)\}$ may be biorthogonalized and normalized provided no linear combination of the functions of either set $u_{-i}(x)$ or $v_{-i}(x)$, is orthogonal to all the functions of the other set*. If the system $\{u_i(x), v_i(x)\}$ is biorthogonal and normal, it follows since

$$\int_a^b u_{-i}(x)v_j(x)dx = \lambda_j \int_a^b v_j(\xi) \int_a^b K(\xi, x)u_{-i}(x)dxd\xi = 0,$$

that the entire system

$$\begin{cases} u_{-m}(x), \cdots, u_{-1}(x), u_0(x), u_1(x), \cdots, \\ v_{-m}(x), \cdots, v_{-1}(x), v_0(x), v_1(x), \cdots, \end{cases}$$

is biorthogonal and normal. We shall call this the extended system $\{u_n(x), v_n(x)\}$, and shall call the entire set of functions $u_n(x)$, $n = -m, (-m+1), \cdots$ the extended set $\{u_n(x)\}$.

THEOREM III. *If the integral equation* (8) *and its associated equation admit as solutions for* $\lambda = \infty$ *the functions of a biorthogonal normal system* $\{u_{-i}(x), v_{-i}(x)\}$, *and if the*

* Goursat, *Cours d'Analyse Mathématique*, 3d ed. (1923), vol. 3, p. 393.

extended set $\{u_n(x)\}$ *is not closed but involves only a finite lack of closure, then there always exists at least one continuous function* $F(x) \not\equiv 0$ *which is linearly independent of the functions* $v_{-i}(x)$, *and which is orthogonal to the kernel, that is,*

$$\int_a^b K(x,\xi)F(x)dx = 0.$$

By hypothesis there exist k functions $f_i(x)$, $i=1, 2, \cdots, k$, which satisfy (9), where n runs over all values in the extended set $\{u_n(x)\}$. Precisely as in the preceding proof it follows that $\theta_i(x)$ in (10) is orthogonal to the functions $u_n(x)$ for $n=0, 1, 2, \cdots$. But further

$$\int_a^b \theta_i(x)u_{-j}(x)dx = \int_a^b f_i(\xi)\int_a^b K(\xi,x)u_{-j}(x)dxd\xi = 0,$$

since $u_{-j}(x)$ is a characteristic function for $\lambda=\infty$. Hence again $\theta_i(x)$ is orthogonal to $u_n(x)$ for all n. As in the preceding proof we may construct now a function $F(x)$ by formula (12) which satisfies equation (14). Moreover the previous reasoning shows that $\rho_1=0$, and hence that $F(x)$ is orthogonal to $K(x, \xi)$. To show that $F(x)$ is linearly independent of the functions $v_{-i}(x)$, we assume the contrary, namely that

$$\gamma_0 F(x) + \sum_{i=1}^{m}\gamma_i v_{-1}(x) \equiv 0.$$

Since the extended set $\{u_n(x), v_n(x)\}$ is biorthogonal and normal, while $F(x)$ is orthogonal to every function $u_n(x)$, it follows, upon multiplying the relation by $u_{-j}(x)$ and integrating, that $\gamma_j=0$, $j=1, 2, \cdots, m$. This implies that $F(x)\equiv 0$ which involves a contradiction.

COROLLARY. *If the system of all linearly independent solutions of equation* (8) *and its associated equation for* $\lambda=\infty$ *may be made biorthogonal and normal, and if any subset of the extended set* $\{u_n(x)\}$ *involves only a finite lack of closure, then the extended set* $\{u_n(x)\}$ *is closed.*

If a subset involves only a finite lack of closure, the same must be true of the entire set.

4. *Examples.* The following examples are given to illustrate the necessity of certain parts of the hypotheses which have been made. Thus in Theorem II the case of an infinite lack of closure is excluded. Such a case is represented by the Volterra equation in which $a=0$, $b=1$, and

$$K(x,\xi) = \begin{cases} 0 \text{ for } \xi > x, \\ 1 \text{ for } \xi \leqq x. \end{cases}$$

The equation admits of no characteristic values other than 0, so the lack of closure is infinite. Since

$$\int_0^1 K(x,\xi)f(x)dx = \int_0^x f(x)dx = 0$$

implies $f(x)\equiv 0$, the assertion of the theorem clearly does not hold.

In the Corollary to Theorem III, the case in which the totality of solutions for $\lambda=\infty$ cannot be made biorthogonal is excluded. This case is illustrated by the example*

$$K(x,\xi) = K_1(x,\xi) + K_2(x,\xi),$$

where

$$K_1(x,\xi) = \phi_1(x)\phi_2(\xi),$$

$$K_2(x,\xi) = \sum_{n=3}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n},$$

the set $\{\phi_n(x)\}$ being any closed normalized orthogonal set and the series for $K_2(x, \xi)$ converging uniformly. The functions $K_1(x, \xi)$ and $K_2(x, \xi)$ are orthogonal, and hence the set of characteristic functions for $K(x, \xi)$ is composed of those for $K_1(x, \xi)$ and $K_2(x, \xi)$.† The characteristic functions for $K_2(x, \xi)$ are obviously $\phi_n(x)$, $n=3, 4, \cdots$. On the other hand equation

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* This example was given to me by Professor J. Tamarkin of Dartmouth College.

† Goursat, loc. cit., p. 402.

$$u(x) = \lambda \int_a^b K_1(x,\xi)u(\xi)d\xi$$

is of the form

$$u(x) = \lambda \int_a^b \phi_2(\xi)u(\xi)d\xi\phi_1(x) = c\phi_1(x),$$

whereas the substitution of this value in the equation yields

$$c\phi_1(x) = \lambda c\phi_1(x) \int_a^b \phi_2(\xi)\phi_1(\xi)d\xi = 0.$$

Hence $k_1(x, \xi)$ admits of no characteristic functions. Further if $f(x)$ is a characteristic function for $\lambda = \infty$ we have upon setting

$$\int_a^b f(\xi)\phi_1(\xi)d\xi = f_i$$

the relation

$$\int_a^b K(x,\xi)f(\xi)d\xi = f_2\phi_1(x) + \sum_{n=3}^{\infty} \frac{f_n\phi_n(x)}{\lambda_n} = 0.$$

Multiplying this by $\phi_j(x)$ and integrating we find that $f_j = 0$ for $j = 2, 3, \cdots$, whence $f(x)$ must be of the form $f(x) = c\phi_1(x)$ and $\phi_1(x)$ is the only characteristic function for $\lambda = \infty$. In similar fashion it is seen that $\phi_2(x)$ is the only characteristic function of the associated equation for $\lambda = \infty$. Since on the one hand $\phi_1(x)$ and $\phi_2(x)$ are orthogonal the hypothesis of the corollary cannot be satisfied. On the other hand since the entire set of characteristic functions

$$\phi_1(x), \ \phi_3(x), \ \phi_4(x), \cdots$$

is not closed while the subset $\phi_i(x)$, $i = 3, 4, \cdots$, involves only a two-fold lack of closure the assertion of the theorem clearly fails.

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A CONNECTED AND CONNECTED IM KLEINEN POINT SET WHICH CONTAINS NO PERFECT SUBSET

BY B. KNASTER AND C. KURATOWSKI

1. *Introduction.* Professor R. L. Moore has shown in this Bulletin (vol. 32, p. 331) that there exist point sets connected and connected im kleinen* which contain no arc. We shall prove in this paper the existence of such a set containing no *perfect* subset. The set has the additional property that it is contained in a *regular curve* (in the sense of K. Menger)†.

2. *The Sierpinski Regular Curve.* Let R be the Sierpinski regular curve,‡ defined as follows. Let T be an equilateral triangle. Divide T in 4 equal triangles. Let T_0, T_1, T_2 denote those three triangles which have a common vertex with T. Similarly divide each of the triangles T_0, T_1, T_2 in 4 equal triangles and let T_{00}, T_{01}, T_{02}, T_{10}, $\cdots$, T_{22} denote those having a common vertex with T_0 or T_1 or T_2; and so on *ad inf.*

The point set formed by the boundaries of all the triangles $T_{\alpha_1\alpha_2\cdots\alpha_k}(\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_i = 0, 1, \text{or } 2)$ and all their limit points is the regular curve R.

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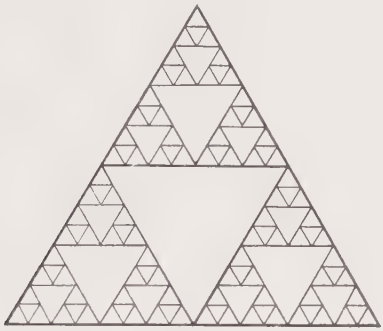
* A point set M is said to be *connected im kleinen* (or to be regular) if, for every point p and every positive number e, there exists a positive number d such that if x is any point of M at a distance from p less than d then x and p both lie in some connected subset of M of diameter less than e.

† A continuum C is called a regular curve if, for every positive number e, C can be expressed as the sum of a finite number of continua each of diameter less than e, each pair of continua having at most a finite number of points in common (see K. Menger, Mathematische Annalen, vol. 95 (1925), p. 300). Every regular curve is a continuous curve whose every subcontinuum is a continuous curve (see H. M. Gehman, Annals of Mathematics, vol. 27 (1925), p. 42).

‡ Prace Matematyczno-Fizyczne, vol. 27 (1915). The Sierpinski curve R contains three points of degree 2, a countable set of points of degree 4, the remainder being composed of points of degree 3. See also Comptes Rendus, vol. 160 (1915), p. 302.

3. *Properties of the Sierpinski Curve.* The complementary set of R is composed of a countable set of regions. Let I_0 denote the unbounded region (exterior to T) and $I_1, I_2, \cdots, I_n, \cdots$ the bounded regions (for $n>0$, I_n forms the interior of a triangle). Let B_n denote the boundary of I_n and V the countable point set of all the vertices of the triangles $T_{\alpha_1\alpha_2\cdots\alpha_k}$. The following properties of the curve R are to be noticed:



PROPERTY 1. *Given a positive integer k, no point of R belongs to more than two of the triangles $T_{\alpha_1\alpha_2\cdots\alpha_k}$.*

PROPERTY 2. *If $m\neq n$, then $\bar{I}_m \cdot \bar{I}_n \subset V$*.*

PROPERTY 3. *Given two points a and b of R and an index n, there exists a subcontinuum C of R containing a and b and such that $C \cdot B_n \subset V+a+b$.*

In a similar manner we have

PROPERTY 4. *Given a triangle $T_{\alpha_1\alpha_2\cdots\alpha_k}$ and an index n, any two points a and b of $R \cdot T_{\alpha_1\alpha_2\cdots\alpha_k}$ may be joined by a subcontinuum C of $R \cdot T_{\alpha_1\alpha_2\cdots\alpha_k}$ such that $C \cdot B_n \subset V+a+b$.*

4. *Two Lemmas.* We shall now prove two lemmas.

LEMMA I. *If K is a bounded continuum such that $K \cdot V=0$ and there exists no index n such that $K \subset \bar{I}_n$, then the set $K \cdot R$ contains a perfect subset.*

* The symbol $\overline{X}'$ denotes the set $X+$ all its limit points. The symbol $X \subset Y$ means that X is contained in Y.

PROOF. It is evident that

$$K = K \cdot R + K \cdot I_0 + K \cdot I_1 + \cdots + K \cdot I_n + \cdots ;$$

hence

$$(1) \qquad K = K \cdot R + K \cdot \bar{I}_0 + K \cdot \bar{I}_1 + \cdots + K \cdot \bar{I}_n + \cdots .$$

Suppose $K \cdot R$ does not contain any perfect subset. As $K \cdot R$ is closed, it follows that $K \cdot R$ is a finite or countable point set.

Let S_n denote the set $K \cdot \bar{I}_n$ and let Q denote the set $K \cdot R - (S_0 + S_1 + \cdots + S_n + \cdots)$. Hence Q is a finite or countable set of points: $p_1, p_2, \cdots$. It follows by (1) that

$$(2) \qquad K = p_1 + p_2 + \cdots + S_0 + S_1 + \cdots + S_n + \cdots .$$

The right-hand side of the identity (2) is composed of mutually exclusive sets, since by Property 2, if $m \neq n$, then $K \cdot \bar{I}_m \cdot K \cdot \bar{I}_n \subset K \cdot V = 0$. But this contradicts a theorem of Sierpinski's* to the effect that if K is a sum of a finite ($\geqq 2$) or countable number of mutually exclusive closed point sets, then K is not a bounded continuum. Thus the existence of a perfect subset of $K \cdot R$ is established.

LEMMA II. *If Z is a subset of R such that each perfect subset of R contains a point belonging to Z, then $Z+V$ is connected and connected im kleinen.*

PROOF. Suppose that the set $Z+V$ is not connected. Then† there exist two points a and b of $Z+V$ and a bounded continuum K which separates a from b and is such that $K \cdot (Z+V) = 0$. Therefore

$$(3) \qquad K \cdot (V + a + b) = 0.$$

Since $K \cdot Z = 0$, it follows, by hypothesis, that the set $K \cdot R$ does not contain any perfect subset. By Lemma I, there exists an index n such that $K \subset \bar{I}_n$. As $\bar{I}_n = I_n + B_n$ and $R \cdot I_n = 0$, it follows that

* Tôhoku Mathematical Journal, vol. 13 (1918), p. 300.

† See our paper *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), p. 233, Theorem 37.

$$K \cdot R \subset B_n. \tag{4}$$

By Property 3, the points a and b may be joined by a subcontinuum C of R such that

$$C \cdot B_n \subset V + a + b. \tag{5}$$

Since $K \cdot C = K \cdot R \cdot C$, it follows from (4) and (5) that

$$K \cdot C \subset B_n \cdot C \subset V + a + b.$$

Hence by (3) $K \cdot C = 0$, contrary to the assumption that K separates a from b. Thus the supposition that $Z + V$ is not a connected point set leads to a contradiction.

Now let H denote any one of the triangles $T_{\alpha_1 \alpha_2 \cdots \alpha_k}$. Each perfect subset of $R \cdot H$ has a common point with $Z \cdot H$. By an argument similar to that used above it may be proved (with the help of Property 4 instead of Property 3) that the set $(Z+V) \cdot H$ is connected. It follows by Property 1 that $Z+V$ is connected im kleinen.

5. *Conclusion.* We may now state the following theorem.

THEOREM. *There exists in the regular curve R of Sierpinski's a connected and connected im kleinen point set which contains no perfect subset.*

PROOF. By a theorem due to F. Bernstein* the plane may be decomposed into two mutually exclusive subsets E and F such that each perfect set contains points of both of them. It follows that each perfect subset of R has a common point with $E \cdot R$. By Lemma II the set $M = E \cdot R + V$ is connected and connected im kleinen.

The set M contains no perfect subset. For suppose P is a perfect subset of M. As the set V is countable, $P - V$ contains a perfect subset P_1. Hence $P_1 \subset E$, contrary to the assumption that P_1 has a common point with F. Thus M is a connected and connected im kleinen point set which contains no perfect subset.

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* Leipziger Berichte, vol. 60 (1908).

SCHLESINGER ON LEBESGUE INTEGRALS

Lebesguesche Integrale und Fouriersche Reihen. By L. Schlesinger and A. Plessner. Berlin and Leipzig, Walter de Gruyter Company, 1926. viii+229 pp.

This book is an outgrowth of a course of lectures by Professor L. Schlesinger at the University of Giessen in the winter semester of 1921–22 and was written with the cooperation of Dr. A. Plessner, who was at that time a student in the course, and later Professor Schlesinger's assistant. As the title indicates the subject is Lebesgue integrals and Fourier expansions from the point of view of Lebesgue's definition of integration.

Necessarily the first part is a study of the properties of real functions defined on point sets in a space of m dimensions. The first five chapters or 180 pages form a masterly exposition of the essential facts of real variable theory, particularly integration, and the sixth chapter of forty-two pages deals with Fourier series.

The headings give a sufficiently definite idea of the contents of each chapter. I. Fundamental concepts of points sets; II. Measure of point sets; III. Concerning functions of real variables; IV. The Lebesgue integral; V. Functions of one and two variables; VI. Fourier series.

Particular attention is directed to Chapter II in which the Borel-Lebesgue definition of measure is developed and compared with the older Peano-Jordan definition of content, and to Chapter IV where the Lebesgue integral is defined following Lebesgue's original geometric point of view. In the last section of this chapter (p. 129 ff.) attention is called to other useful definitions of the L-integral, due to Lebesgue himself, W. H. Young, J. Pierpont, and F. Riesz. This last definition will be discussed by the reviewer a little further on in greater detail, but at this point, it is well to note that mention should have been made along side of these of a two-way infinite series definition of a Lebesgue integral due to M. B. Porter.* By means of this definition the properties of the L-integral and the theorems of Lusin, Egoroff, and other important results are almost immediate.

Chapter VI contains an elegant and concise treatment of the most important newer results for Fourier series, notably those beginning with Lebesgue's *Leçons sur les Séries Trigonométriques*, 1906, which include the classic results of Riemann, Dini, Lipschitz, Dirichlet, Jordan, and others. The chapter closes with theorems on summability and convergence in the mean of Fejér, Cesàro, E. Fischer, and F. Riesz.

At the close of the preface is found a list of references to monographs, treatises, and encyclopedia articles whose content is closely related to the present text. Scattered throughout the book are to be found in footnotes, full references to the original memoirs bearing on all phases of the field.

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* This Bulletin, vol. 28 (1922), pp. 105–8.

The feature which, aside from the excellent form of the entire book, seems of foremost importance to the writer, receives no notice whatever in the table of contents and but one citation in the index. The number of references, however, is, in reality, six. These occur on pages 114, 123, 131, 194, 215, 221. This feature is the use of a sequence of step-functions (*Treppenfunktionen*) or *horizontal functions* in certain questions of convergence. The reviewer* has given the following definition:

A *horizontal function*, $h_n(x)$, *of index* n on an interval $I \equiv a \leqq x \leqq b$, associated with a subdivision of I into n subintervals, is one such that in the interior of the ith subinterval $h_n(x)$ is constant, i.e., $h_n(x) \equiv h_{in}$. The function need not be defined at the end points of the subintervals. Clearly $h_n(x)$ is represented for each value of n by n horizontal segments. Such functions are implicit in Riemann's and Lebesgue's (analytic) definition of an integral. In their explicit use, F. Riesz who called them "*fonctions simples*" seems to have been the first (references in the footnote, p. 131, in the present book under review). His use of them to give an alternate definition of an L-integral will be stated.

Concerning these horizontal functions, the following theorems are immediate. They will be stated without proof.

THEOREM I. *For every continuous function $f(x)$ on an interval I, there exists a sequence of horizontal functions $f_n(x)$ which has $f(x)$ for its limit function for every x in I. Furthermore, the sequence can be chosen so that the approach to the limit function will be monotone and uniform.*

THEOREM II. *For every bounded Riemann integrable function $f(x)$ on I there exists a sequence of horizontal functions $f_n(x)$ which has $f(x)$ for its limit function for every x in I save for a null set. Furthermore, the sequence can be chosen so that the approach to the limit function will be monotone.*

In the above two theorems, the set of horizontal functions may be built up in terms of a value of $f(x)$ in each subinterval. In particular, the maximum value (upper limit) or the minimum value (lower limit) may be used in each subinterval. We shall prove the following theorem used by Schlesinger for the simplest case.

THEOREM III. *For every bounded measurable function $f(x)$ on I, there exists a sequence of uniformly bounded horizontal functions $f_n(x)$ which has $f(x)$ for its limit function for every value of x on I save a null set.*

Proof. By hypothesis $f(x)$ has an indefinite L-integral

$$F(x) = \int_a^x f(t)\, dt,$$

where $F(x)$ has a bounded incremental ratio on I and $F'(x) = f(x)$ almost everywhere on I. Let $x_{in} = i(b-a)/n$ and let N be the number of sub-

* H. J. Ettlinger, *Note on a fundamental lemma concerning the limit of a sum*, this Bulletin, vol. 32 (1926), p. 69. See also H. J. Ettlinger, *On multiple iterated integrals*, American Journal, vol. 48 (1926), pp. 215-222.

intervals defined for every value of n by each pair of consecutive points $x_{ij}, i=1, 2, \cdots, j; j=1, 2, \cdots, n$. Let

$h_N(x) = [F(x_{i+1,N}) - F(x_{iN})]/[x_{i+1,N} - x_{iN}]$. Then $\lim_{n\to\infty} h_N(x) = f(x)$

for every x on I save a null set. Schlesinger proves this theorem for a bounded measurable function of m variables defined on a measurable point set.

The following theorem is in effect the equivalent of Riesz's definition of an L-integral. It may be regarded as a converse of Theorem III.

THEOREM IV. *If a uniformly bounded sequence of horizontal functions $f_n(x)$ on I has a limit function $f(x)$ almost everywhere on I, then $f(x)$ is bounded and measurable on I, i.e., is L-integrable on I.*

In terms of horizontal functions it is easy to establish the substantial equivalence of a *principle* used by the reviewer* on various occasions and named the Duhamel-Moore theorem,† and a theorem due to Lebesgue‡ on the integral of the limit of a sequence of functions. The Duhamel-Moore theorem (DM) may be stated for an interval I as follows:

HYPOTHESIS. 1. *The set of n non-overlapping subintervals I_{in}, $i=1, 2, \cdots, n$, of length l_{in} are obtained by subdividing the interval I. The sets of numbers r_{in}, r'_{in} are such that $|r_{in} - r'_{in}|$ is uniformly bounded.*

2. *If P is a point of I then $\lim_{n\to\infty}(r_{iP_n}n - r_{i'P_n}n) = 0$ almost everywhere on I.*

3. $\operatorname{Lim}_{n\to\infty} \sum_1^n r_{in}l_{in}$ *exists.*

CONCLUSION

1. $\operatorname{Lim}_{n\to\infty} \sum_1^n r_{in}'l_{in}$ *exists.*

2. $\operatorname{Lim}_{n\to\infty} \sum_1^n r_{in}l_{in} = \operatorname{Lim}_{n\to\infty} \sum_1^n r_{in}'l_{in}$.

Lebesgue's theorem (L) may be stated as follows:

HYPOTHESIS. 1. *$f_n(x)$ is a uniformly bounded sequence of integrable functions on I.*

2. $\operatorname{Lim}_{n\to\infty} f_n(x) = f(x)$ *almost everywhere on I.*

CONCLUSION

1. *$f(x)$ is bounded and integrable on I.*

2. $\operatorname{Lim}_{n\to\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt$.

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* See my paper entitled *On multiple iterated integrals*, loc. cit., for complete references.

† R. L. Moore, *On Duhamel's theorem*, Annals of Mathematics, (2), vol. 13 (1912), pp. 162–3.

‡ *Leçons sur l'Intégration*, etc. (Borel monograph.) Paris, Gauthier-Villars, 1904, p. 114.

By the use of Theorems III and IV, we derive (DM) from (L). Let $r_{in}-r'_{in}$ define a sequence of horizontal functions on I. The limit function is obviously a null function whose integral is zero. Conclusions 1 and 2 of (DM) follow immediately from (L).

Conversely we derive (L) from (DM). By Theorem IV, $f(x)$ is bounded and integrable in I. Let $h_n(x)$ be a set of horizontal functions on I with respect to I_{in} approaching $f(x)$ as a limit function almost everywhere on I. Identify $r_{in}=h_{in}$. Also let $H_n^{(k)}(x)$ be a horizontal function of index n on I associated with I_{in} and having $f_k(x)$ as a limit function almost everywhere on I. Identify $r'_{in}=H_{in}^{(nk)}$ and Conclusion 2 of (L) follows from (DM).

It is worthwhile emphasizing the role which horizontal functions, together with the principle isolated by R. L. Moore, are destined to play in the theory of functions of a real variable. They may be made the basis for a concise and elegant treatment of the greater part of the theory. The bulk of a book like Hobson's *Theory of Functions of a Real Variable*, volume I, may be reduced to one-third or less by their use. Schlesinger in the book under review has made a beginning in this direction.

H. J. Ettlinger

FUBINI AND ČECH, PROJECTIVE DIFFERENTIAL GEOMETRY

Geometria Proiettiva Differenziale, Vol. I. By G. Fubini and E. Čech. Bologna, N. Zanichelli, 1926. 388 pp.

"Un nuovo indizio dei sentimenti fraterni che vanno sempre più legando fra loro i vari rami della matematica!"

These words of Segre were chosen by Wilczynski to be inscribed on the title page of his prize memoir *Sur la théorie générale des congruences*. Thus the founder of the American school of projective differential geometry indicated that he was studying the projective differential properties of a geometric configuration by the means of the invariants and covariants of a completely integrable system of linear homogeneous partial differential equations under a certain continuous group of transformations, in the sense of Lie. The same sentiment would be no less appropriate as a motto for the new book by Fubini and Čech, since these distinguished protagonists of the Italian school of projective differential geometry define a configuration by means of differential forms, after the manner of Gauss, and employ the absolute calculus of Ricci.

Those who know the absolute calculus only as it is used in the theory of relativity will be interested to see this geometric application of it. And those who know only Wilczynski's method of attacking a problem in projective differential geometry will be eager to learn this new theory. Wilczynski's method is particularly adapted to certain types of problems and has a power and elegance of its own. But it has some inconveniences. For instance, certain calculations become quite laborious, which are accomplished more easily and efficiently by the tensor analysis.

The first volume of the treatise before us is dedicated to the dean of differential geometers, Luigi Bianchi, and is devoted to the geometry of

curves and surfaces in space of three dimensions. The projective definition of a surface by means of three differential forms of the first order, two quadratic and one cubic, is due to Fubini. Of these, one, F_2, vanishes for the asymptotics, and another, F_3, which is apolar to F_2, vanishes for the curves of Darboux. The ratio $F_3 : F_2$ is called the *linear projective element*. For this Bompiani has furnished a geometric interpretation; Čech has given an algebraic theory of the invariants thereof. The extremals of the integral $\int F_3 : F_2$ are called *pangeodesics*.

Fubini has defined a *projective applicability*, two surfaces being projectively applicable in case they have the same linear projective element. Moreover, for a non-ruled surface, Fubini has obtained a normalization of the proportionality factor of the homogeneous coordinate system such that his *normal coordinates*, if the asymptotic net is parametric, satisfy a certain canonical system of differential equations of the same form in covariant derivatives as Wilczynski's much used canonical equations in ordinary derivatives. Introducing the normalized form of F_2 as a projective substitute for the squared arc element, or first form of Gauss, Fubini defines a *projective normal*, which G. M. Green also discovered independently, projective geodesics, projective lines of curvature, etc. For this projective metric of Fubini, a geometric interpretation was furnished by Wilczynski.

The principle of duality plays its proper role, particularly in the formulation of the differential equations for the determination of a surface when the three forms that define it are given. The authors prefer, when using a local coordinate system at a point of a surface, to define the system so that a point and a plane whose corresponding local coordinates are equal are respectively pole and polar with respect to the quadric of Lie, called by Wilczynski the osculating quadric. With this convention the condition of united position of point and plane does not have its usual simple form.

An entire chapter is devoted to the theory of space curves, another to the theory of ruled surfaces, and still another to the transformation of surfaces by means of Weingarten congruences. The calculation of the integrability conditions for a curved surface in arbitrary parameters, by means of covariant derivatives, deserves special mention, as does also the treatment of various interesting classes of surfaces.

The book contains surprisingly few typographical errors, and these are for the most part easily corrected. One might wish that the type setter had been consistent in his use of italics for mathematical symbols. Occasionally the differentials du and dv and some other symbols are not in italics. The notation is well chosen, although one of my students expressed the wish that the authors had used c_{rst} instead of a_{rst} for the coefficients of F_3, since they had already used a_{rs} for the coefficients of F_2. The style of exposition is clear, concise, and direct.

This treatise fills a long felt want. There is no other of its kind. Hereafter the projective differential geometer must know his Fubini and Čech, and every student of geometry has now a new reason for acquiring at least a reading knowledge of *la lingua italiana*.

E. P. LANE

HOBSON'S SECOND VOLUME

The Theory of Functions of a Real Variable and the Theory of Fourier's Series. By E. W. Hobson. Second Edition, vol. II. Cambridge University Press, 1926. x+780 pp.

It is a long time since I previously had so much delight in examining a book as I have recently experienced in the rapid reading of the second volume of Hobson's *Functions of a Real Variable*. It is characterized by an extraordinary richness of content and by the remarkable indications which it affords of the marvelous vitality and fecundity of the current investigations in the general theory of functions of a real variable and in the theory of Fourier's series in particular. This second volume completes the author's extension and revision of the one-volume work which first appeared in 1907, the earlier part of his revision having been published in 1921 as volume I of the second edition. Almost the whole of the matter for the new volume II has been re-written and much new matter has been added so that the extent of the whole work is now twice as great as that of the first edition. The new matter is largely the fruit of investigations carried out in the last twenty years. The completed work now affords a most effective testimonial to current progress in one division of mathematics as well as the most important exposition yet given of knowledge in that domain.

The volume is beautifully printed in the style for which the Cambridge University Press has justly become famous among mathematicians. The volume is bound so as to be pleasing both to the eye and to the hand when new; but, as experience with earlier books so bound has shown, it has not the durability which one has a right to expect in the case of a volume of such permanent value and one likely to be so frequently used. In some places the author might have improved the appearance of the page by a happier choice of notation or a more suitable arrangement of formulas. A considerable number of minor errors escaped detection in the final proof-reading. But where there is so much excellence in regard to the matters of major importance, it is perhaps ungracious to do so much as to call attention to these minor blemishes.

There can be no doubt that the two volumes of the second edition of this work will be for a long time of constant use to every one who has to do with investigations touching the theory of functions of a real variable and the theory of Fourier series.

The first of the ten chapters of volume II deals with sequences and series of numbers. Into the 98 pages of this chapter the author compresses an excellent exposition of the theory of series with real constant terms, the earlier part of it being abbreviated by the use of some theorems from volume I. The reader will be interested in the compactness of the exposition and the comprehensiveness of the theory so far as the problem of convergence is concerned. It is in no sense a complete summary of what is known about series with constant real terms but it is an account which sets all the main facts in their proper light. Nearly forty pages of the chapter are given to the problem of the summability of series of constant terms.

In this part no effort is made to reach a comprehensive exposition. In fact attention is directed chiefly to an account of some of the main features of summability by the methods of Cesàro, Hölder, and Riesz; and the more general problems connected with the summability of series find here no systematic treatment.

An exhaustive exposition of the general theory of the summability of series has not yet appeared. The theory has now probably attained such a state of development as would render feasible such an exposition. It is one of the important desiderata which must be felt by every one who has to deal with the more delicate questions concerning the convergence of infinite processes. Such a work, if properly prepared, would be of constant use to a large number of investigators in many fields of analysis.

In Chapter II (pp. 99–171) is given a systematic account of the theories of convergence and oscillation of sequences and series of functions of one or more variables. This is done in an elegant and pleasing way and with such a comprehensiveness as one would expect in a general treatise. Special attention is given to the problems associated with uniform convergence and with the various modifications of the notion of uniformity of convergence. The question of the continuity of the sum-function of a series is treated in detail.

An application of the general results of Chapter II is made in Chapter III (pp. 172–227) to the special but important case of power series. Problems relating to interior points of the interval of convergence are disposed of briefly. Much attention is devoted to the behavior of the sum-function near the ends of the interval of convergence and to its relation to the series obtained by giving to the variable its value at an end-point of this interval. In this connection one finds an interesting and illuminating account of the Tauberian theorems which have been the subject of a considerable amount of recent investigation.

Chapter IV (pp. 228–288) contains an account of the theorem of Weierstrass concerning the representation of a continuous function (of one or more variables) by means of sequences of polynomials; of the theory of convergence of sequences on the average; of F. Riesz' classification of summable functions; and of Baire's classification of functions. The fundamental result of Baire, relating to the representation of a function as the limit of a sequence of continuous functions, is proved by a method due to de la Vallée Poussin, but with some modification and extension.

Chapter V (pp. 289–388) is on sequences of integrals. It contains an exposition of those parts of the theory of integration which were not already treated in volume I. In particular, considerable space is given to the theories of integration due to W. H. Young, to Tonelli, and to Perron. The chapter concludes with a very brief account of the summability of integrals, especially by the methods analogous to Cesàro and Hölder summability of series. The author expresses (p. 384) the opinion that Perron's recent definition of an integral may prove to be of great importance in the future developments of the concept of integration.

In Chapter VI (pp. 389–421) is to be found an account of the methods of construction of functions exhibiting assigned peculiarities of behavior

and in particular of functions which are continuous but non-differentiable. An exposition is given of various methods of condensation of singularities.

Chapter VII (pp. 422–475) is devoted to the problem of the representation of functions as limits of integrals. A special feature of this chapter, and indeed of the whole volume, is the prominence which is here given to what the author calls the General Convergence Theorem and to its developments and consequences. This general theorem is treated fully, and with considerable novelty of exposition, in Chapter VII, and various applications of it to Fourier's series and integrals are given in the following chapters.

The remaining three chapters of this volume correspond to the final chapter of the first edition.

Chapter VIII (pp. 476–719) deals with trigonometric series. It is by far the longest chapter in the volume. It is here that we find the amplest evidence of the recent activity of investigators to be found anywhere in the volume,—a volume which is marked throughout with much evidence of rapid recent progress. Most of the recent progress in the development of the theory of trigonometric series has been due to the use of two recently constructed tools—that afforded by the theory of Lebesgue integration and various other more recent definitions of integration and that due to various definitions of the summability of series recently exploited with so much vigor. Though trigonometric series have had a remarkable history covering more than a century and a half, the extraordinary recent progress in our knowledge of them and the questions that still remain unanswered show unmistakably that the subject is capable of much further development.

The theory of trigonometric series has had a nearly continuous and remarkably intimate connection with the development of much of the general theory of functions of a real variable. This alone is sufficient to justify the author in analyzing it with some fulness in such a general treatise as that under review. The connection began early and its intimacy is reflected in much of the most recent development. Concerning the connection the author says (p. 476): "Historically, the questions which have arisen in connection with this theory have influenced the development of the theory of functions of a real variable to an extent which is comparable with the degree in which the theory of functions in general has been affected by the theory of power series. The theory of sets of points, which led later to the abstract theory of aggregates, arose directly from questions connected with trigonometric series. The precise formulation by Riemann of the conception of the definite integral, and the gradual development of the modern notion of a function as existent independently of any special mode of representation by an analytical expression, are further examples of the results of the study of the properties of these series upon Mathematical Analysis."

The chapter contains an adequate treatment both of the earlier and of the later theory of trigonometric series and especially of Fourier series, limitation being made to the case in which the arguments involved in the trigonometric functions are integral multiples of the independent variable.

A reference is made to the fact that there are important trigonometric series in which the arguments are differently determined, but these additional types are not treated. Notwithstanding the great richness of detail in the theory as here expounded, one comes from a reading of the chapter with the feeling that the whole theory is yet in a state of development which is far from complete. It can be predicted that much further development of the theory will take place in the near future, both in the way of deeper penetration into details and in the analysis of general methods of treatment and especially of more general types of summability of the series. The theory of summability is capable of much further development by aid of the theory of functions of a complex variable (and especially the theory of residues), a tool which is nowhere employed in the exposition as given by Hobson.

Chapter IX (pp. 720–752) is devoted to the representation of functions by Fourier integrals, together with the modern theory of Fourier transforms. A significant part of the chapter is given to the summability of Fourier integrals, a subject which appears to be capable of much further development.

The final Chapter X (pp. 753–772) contains a brief treatment of series of normal orthogonal functions, a subject which is included "not only on account of the intrinsic importance of the subject, but also because the processes which have been employed in various recent investigations in this domain afford excellent illustrations of ideas and methods which have been developed earlier in this work." In this chapter no attempt at a comprehensive treatment has been made. In fact the exposition scarcely contains so much as an introduction to this extended subject. Many of the properties of Fourier series can be carried over to certain classes of expansions in orthogonal functions—a fact which is but little more than indicated in the brief account to be found here. Just as the first chapter leaves one with the desire for a full exposition of a subject but partially treated so the final chapter leaves one wishing for an adequate and exhaustive treatise on the theory of expansions in orthogonal and biorthogonal functions. What the author gives is scarcely enough to suggest the vast richness of this subject. In this respect the chapter is different from most of those contained in the volume,—the exposition being in most cases adequate and satisfying.

R. D. Carmichael

SHORTER NOTICES

Versicherungsmathematik. I: *Elemente der Versicherungsrechnung.* By Friederich Boehm. (Sammlung Göschen.) Berlin, W. de Gruyter, 1925. 144 pp.

This is an elementary text on the mathematics of life insurance, the subject being treated particularly from the point of view of German and continental practice. It is divided into four chapters. The first deals with the elements of the theory of compound interest—just enough to carry through the accumulation and discount applications necessary in the theory of life insurance.

The second chapter starts in with the life table and forty pages are devoted to questions concerning expectation of life, life annuities of various kinds and single and annual premiums for ordinary and limited payment life, temporary, and endowment insurance.

The third chapter develops the theory of premium reserves including a discussion of net level premium reserves and Zillmer's formula for reserves; in particular, the full preliminary term method of Zillmer is explained.

Chapter IV deals with annuities and insurance on joint lives, including the calculation of single and annual premiums and reserves. Some little space is given to mixed forms of insurance, widows' pensions prior to remarriage and annuities to orphans.

This little book is evidently intended as an introduction to the theory of life insurance, the formulas being developed in considerable detail and practically every formula immediately illustrated by a numerical example. The author is not familiar with the international code of symbols adopted and carefully adhered to by most actuaries, or else he has intentionally modified these symbols. This is particularly noticeable in the use of the summation sign for the N_x and M_x, the use of the letter R instead of V to designate reserves, and the use of the symbol for endowment insurance for temporary insurance. Similar modifications have been made for the present value and accumulated value compound interest functions. These changes are unfortunate and quite uncalled for and are bound to cause confusion to all students who get their introduction to actuarial theory through this book when they come to read other text books on this subject. The use of the summation sign for the N and M commutation symbols makes the formulas quite cumbersome in appearance. It is to be hoped that if a new edition of this book should appear the author will adopt the symbols of the international code of actuaries. Apart from these criticisms the book is well written and may be regarded as a good introduction to actuarial theory.

J. W. Glover

Traité de Balistique Extérieure. By P. Charbonnier. Vol. I. Paris, Gauthier-Villars, 1921. ix+637 pp.

For a number of years, General Charbonnier has been preparing an encyclopedic treatise on exterior ballistics in six volumes, of which this is the first.

The first part of this volume contains the most exhaustive treatment of the trajectory in vacuum which has yet been written.

The second part deals with the rectilinear motion of a particle in a medium where the resistance depends on the velocity alone; the Siacci ballistic functions are introduced and their simplest properties derived.

The third part is concerned with the general case of the trajectory of a heavy particle, the resistance being a function of the velocity. Here, as in the second part, the change in resistance due to the altitude variation in the density of the atmosphere, is neglected. This part begins with a very detailed study of the qualitative properties of the trajectory profusely illustrated by well-drawn diagrams. This investigation is of particular interest to the mathematician, and might be used as a source of examples of the behavior of the integral curves in an advanced course in differential equations. As far as the reviewer has examined the theorems in this section, they are correct, but the proofs are frequently lacking in rigor and occasionally contain actual errors, which are, however, not very difficult to detect and remedy.

This section also contains a brief and very readable exposition by A. Denjoy of Drach's recent determination of all the forms of the resistance function for which the equations of the trajectory are integrable by quadratures.

The further treatment of the trajectory follows the customary lines of the Siacci and similar theories, and the volume ends with a chapter on the use of power series expansions for computing a small arc of the trajectory.

The modern methods in ballistics are not dealt with here, but are reserved for the later volumes, none of which has yet appeared.

T. H. Gronwall

Séries Analytiques. Sommabilité. (Mémorial des Sciences Mathématiques, No. 7.) By A. Buhl. Paris, Gauthier-Villars, 1925. 55 pp.

This monograph deals with analytic extension by means of various methods of summation applied to the Taylor's series of a function in regions where this series is divergent. The results obtained are those due to Borel and Mittag-Leffler and certain other investigators who have taken their methods as a point of departure. The various theories are unified by basing the whole discussion on a double integral formula which is one of the important generalizations of the well known integral formula of Cauchy. By this synthetic form of treatment Professor Buhl has found it possible to include an unusually large number of interesting results in the fifty odd pages of the monograph.

C. N. Moore

Lehrbuch der Ballistik. By C. Cranz. Berlin, Julius Springer, 1925. xiv+ 711 pp.

This is the fifth edition of the author's first volume on ballistics, and it is confined to exterior ballistics. It is dedicated with appropriate courtesy to the late General A. von Kersting, the founder and first director of the Military Academy (1903–1912) who, during the first two years of the World War, rendered valuable service to German military technique as President of the Artillery Proving Commission. Proper acknowledgements are made to Major Becker for his work on ballistic wind and ballistic density, and to Professor O. von Eberhard who was the first to prove by actual computations the possibility of a trajectory of more than 100 km. in range.

Whatever might have been in the mind of Dr. Cranz in selecting the title of the book, the importance of its contents may be best described to the American reader by giving it the title *A Compendium of Exterior Ballistics*. The reviewer knows of no other volume on this subject that contains so much information together with such a thorough and masterly treatment of the subject. There are 539 pages of text and 267 pages of bibliography, ballistic tables, and ballistic diagrams. The references to literature are carefully assified as having special bearing upon definite articles. The bibliography alone makes the book highly desirable for a library.

In the first part of the book the author discusses in detail the trajectory in vacuo with numerous illustrations of the path of a projectile fired from an elevation and from an inclined plane. These are interesting and, of course, elementary. If the reviewer is at liberty to give his real impressions of the contents of a book, possibly I may be pardoned for saying that one of these examples struck me with surprise. The problem is as follows: "Is it possible to throw a stone from the top of the Great Pyramid out over the base." After reading that its altitude is 137.2 m. and that the side of its square base is 227.5 m. one remains somewhat at a loss to know what the problem really is until he meets the statement that the velocity of throwing from the hand is assumed to be 24m/s which, he explains parenthetically, is the mean of thirty tests made upon as many different persons. He does not state how these tests were conducted, nor does he describe the physical characteristics of the persons who did the throwing. He shows that it is possible to clear the pyramid with an initial velocity of about 20m/s provided that the stone is thrown at the proper angle and he concludes that it is possible to throw a stone clear of the pyramid with " a certain amount of skill." Merely to throw a stone 20 m/s requires very little skill. Certainly the thirty persons would be poor candidates for a baseball team if the average velocity with which they throw a stone is about 65 f/s. The baseball player, I should think, throws with a velocity of 150 f/s to 165 f/s. Skill in this case consists in throwing at 20 m/s from the very restricted area at the top of the pyramid.

He follows the usual course of deriving the differential equations of the trajectory and then presenting the different methods of integrating these equations and of constructing the trajectory graphically. A number of graphical processes are described beginning with that of Poncelet in 1848

and coming down to those developed during the World War and later. Two of these, one by Vahlen in 1918 and another by Brauer in 1918, call forth the comment by Dr. Cranz that it would be a worthwhile problem for the investigator in ballistics to substantiate the sufficiency and utility of both of these graphical processes. The Vahlen method depends upon vectorial addition; that of Brauer is based upon the hodograph.

The author's presentation of satellite, parabolic, and hyperbolic velocities is of more than usual interest and the figures that accompany this discussion are especially attractive. The effect of yaw is considered in detail both from the theoretical and practical standpoints. This is followed by the fundamentals in projectile design for all types of projectiles.

The subjects of vertical, near-vertical, and long-range firing are treated under separate heads. The first two of these, of course, apply to anti-aircraft fire. The third is of very great interest to all those who are interested in modern ballistics. This is the portion of the book for which Dr. Cranz gives credit to Professor Eberhard. Comments are made upon phenomena observed abroad in 1914 which have been observed in this country since that time. A projectile was fired under initial conditions which, according to Siaccian methods of computing the trajectory, would have given a range of 38 Km.; the observed range was 49 Km. The angle of elevation which gave the greatest range was in the neighborhood of 55°. In this country unusually long ranges, viewed from the standpoint of Siaccian ballistics, have been obtained by firing at high angles of elevation, the angle of elevation giving the greatest range being above 45°, and comparatively little difference in range for angles between 45° and 55° elevation. This is explained by the fact that the projectile encounters less resistance as it rises through lighter and lighter strata than if it had passed through only comparatively denser air as in lower-angle fire.

Probably a better illustration of the effect of the rotation of the earth upon a projectile might have been given than the one selected by the author. Certainly it is not usual to fire with a velocity of 5000 f/s (1600 m/s). Until more firing has been done at such high velocity and the results observed, his illustration is more in the nature of the prediction of the pure mathematician than that of the conservative military engineer. If this particular example is given to register surprise, others with smaller initial velocities should have been given.

The arrangement of the material and the clearness of presentation are such that the book might be made the basis of a course of lectures on ballistics. Such a course could be made very elementary and involve merely an intelligent use of the tables and formulas, but by delving into the more theoretical portions of the book, especially the parts relating to the rotating projectile, it may be made exceedingly difficult. The volume is not only a *compendium* of facts and methods relating to exterior ballistics, written by one whose pen has enriched the literature on the general subject of ballistics for more than thirty years and filled with data that could be assembled and classified only by a lifelong interest in the subject, but it is also the product of a high type of thorough and mature scholarship.

J. E. Rowe

NOTES

Professor J. H. M. Wedderburn, of Princeton University, has been appointed an associate editor of this Bulletin. Professors R. W. Brink, of the University of Minnesota, and J. D. Tamarkin, of Dartmouth College, have been appointed associate editors of the Transactions of this Society.

The concluding number of volume 28 of the Transactions of this Society (October, 1926) contains the following papers: *A boundary value problem for a system of ordinary linear differential equations of the first order*, by G. A. Bliss; *On the theory of integral equations with discontinuous kernels*, by R. E. Langer; *The figuratrix in the calculus of variations*, by P. R. Rider; *On the existence of fields in boolean algebras*, by B. A. Bernstein; *Asymmetric displacement of a vector*, by J. M. Thomas; *Tensors determined by a hypersurface in Riemann space*, by H. Levy; *A comparison of the series of Fourier and Birkhoff*, by M. H. Stone; *Analytic approximations to topological transformations*, by P. Franklin and N. Wiener.

The concluding number of volume 48 of the American Journal of Mathematics (October, 1926) contains: *Regular maps on an anchor ring*, by H. R. Brahana; *Determination of the type of the tricrunodal quartic by means of its invariants*, by L. T. Moore; *Subgroups of index p^2 contained in a group of order p^m*, by G. A. Miller; *Mapping by means of linear systems of curves invariant under Cremona involutions*, by T. L. Bennett; *Algebraic and transcendental equations connected with the form of stream lines*, by H. Bateman.

The Journal für die reine und angewandte Mathematik, founded in 1826 by August Leopold Crelle, is celebrating its centenary by the publication of a special Jubiläumsband.

On the occasion of the two hundredth anniversary of the death of Sir Isaac Newton, which occurs on March 20, 1927, a meeting will be held under the auspices of the (British) Mathematical Association at Grantham, near Newton's birthplace. Addresses will be delivered by Sir J. J. Thomson and others, on various aspects of his work.

Dr. Horace Lamb has been elected president of the Cambridge Philosophical Society.

Among the officers elected at the Philadelphia meeting of the American Association for the Advancement of Science are the following: Professor A. A. Noyes, of the California Institute of Technology, president; Professor Dunham Jackson, of the University of Minnesota, vice-president of Section A (mathematics); Professor A. H. Compton, of the University of Chicago, vice-president of Section B (physics); Dr. W. S. Adams, of Mount Wilson Observatory, vice-president of Section D (astronomy). Professor L. E. Dickson was elected a member of the Council, and Professor Oswald Veblen a member of the Committee on Grants.

Professor K. T. Compton, of Princeton University, has been elected president of the American Physical Society.

Nobel prizes in physics have been awarded as follows: for 1925, jointly to Professors J. Franck, of Göttingen, and G. Hertz, of Halle; for 1926, to Professor Jean Perrin, of the Sorbonne.

The prize of the American Association for the Advancement of Science for a paper read at its annual meeting of 1926, was awarded to Professor G. D. Birkhoff for his address as retiring president of the American Mathematical Society, delivered at a joint session of this Society with Section A of the Association. The address, entitled *A mathematical critique of some physical theories*, will appear in the next issue of this Bulletin.

Dr. Wolfgang Krull, of the University of Freiburg i. B., has been promoted to an associate professorship of mathematics.

Professor William H. Bussey has resigned the chairmanship of the Department of Mathematics at the University of Minnesota, in the College of Science, Literature and the Arts, so that he may devote more time to his duties as assistant dean for the junior college, and as editor-in-chief of the American Mathematical Monthly. Professor William L. Hart has been appointed chairman of the Department of Mathematics.

Dr. Selig Brodetsky, professor of applied mathematics at the University of Leeds, has arrived in the United States to make a tour of the country in the interest of the University of Jerusalem, of which he is one of the governors. He will lecture on *The Einstein theory of relativity* at Clark University on January 10; later he will give a series of lectures at Harvard University.

Associate Professor M. G. Gaba, of the University of Nebraska, has been promoted to a full professorship of mathematics.

Dr. Mayme I. Logsdon, of the University of Chicago, has been promoted to an assistant professorship of mathematics.

Assistant Professor F. S. Nowlan, of the University of Manitoba, has been appointed professor of mathematics at the University of British Columbia.

Professor E. Blaschke, of the Vienna Technical School, died October 30, 1926, at the age of seventy.

Professor Alexander Tschuprow, formerly of the Technical Institute at Petrograd, died April 19, 1926, at the age of fifty-two. He was known for his work in mathematical statistics.

Professor Louis Siff, of Louisville University, died December 25, 1926, at the age of fifty-seven.

Professor J. H. Tudor, of Pennsylvania State College, died June 7, 1926, at the age of sixty-nine. Professor Tudor was a member of this Society.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

BOULIGAND (G.) et RABATÉ (G.). Initiation aux méthodes vectorielles et aux applications géométriques de l'analyse. Paris, Vuibert, 1926. 2+215 pp.

CRUMLEY (T.). Logic, deductive and inductive. New York, Macmillan, 1926. 442 pp.

DURELL (C. V.). Projective Geometry. London, Macmillan, 1926. 10+222 pp.

EUCLID. See HEATH (T. L.).

FANO (G.). Lezioni di geometria descrittiva. Torino, Paravia, 1926. 19+461 pp.

GONSETH (F.). Les fondements des mathématiques. De la géométrie d'Euclid à la relativité générale et à l'intuitionisme. Paris, Blanchard, 1926. 16+243 pp.

VAN HAAFTEN (—.). Reziproken aller ganzen zahlen van 1 bis 10000. Aufgabe F von Noordhoff's Tafeln. Groningen, Noordhoff, 1926. 23+50 pp.

HADAMARD (J.) et MANDELBROJT (—.). La série de Taylor et son prolongement analytique. 2e édition, revue et mise au courant des progrès récents. (Scientia, No. 41.) Paris, Gauthier-Villars, 1926. 87 pp.

HASSE (H.). Höhere Algebra. I: Lineare Gleichungen. (Sammlung Göschen.) Berlin, de Gruyter, 1926. 160 pp.

HEATH (T. L.). The thirteen books of Euclid's elements translated from the text of Heiberg with introduction and commentary. 2d edition, revised with additions. 3 volumes. Cambridge, University Press, 1926. 432+436+546 pp.

HENDERSON (J.). Bibliotheca tabularum mathematicarum; being a descriptive catalogue of mathematical tables. Part 1: Logarithmic tables. (Tracts for Computers, No. 13.) Cambridge, University Press, 1926. 6+208 pp.

HOPF (E.). Ueber die Zusammenhänge zwischen gewissen höheren Differenzen-Quotienten reeller Funktionen einer reeller Variablen und deren Differenzierbarkeitseigenschaften. (Dissertation.) Berlin, Norddeutsche Buchdruckerei, 1926.

KRAITCHIK (M.). Théorie des nombres. Tome II: Analyse indéterminée du second degré et factorisation. Paris, Gauthier-Villars, 1926. 4+251 pp.

LAMOUCHE (A.). Le principe de simplicité dans les sciences mathématiques et dans les sciences physiques. Brest, La Dépêche, 1926.

LECAT (M.). Coup d'œil sur la théorie de la multiplication des déterminants supérieurs dans son état actuel. Louvain, Ceuterick, 1926. 40 pp.

LITTLEWOOD (J. E.). The elements of the theory of real functions. 2d (revised) edition. Cambridge, Heffer, 1926. 7+60 pp.

Lötzbeyer (P.). Vierstellige Tafeln zum logarithmischen und natürlichen Rechnen. Ausgabe B. Leipzig, Teubner, 1926.

Mair (D. B.). Fourfold geometry. Being the elementary geometry of the four-dimensional world. London, Methuen, 1926. 8+183 pp.

Mandelbrojt (—.). See Hadamard (J.).

Muika (I.). Das Fermatsche Theorem. Bukarest, F. Göhl Söhne, 1926. 3 pp.

Neugebauer (O.). Die Grundlagen der ägyptischen Bruchrechnung. Berlin, Springer, 1926. 46 pp.

Paraf-Javal (—.). Géométrie à toutes dimensions. Principes de géométric physique. Paris, Groupe d'Etudes Scientifiques, 1926.

Rabaté (G.). See Bouligand (G.).

Saint-Laguë (A.). Les réseaux (ou graphes). (Mémorial des Sciences Mathématiques, No. 18.) Paris, Gauthier-Villars, 1926.

Wolf (A.). Excercises in logic and scientific methods. New edition, revised and enlarged. London, Allen and Unwin, 1926. 125 pp.

Woods (F. S.). Advanced calculus. Boston, Ginn, 1926. 9+397 pp.

PART II. APPLIED MATHEMATICS

Arsovitch (R.). Pascal et l'expérience du Puy-de-Dome. Montpelier Imprimerie Firmin et Montane, 1925. 76 pp.

Barkhausen (H.). Elektronen-Röhren. Band 2. Leipzig, Hirzel, 1925. 6+168 pp.

Beaver (J. L.). Elements of alternating currents and alternating current apparatus. London, Longmans, 1926. 370 pp.

Betz (A.). Wind-Energie und ihre Ausnutzung durch Windmühlen. Göttingen, Vandenhoeck und Ruprecht, 1926. 70 pp. +4 tables.

Bleich (F.). Die Berechnung statisch unbestimmter Tragwerke nach der Methode des Viermomentensatzes. Berlin, Springer, 1925. 5+220 pp.

Breed (C. B.) and Hosmer (G. L.). The principles and practice of surveying. 3d edition, revised. Volume 2. London, Chapman and Hall, 1926. 507 pp.

Brunet (P.). Les physiciens hollandais et la méthode expérimentale en France au XVIIIe siècle. Paris, A. Blanchard, 1926. 153 pp.

Carstangen (M.). See Koch (A.).

Corps (—.). Les origines de l'erreur relativiste. La nouvelle expérience de M. Michelson. Paris, Les Presses Universitaires de France, 1926. 20 pp.

Corral (J. I.). Nueva solución del problema de Lord Kelvin sobre ecuaciones de coeficientes reales. Madrid, Imprenta Clasica Española, 1926.

Crandall (I. B.). Theory of vibrating systems and sound. New York, Van Nostrand, 1926. 10+272 pp.

Crehore (A. C.). The progress of atomic theory. London, Taylor and Francis, 1926.

Eyraud (H.). Les équations de la dynamique de l'éther, avec une note sur la technique du repérage de l'espace et du temps dans ses rapports avec la relativité. Paris, A. Blanchard, 1926. 68 pp.

Forestier (A.). L'énergie rayonnante. 2e édition, augmentée. Paris, A. Blanchard, 1926. 76 pp.

Franck (J.) und Jordan (P.). Anregung von Quantensprüngen durch Stösse. Berlin, Springer, 1926. 8+312 pp.

Garrett (H. E.). Statistics in psychology and education. New York, Longmans, 1926. 14+317 pp.

Geiger (H.) und Scheel (K.), herausgegeben von. Handbuch der Physik. Band 1: Geschichte der Physik. Vorlesungstechnik. Redigiert von K. Scheel. Band 2: Elementare Einheiten und ihre Messung. Redigiert von K. Scheel. Band 9: Theorien der Wärme. Redigiert von F. Henning. Band 17: Elektrotechnik. Redigiert von W. Westphal. Band 23: Quanten. Redigiert von H. Geiger. Berlin, Springer, 1926. 6+404+8+522+8+616+399+9+782 pp.

Gibson (A. H.). Hydraulics and its applications. 3d edition. London, Constable, 1925. 15+801 pp.

Glauert (H.). The elements of aerofoil and airscrew theory. Cambridge, University Press, 1926. 8+228 pp.

Haas (A.). See Wien (W.).

Harms (F.). See Wien (W.).

Henning (F.). See Geiger (H.).

Hosmer (G. L.). See Breed (C. B.).

Ilberg (W.). Drahtlose Telegraphie und Telephonie in ihren physikalischen Grundlagen. Leipzig, Teubner, 1925.

Joly (G.). Nouvelle étude physique de l'expérience de MM. Michelson et Morley qui sert de base à la théorie de la relativité. Paris, Editions Spes, 1926. 8 pp.

Jordan (P.). See Franck (J.).

von Jüptner (H.). Gas, Dampf und Flüssigkeit. Leipzig, O. Spammer, 1925.

Kafka (H.). Die ebene Vektorrechnung und ihre Anwendungen in der Wechselstromtechnik. Teil 1: Grundlagen. Leipzig, Teubner, 1926. 8+132 pp.

Kapp (G.). Transformers for single and multiphase currents. A treatise on their theory, construction and use. 3d edition, revised by R. O. Kapp. London, Pitman, 1925.

Kapp (R. O.). See Kapp (G.).

Koch (A.). Von der Bewegung des Wassers und den dabei auftretenden Kräften, nach Arbeiten von Staatsrat. Herausgegeben von M. Carstangen. Berlin, Springer, 1926.

Lepape (A.). See Millikan (R. A.).

Lewitt (E. H.). The rigid airship. A treatise on the design and performance. London, Pitman, 1925. 10+283 pp.

Limburg (H.). Onderzoekingen over emulsies. (Dissertation, Delft.) Delft, 1924. 126 pp.

Loney (S. L.). Solutions of the examples in a treatise on dynamics of a particle and of rigid bodies. Cambridge, University Press, 1926. 6+240 pp..

Maney (G. A.). See Parcel (J. I.).

Marcus (H.). Die vereinfachte Berechnung biegsamer Platten. Berlin, Springer, 1925. 92 pp.

Metz (A.). Les nouvelles théories scientifiques et leurs adversaires. La relativité. 15e édition. Paris, Chiron, 1926. 270 pp.

Millikan (R. A.). L'électron. Traduit sur la 2e édition américaine par A. Lepape. Paris, Alcan, 1926. 346 pp.

O'Rourke (C. E.). See Urquhart (L. C.).

Parcel (J. I.) and Maney (G. A.). An elementary treatise on statically indeterminate stresses. New York, Wiley, 1926. 368 pp.

Payne (C. H.). Stellar atmospheres. Cambridge, Harvard University Press, 1925. 9+215 pp.

Peake (E. G.). An academic study of some money market and other statistics. 2d edition. London, King, 1926. 1+105 pp.

Pertsch (J. G.). Electrical engineering problems. Part 1. New York, McGraw-Hill, 1925. 8+213 pp.

Petitet (A.). See Rousselet (L.).

Pöschl (T.), herausgegeben von. Berechnung von Behältern nach neueren analytischen und graphischen Methoden. 2te Auflage. Berlin, Springer, 1926.

Popoff (K.). Les méthodes d'intégration de Poincaré et le problème général de la balistique extérieure. Paris, Gauthier-Villars, 1925.

Pritchard (J. L.). Broadcast reception in theory and practice. London, Chapman and Hall, 1926. 11+259 pp.

Rousselet (L.) et Petitet (A.). Stabilité des constructions usuelles. Calculs algébriques et graphiques des efforts dans l'ossature principale. 2e édition, revue, corrigée et augmentée. Paris, Dunod, 1926. 6+329 pp.

Scheel (K.). See Geiger (H.).

Schreiber (P.). Wärmemechanik wasserhaltiger Gasgemische mit den Hilfsmitteln der Flächennomographie bearbeitet. Braunschweig, Vieweg, 1925. 8+195 pp.

Spreen (W.). Formeln und Tabellen aus dem Gebiete der Funktechnik. Berlin, Springer, 1925. 80 pp.

Stuyvaert (M.). Statique et dynamique: 2e édition. Gand, Van Ryssel-berg et Rombaut, 1926. 208 pp.

Thomas (P.). Calcul des constructions continues à éléments droits. Application aux arcs hyperstatiques. Paris, Béranger, 1925.

Urquhart (L. C.) and O'Rourke (C. E.). Steel structures. Stresses in simple structures. New York, McGraw-Hill, 1926. 9+278 pp.

Westphal (W.). See Geiger (H.).

Wien (W.) und Harms (F.). Handbuch der Experimentalphysik. Band 2: Mechanik der Massenpunkte und der starren Körper, von Arthur Haas. Leipzig, Akademische Verlagsgesellschaft, 1926. 14+355 pp.

THE THIRTY-THIRD ANNUAL MEETING OF THE SOCIETY

The thirty-third Annual Meeting of the American Mathematical Society was held at the University of Pennsylvania, from Tuesday to Thursday, December 28–30, 1926, inclusive, in conjunction with the meetings of the Mathematical Association of America and the American Association for the Advancement of Science. The mornings and afternoons of Tuesday and Wednesday were devoted to the regular scientific sessions of the Society; the morning sessions were in sections, on Tuesday (i) Applications and Analysis, and (ii) Geometry, and on Wednesday (i) Analysis and (ii) Point Sets and Geometry. The Gibbs Lecture was delivered on Tuesday afternoon. On Thursday morning a joint session of the two mathematical organizations and Section A of the American Association for the Advancement of Science was held. The joint dinner on Wednesday evening at the Aldine Hotel was attended by about one hundred seventy-five mathematicians and their friends; Professor E. V. Huntington, Vice-President of Section A, presided.

At the joint session on Thursday morning, it was voted to express to the University of Pennsylvania and its Department of Mathematics and also to the Drexel Institute and other inviting organizations the thanks of the Society for their hospitality during the meeting, the largest in the history of the Society.

The attendance included the following two hundred four members of the Society:

C. R. Adams, Archibald, Atchison, W. L. Ayres, R. W. Babcock, Bacon, Barney, F. W. Beal, A. A. Bennett, Benton, William James Berry, Bill, Birkhoff, G. A. Bliss, Blumberg, Bower, Bradshaw, Brasefield, B. H. Brown, H. S. Brown, Margaret Buchanan, R. W. Burgess, Caldwell, W. B. Campbell, Caris, Carlen, Chapelon, Clawson, Clutz, Coble, Abraham Cohen, Coleman, J. T. Colpitts, Comegys, Crane, Crawley, Currier, Curry, Curtiss, Dadourian, Daniells, J. E. Davis, Decker, Dederick, Dillingham, Dimick, Douglas, Edmondson, Eiesland, Eisenhart,

W. W. Elliott, Eshleman, H. B. Evans, Everett, Feinler, Finkel, Fite, Focke, Foraker, W. B. Ford, C. H. Forsyth, Fort, M. C. Foster, Philip Franklin, Fry, C. A. Garabedian, Garretson, Gehman, D. C. Gillespie, Gleason, Glenn, Gravatt, C. C. Grove, Guggenbühl, Hallett, Hancock, M. G. Haseman, Haskins, Hazlett, Hedlund, E. R. Hedrick, H. B. Hedrick, Archibald Henderson, Robert Henderson, Horton, Hosford, Huber, J. C. Hughes, Huntington, Hurwitz, F. K. Hyatt, Louis Ingold, Ingraham, Dunham Jackson, M. I. Johnson, R. A. Johnson, O. D. Kellogg, Knebelman, Kormes, Lambert, Lamond, Langer, Langford, Lefschetz, Lehr, Harry Levy, F. P. Lewis, Littauer, Locke, Logsdon, McGiffert, McMackin, MacColl, MacInnes, E. N. Martin, Mears, Meder, Miser, H. H. Mitchell, Molina, R. L. Moore, Richard Morris, Marston Morse, Murnaghan, F. H. Murray, Nassau, Neelley, Olds, F. W. Owens, H. B. Owens, B. C. Patterson, Paxton, Pehrson, Pepper, E. C. Phillips, Pierpont, Polley, R. G. Putnam, Ragsdale, Rau, Raynor, C. J. Rees, Reid, C. N. Reynolds, J. N. Rice, R. G. D. Richardson, H. L. Rietz, Ritt, E. D. Roe, Roever, Root, Rorer, Rowe, Rowland, Safford, Sakellariou, Seely, Seidlin, Shewhart, Shugert, Silverman, W. G. Simon, L. G. Simons, Slaught, Smail, A. W. Smith, W. M. Smith, Virgil Snyder, Starke, Stokes, Swartzel, Swift, Tamarkin, Thurston, Torrey, B. M. Turner, Tyler, Uhler, Vandiver, Veblen, Vivian, Waddell, G. W. Walker, J. L. Walsh, Wedderburn, Weida, V. H. Wells, A. P. Wheeler, A. M. Whelan, R. A. Whelan, Whited, Widder, C. E. Wilder, F. B. Williams, K. P. Williams, W. L. G. Williams, A. H. Wilson, E. W. Wilson, R. G. Wood, Wyant, Yeaton, J. W. Young, M. M. Young, Zippin.

The Council elected the following sixteen persons to membership in the Society:

Mr. Ross Harvey Bardell, University of South Dakota;
Professor Stanley Eugene Brasefield, Rutgers University;
Professor Ona Kenneth DeFoe, College of the Ozarks;
Mr. Victor C. d'Unger, Little Rock, Arkansas;
Mr. Aurelius Augustus Evans, Columbia University;
Professor Hemphill Moffett Hosford, Southern Methodist University;
Miss Olive Allan Kee, Teachers College, Boston;
Mr. Eugene A. Kholodovsky, Columbia University;
Mr. Edward Dennis McCarthy, Pennsylvania State College;
Mr. Guerdon David Nichols, Colorado School of Mines;
Miss Katharine Elizabeth O'Brien, College of New Rochelle;
Mr. Thomas Howard Rawles, Yale University;
Mr. George Wesley Riddle, Lehigh University;
Mr. Homer Morgan Rutherford, Houston, Texas;
Mr. Albert Edward Staniland, University of Pittsburgh;
Professor William John Webber, University of Toronto.

The following four persons were elected to membership as nominees of Allyn and Bacon:

Mr. H. V. Craig, University of Wisconsin;
Mr. Lawrence Sanford Kennison, Brown University;
Miss Rose Alice Whelan, Bryn Mawr College;
Mr. Arthur Simeon Winsor, Johns Hopkins University.

The following three persons were elected to membership as nominees of the National Life Insurance Company of the United States of America:
Mr. George Cramer, University of Missouri;
Miss Florence Marie Mears, Cornell University;
Mr. Aubrey Henderson Smith, Brown University.

As a nominee of the University of Pennsylvania was elected:
Mr. Leo Zippin.

The ordinary membership in the Society is now 1692, including 178 nominees of sustaining members and 82 life members. There are also 38 sustaining members. The total attendance of members at all meetings during the past year was 729; the number of papers read was 361. The number of members attending at least one meeting was 438. At the annual election 275 votes were cast.

The reports of the Treasurer and of the auditors (Mr. S. A. Joffe and Professor H. W. Reddick) were received, showing a balance of $4214.93, exclusive of the balances in the Bulletin, Transactions, Colloquium and special funds, and in the life membership reserve. A vote of thanks was tendered Mr. Joffe for his splendid services extending over several years.

The Society's Endowment Fund now has securities of par value $64,000, yielding an annual income of $2975; sustaining membership fees for the year amounted to $5000. The amount received from sales of the Society's publications was $4849.30.

The Board of Trustees adopted a budget for 1927 showing expenditures and receipts as $31,898.56 and $29,473.56 respectively. The deficit for the year can be met from the balance brought forward from 1926.

The Librarian reported that the Library of the Society now contains 7239 volumes. A catalogue of the Library was printed and distributed early in 1926.

At the annual election, which closed on Tuesday afternoon, the following trustees and officers and other members of the Council were chosen:

Board of Trustees, Professor W. B. Fite, Mr. Robert Henderson, Professors R. G. D. Richardson, Virgil Snyder, Oswald Veblen.

President, Professor Virgil Snyder.

Vice-President, Professor O. D. Kellogg.

Assistant Secretary, Professor Arnold Dresden.

Librarian, Professor R. C. Archibald.

Member of the Editorial Committee of the Bulletin, Professor Arnold Dresden.

Member of the Editorial Committee of the Transactions, Professor Edward Kasner.

Members of the Council: for one year, Professor Daniel Buchanan; for three years, Professors A. A. Bennett, J. W. Glover, James Pierpont, H. S. Vandiver, G. E. Wahlin.

The tellers appointed by President Birkhoff to count the ballots were Professors C. R. Adams and R. G. Putnam.

The following appointments were reported: to represent the Society on the Sectional Committee on Standards for Graphic Presentation sponsored by the American Society of Mechanical Engineers, Professor W. H. Roever; to represent the Society at the American Mining Congress to be held in Washington, D. C., December 7–10, 1927, Mr. C. E. Van Orstrand, of the Geological Survey; as Committee on Arrangements for the Annual Meeting of 1927 in Nashville, Professors C. M. Sarratt (chairman), W. L. Miser, G. D. Evans, Archibald Henderson, Arnold Dresden, and R. G. C. Richardson; to represent the Society in collaboration with the History of Science Society, the American Physical Society, and the American Astronomical Society, in the celebration in New York City, during the coming spring, of the bicentenary of Newton's death, Professors R. C. Archibald and E. W. Brown.

An account of the agreement between the Society and the Johns Hopkins University regarding the conduct of the

American Journal of Mathematics appears on page 1 of the January-February issue of this Bulletin.

At the meeting of the Council, Professor G. C. Evans was named to succeed Professor H. F. Blichfeldt as representative of the Society in the National Research Council. Professors Arnold Dresden and Tomlinson Fort were appointed to represent the Society on the Council of the American Association for the Advancement of Science for the year 1927. As the Committee on the Award of the Frank Nelson Cole Prize, which will take place at the Annual Meeting of 1927, the following were appointed: Professors H. H. Mitchell (chairman), E. T. Bell, and H. F. Blichfeldt.

At the request of Professor J. H. M. Wedderburn, the Society named three associate editors on the board of the Annals of Mathematics for a period of three years. These persons are: Professors D. C. Gillespie, W. L. Hart, and R. E. Langer.

It was reported that the Committee on Colloquia had accepted for the Colloquium Series a book by Professor G. C. Evans entitled Potential Theory.

It was reported that Professor Coble has accepted the invitation of the Council to give a colloquium lecture at the summer meeting of 1928 at Amherst, and that his subject is *The space sextic of genus four*. Professor R. L. Moore was invited to give a colloquium lecture at the Boulder meeting in the summer of 1929.

The fourth Josiah Willard Gibbs Lecture was delivered on Tuesday afternoon by Dr. H. B. Williams, Dalton professor of Physiology at Columbia University. This lecture, which will appear in full in the May-June issue of this Bulletin, was entitled *Mathematics and the biological sciences*. The fifth Josiah Willard Gibbs Lecture will be given at Nashville in December, 1927. Professors Tomlinson Fort (chairman), O. D. Kellogg, and M. B. Porter were appointed to recommend to the Council a lecturer.

At the joint session of the Society, the Mathematical Association, and Section A, held on Thursday morning, the following papers were read:

I. *A mathematical critique of some physical theories*, by Professor G. D. Birkhoff, retiring President of the Society. This paper appears in full in the present issue of this Bulletin.

II. *The weight field of force of the earth*, by Professor W. H. Roever, retiring Vice-President of Section A.

III. *The duty of exposition (with special reference to the Cauchy-Heaviside expansion theorem)*, by Professor F. D. Murnaghan. (Address delivered at the request of the Mathematical Association of America.)

An event of the highest significance was the award to President Birkhoff, for his retiring address, of the fourth American Association for the Advancement of Science Prize of one thousand dollars. This prize is awarded to some person presenting at the annual meeting a notable contribution to the advancement of science. It will be recalled that the first prize was awarded to Professor L. E. Dickson three years ago.

Titles and abstracts of the papers read at the regular sessions of the Society follow below. On Tuesday morning, Professor J. H. M. Wedderburn presided over the Section of Applications and Analysis, relieved by Professor H. H. Mitchell, and Professor F. D. Murnaghan over the Section of Geometry; retiring President Birkhoff presided on Tuesday afternoon. Professor Marston Morse presided over the Section of Analysis on Wednesday morning, and Professor R. L. Moore over that of Point Sets and Algebra; President Virgil Snyder presided on Wednesday afternoon. Professor Sibert's paper was communicated by Professor Hancock. The papers of Blumberg (first paper), B. H. Camp, C. C. Camp, Evans, Fort, Franklin (first paper), Glenn, Graustein, Hazlett (second paper), Hollcroft, Koopman, Pierpont (first paper), Schwatt, Vandiver (first paper), Wells, Widder (second and third papers), Wilder, and Zeldin were read by

title. Dr. Mandelbrojt was introduced by Professor Evans, and Professor Sibert by Professor Hancock.

1. Professor C. H. Forsyth: *The yield of a dividend paying venture.*

The author offers a solution of the oft occurring problem of finding the yield of property, bought at one price and sold at another, taking into consideration any regular dividend. Two rates of interest are used, in addition to the dividend rate, to give a direct solution, not an approximated result. The solution applies as well, of course, to deals in stocks and even in bonds.

2. Professor J. R. Musselman: *On the linear correlation ratio in the case of certain symmetrical frequency distributions.*

The author has extended the idea of a linear correlation ratio, which is of use when one of the characters is in broad categories, to the frequency surface, with linear regression, for which the β_1's in the case of both marginal totals are zero, and the β_2's are equal but arbitrary. This article has been accepted by Biometrika.

3. Professor C. A. Garabedian: *Correction of certain results on the flexure of a thick circular plate given by de Saint-Venant in the celebrated "Note finale du §45" of the translation of Clebsch.*

In this examination of de Saint-Venant's note on a method for dealing with the flexure of a thick circular plate (1883), it is shown that all the solutions obtained are in error in so far as terms of the second order are concerned. The suppression of these terms would reduce the formulas to the thin plate solutions given by Poisson in the memoir of 1828. Although de Saint-Venant's results were questioned by Karl Pearson in 1893 (*History of the Theory of Elasticity*, vol. 2, part 1), they continued to be accepted, and are still to be found in literature of the present day. By extension of methods of analysis developed in the original note, the present paper obtains the correct solutions, and thus removes certain contradictions that de Saint-Venant attempted to explain away by a physical argument.

4. Professor C. R. Adams: *On the series formally satisfying a difference equation whose characteristic equation has irregular roots.*

A root of the characteristic equation is said to be regular when it is simple, finite, and non-zero. In the present paper formal series are found for the linear difference equation of nth order with rational coefficients in all possible irregular cases subject only to the restriction that when a root is multiple a certain secondary condition is not satisfied. When multiple roots occur, the series are in general analogous to the anormal series of differential equations; i.e., to a root of multiplicity m there correspond in

general m series in $1/x^{1/m}$, multiplied by suitable factors. These results extend considerably beyond those previously obtained in this field, which are limited to the case in which the only irregularity is one zero root (Horn, 1910) and to the irregular cases of a second-order equation (Batchelder, 1913).

5. Professor C. R. Adams: *Analytic solutions of the linear difference equation in the irregular case.*

The formal series found in the preceding paper are here made the basis for a study of the existence and properties of analytic solutions; the general method employed is that used by Birkhoff in the regular case. The coefficients being assumed rational, we show that in many cases there exist two sets of n solutions each. In most of these cases we find that one solution from each set is meromorphic over the entire finite plane, and is asymptotically represented in a certain sector ($>\pi$) by one of the formal series; the remaining $n-1$ solutions of each set are meromorphic in the portion of the plane above (or below) some parallel to the axis of reals and are represented asymptotically by the remaining $n-1$ formal series in the sector $\epsilon \leqq \arg x \leqq \pi-\epsilon$ (or $-\pi+\epsilon \leqq \arg x \leqq -\epsilon$), ϵ being an arbitrarily small positive number. In certain cases, however, the situation is more like the regular case, in that there exist two full sets of solutions which are meromorphic over the entire finite plane and asymptotically represented, in certain sectors, by the formal series.

6. Professor R. E. Langer: *Three theorems on closure of biorthogonal systems of functions.*

This paper has appeared in the January-February issue of this Bulletin.

7. Professor R. E. Langer and Mrs. Eleanor P. Brown: *On the theory of integral equations with discontinuous kernels.*

In an earlier paper by one of the authors, the theory of the integral equation $u(x)=\lambda\int_a^b K(x,\ \xi)u(\xi)d\xi$, with a kernel which is discontinuous along $\xi=x$, was developed. The present paper considers the case in which the kernel is continuous while a first partial derivative possesses a finite non-vanishing discontinuity along $\xi=x$. As in the earlier paper, the method used centers on a transformation of the given equation into an integro-differential system. It is proved under hypotheses of a general nature that there exist infinitely many characteristic values of the parameter λ, and asymptotic expressions for these values and for the characteristic functions are obtained. The closure of the set of solutions is established, and the problem of the expansion of an arbitrary function in a series of solutions is discussed.

8. Professors R. E. Langer and J. D. Tamarkin: *On the theory of integral equations with discontinuous kernels.*

This paper deals with the integral equation $u(x)=\lambda\int_a^b K(x,\ \xi)\ u(\xi)d\xi$, whose kernel $K(x,\ \xi)$ possesses partial derivatives of order n, which are

discontinuous along the line $\xi = x$, while the kernel itself and its partial derivatives of order $(n-1)$ are continuous. Under suitable restrictions of more or less general character, the existence of infinitely many characteristic values is established, and asymptotic expressions for these values and for the characteristic functions are found. A theorem on the expansion of an arbitrary function in a series of characteristic functions is proved. The method used is a combination of methods used by the authors individually in previous papers on integral and integro-differential equations.

9. Professor J. D. Tamarkin: *The notion of the Green's function in the theory of integro-differential equations.* Second paper.

The subject of this paper is the integro-differential boundary problem $u^{(n)} + p_1(x, \rho)u^{(n-1)} + \cdots + p_n(x, \rho)u = f(x) + \int_a^b H(x, \xi, \rho)u(\xi)d\xi$; $L_i(u) = 0$, $i = 1, 2, \cdots, n$, where $p_i(x, \rho)$ is a polynomial in ρ of the ith degree and $H(x, \xi, \rho)$ is of the nth degree in ρ. The operators $L_i(u)$ are of the type involving integrals of u as well as the functional values at the end points. The case in which $H(x, \xi, \rho)$ is of the $(n-1)$th degree in ρ has been treated in an earlier paper by the author. The discussion in the present paper is based upon an elementary lemma of the theory of the Fredholm integral equation, which generalizes known properties of orthogonal kernels. All the results of the first paper concerning the existence and asymptotic expressions of characteristic values and the equiconvergence theorem are extended to the case in hand. The equiconvergence theorem is not valid unless the usual Birkhoff integral is replaced by a more complicated one.

10. Dr. B. O. Koopman: *On multiform integrals of differential equations.* Preliminary report.

Consider the system of equations $dx_1/X_1 = \cdots = dx_n/X_n$, either in the real or the complex domain. The X's are single-valued analytic functions of the x's, nowhere all zero, in a certain region R. Suppose that the allied partial differential equation (1): $X_1(\partial f/\partial x_1) + \cdots + X_n(\partial f/\partial x_n) = 0$ admits a system of integrals $f_1, \cdots, f_{n-1}$, each of which allows of indefinite analytic continuation throughout R, and suppose that at every point of R the f's form an independent system. If we start from a point P with a given determination of $f_1, \cdots, f_{n-1}$, and continue the functions analytically back to P along a loop Γ which cannot be shrunk to a point in the region R, we shall obtain a second system of branches, $f_1', \cdots, f_{n-1}'$, in general distinct from the first. But since there can be no more than $n-1$ independent integrals of (1) at P, we shall have (2): $f_k' = \theta_k(f_1, \cdots, f_{n-1})$, $k = 1, 2, \cdots, n-1$. It is shown that we have here a single-valued analytic transformation (independent of Γ) of the variables $f_1, \cdots, f_{n-1}$, having a single-valued analytic inverse. Further, if Γ is followed a second time, the next determinations are given by iterating (2). But if a curve Γ' be used which cannot be deformed into Γ in R, we shall have a second transformation distinct from (2). In this paper, the idea of the transformation (2) is exploited, and applied to analysis, differential equations, and dynamics.

11. Dr. Harry Levy (National Research Fellow): *On nets of Tchebycheff in Riemann space.*

The author shows that a necessary and sufficient condition that two space-filling congruences of curves have the property that each forms a system of parallels (in the sense of Levi-Civita) along the other is, first, that the two given congruences generate a space-filling family of surfaces, and, second, that they form nets of Tchebycheff on these surfaces. Further study is made of the case of p congruences of curves enjoying the above properties, and similar results are obtained.

12. Professor F. D. Murnaghan: *A wave-theory formulation of a geometry of paths.* Preliminary communication.

The particular geometry of paths known as Riemannian geometry is visualized by a corpuscular theory of light or matter, but the paths may also be regarded as the extremal curves for the action integral $A = \int p_1 dq' + \cdots + p_n dq^n - H\,dt$ in the state space of $2n+1$ dimensions (p, q, t). The equations of the paths are $a_{r\alpha}x^\alpha = 0$, where a_{rs} is the curl of the tensor $(x_1, \cdots, x_{2n+1}) \equiv (p_1, \cdots, p_n, 0, \cdots, 0, -H)$. Riemannian geometry in the coordinate space of n dimensions (q) is secured when H is a homogeneous quadratic function of the p's. It is proposed to introduce a geometry of paths in the n-dimensional coordinate space (q) by integrating the Hamiltonian equations $dq/(\partial H/\partial p) = dp/(-\partial H/\partial q) = dt$ (H being an arbitrary function of q, p, t) and defining the paths by the equations $q = q(q^0, p^0, t)$. By varying the initial values p^0 we get ∞^{n-1} paths through any point q^0. It is well known that the Hamiltonian equations define a contact transformation, so that a point q^0 and the directions p^0 through it go over into the points $q = q(q^0, p^0, t)$ of a spread of $n-1$ dimensions (here t is constant, as are the q^0, and the ratios of the p^0 are the variables). This is the wave front which is to be the fundamental element of the proposed geometry, the paths being secondary.

13. Professor T. R. Hollcroft: *Conditions for singularities of algebraic curves.*

Conditions determining singularities of algebraic curves are either metric or projective. Metric conditions determine both the nature and position of a singularity, while projective conditions determine the nature only. Metric and projective conditions do not exist simultaneously for a given singularity. However, when a singularity is defined by metric conditions, the projective conditions that it would have if its position were not given must be considered as latent, since they appear if the singularity is projected to a general position. The number of projective conditions associated with a given singularity is the number of invariant relations among the coefficients of the equation of a curve of any order which are sufficient for the curve to have this singularity. In this paper, the various types of singularities of algebraic curves are built up and the number of projective conditions associated with each singularity is obtained consider-

ing the curve first as a locus and second as an envelope. Finding the number of projective conditions on the class curve occasioned by a given singularity makes unnecessary the separate consideration of the singularity reciprocal to the one given. Limits are found for the number of certain singularities on curves of given order.

14. Mr. J. H. Neelley: *Compound singularities of the plane rational quartic curve.* Second paper.

This paper gives a classification of tacnodes and oscnodes, and theorems relating to the duality of these compound singularities. It also considers special quartic surfaces whose plane sections give all types of the plane rational quartic with compound singularities of all kinds.

15. Professor M. C. Foster: *Congruences of lines of special orientation relative to a surface of reference.*

Through every point M, of a surface S, consider a line l whose direction cosines α, β, γ relative to the moving trihedral at M are either special functions of the parameters u and v, or constant. This paper is concerned with a study of the properties of the rectilinear congruence formed by the lines l of this special orientation.

16. Professor V. H. Wells: *Systems of conics associated with loci in five-dimensional space.*

In this paper the author considers the set of all conics as a five-parameter system, and regards the six coefficients of the equation of such a conic as the homogeneous coordinates of a point in five-dimensional space S_5, as suggested by Veblen and Young (*Projective Geometry*, vol. 2, p. 207). The conics of one-, two-, three-, and four-parameter systems are represented in S_5 by points of S_1, S_2, S_3, and S_4 respectively, where S_1 is a line, S_2 a plane, etc. Sub-systems of an n-parameter system are represented by loci in the corresponding n-space. For example, in a two-parameter system, the sub-systems of degenerate conics and of conics through three points and tangent to one line are represented in S_2 by a cubic and a conic respectively. The two sub-systems have three degenerate conics in common.

17. Professor F. A. Foraker: *The nepolar reciprocal with respect to a central conic.*

The nepolar line of a point with respect to a central conic is defined to be that straight parallel to the polar line of the point which cuts the diameter through the given point in the point which is the neharmonic conjugate of the given point. The nepolar line of the point (y_1, y_2, y_3) with respect to the central conic $b_{11}x_1^2+b_{22}x_2^2+b_{33}x_3^2+2b_{12}x_1x_2+2b_{23}x_1x_3+2b_{31}x_3x_1=0$ is the line $(b_{11}y_1+b_{12}y_2+b_{13}y_3)x_1+(b_{21}y_1+b_{22}y_2+b_{23}y_3)x_2+(b_{31}y_1+b_{32}y_2+(b_{33}-2\Delta/\Sigma)y_3)x_3=0$, where Δ and Σ have their usual meaning. The nepolar reciprocal of the point (y_1, y_2, y_3) with respect to the conic $x_1^2+x_2^2-x_3^2=0$ is the line $y_1x_1+y_2x_2+y_3x_3=0$. That of the

conic $f(x_1, x_2, x_3) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1 = 0$ with respect to the conic $x_1^2 + x_2^2 - x_3^2 = 0$ is the conic $f(u_1, u_2, u_3) = 0$.

18. Dr. S. D. Zeldin: *Contact transformations in intrinsic geometry.*

In this paper the author determines the element of arc, the lowest differential invariant, and the covariant coordinates for each of the following contact groups of the plane: $G_6 : [p, q, xq+q', \frac{1}{2}x^2q+xq', xp-y'q', y'p+\frac{1}{2}y'^2q]$; $G_7 : [G_6, xp+2yq+y'q']$; $G_{10} : [G_7, (y-xy')p-\frac{1}{2}xy'^2q, \frac{1}{2}x^2p+xyq+yq', (xy-\frac{1}{2}x^2y')p+(y^2-\frac{1}{4}x^2y'^2)q+(yy'-\frac{1}{2}xy'^2)q']$.

19. Professor F. W. Beal: *Orthogonal systems of curves on surfaces.*

The fundamental equations are expressed in terms of α_i, β_i, λ_i, and ω_i, respectively, the first and second curvatures and the derivatives with respect to their parameters of the arcs of the curves of an orthogonal parametric system, and the angles the principal normals of these curves make with the normal to the surface. It is easily shown that if any pair except α_1 and α_2, λ_1 and α_2, or λ_2 and α_1 are given functions of the parameters, the general existence theorems apply. In the exceptional cases, the quantities are assumed to be functions of other quantities, and, when desired, of certain derivatives. For example, to show that the parameters $u=c$ and $v=c$ may be congruent helices, it is desirable to let $\partial\omega_1/\partial u$ and $\partial\omega_2/\partial v$ occur. In certain cases solutions unique to within allowable transformations of parameters are found.

20. Professor W. C. Graustein: *On the determination of the linear element of a Riemannian V_n from the Christoffel symbols of the second kind.*

The author establishes the following theorem for the case $n=2$: There exists a positive definite quadratic form $g_{ab}dx^a dx^b$, which has prescribed values for the Christoffel symbols $\{{}_a{}^c{}_b\}$ of the second kind if, and only if, $\{{}_a{}^c{}_b\}$ and the Ricci tensor R_{ab}, formed from $\{{}_a{}^c{}_b\}$, are symmetric in a and b, $R_{11}R_{22}-R_{12}^2>0$, and the covariant derivative, $R_{ab,c}$, taken with respect to $\{{}_a{}^c{}_b\}$, is the product of R_{ab} and a tensor of the first order: $R_{ab,c}=X_cR_{ab}$; the linear element can then be found by quadratures. Generalizations of the theorem to the case $n=n$ are given for spaces of constant curvature and, more generally, for Einstein spaces.

21. Professor Dunham Jackson: *Some non-linear problems of approximation.*

This paper presents theorems of convergence, of which the following is illustrative. Let $f(x)$ be a given positive function of period 2π having a continuous derivative. Among all trigonometric sums of the nth order, let $T_n(x)$ be one which minimizes the integral of $\{f(x)-[T_n(x)]\}^2$, extended

over a period. Then $[T_n(x)]^2$ converges uniformly toward $f(x)$ as n becomes infinite. A corresponding theorem is obtained if $[T_n(x)]^2$ is replaced by $[T_n(x)]^{1/2}$. The proofs involve a generalization of Bernstein's theorem on the derivative of a trigonometric sum.

22. Professor F. D. Murnaghan: *On Whittaker's method for the numerical solution of integral equations.*

Whittaker has given (Proceedings of the Royal Society, 1918) a method for the numerical solution of integral equations of Volterra's type based on the approximation to the kernel, which is assumed to be a function of the difference of its arguments, by a sum of n exponential functions of this difference, or by a polynomial in this difference. It is shown in the present paper that the problem is equivalent to the solution of an ordinary linear differential equation of order n with constant coefficients, and Whittaker's formulas follow at once. The Fredholm equation is treated similarly, when the kernel is a function of the difference of its arguments.

23. Professor R. L. Moore: *Abstract sets and the foundations of analysis situs.*

Axioms 1, 2, and 4 and Theorem 4 of the author's *Foundations of plane analysis situs* hold true in euclidean space of any finite number of dimensions. But Axiom 4 does not hold true in a Hilbert space or a space E_ω, and Axiom 1 does not hold in the non-separable space D_ω. Axiom 2, Theorem 4, and Axiom 1′ stated below hold true in all these spaces, and form a basis for a considerable body of theorems, in particular, Theorems 1–10, and 15, of the paper cited above. Axiom 1′: There exists a countable sequence $G_1, G_2, G_3, \cdots$ such that (a) for each n, G_n is a collection of regions covering space, (b) if P_1 and P_2 are distinct points of a region R, there exists an integer δ such that if $n > \delta$ and K_n is a region containing P_1 and belonging to G_n, then K_n' is a subset of $R - P_2$, (c) if $R_1, R_2, R_3, \cdots$ is a sequence of regions such that, for each n, R_n belongs to G_n and such that, for each n, $R_1, R_2, \cdots, R_n$ have a point in common, then there exists a point common to all the point sets $R_1', R_2', R_3', \cdots$.

24. Professor L. L. Smail: *A simplified derivation of the fundamental properties of double sequences.*

In Pringsheim's classical paper on double sequences and series in Mathematische Annalen, vol. 53, reprinted with slight modifications in his *Vorlesungen über Zahlen- und Funktionenlehre*, and also in London's paper on the same subject in Mathematische Annalen, vol. 53, the derivation of many of the fundamental properties of double sequences is based on a study of the simple sequences contained in the double sequence. By methods explained in the present paper, based on the use of the double sequence only, these fundamental properties may be obtained in a very much simpler way, by use of arguments analogous to those used recently for simple sequences by several writers on the subject of infinite processes.

25. Professor W. W. Elliott: *Generalized Green's functions for compatible differential systems.*

The theory of Green's functions for incompatible differential systems which consist of a linear differential equation of the nth order and n linearly independent boundary conditions involving values of the function and its $n-1$ derivatives at the two end points of a given interval is well known. The existence of generalized Green's functions for certain special types of compatible differential systems has also been shown. In this paper it is shown that the generalized Green's functions for general compatible systems exist, and their characteristic properties and some applications are discussed.

26. Professor L. L. Silverman: *On the omission of terms in certain summable series.*

It is well known that a definition of summability does not in general permit the dropping of a term in a series. In the present paper, a sufficient condition is obtained for the permissibility of dropping a term in the case of a certain class of definitions of summability. This result contains as special cases the results previously obtained by Hurwitz and Silverman, Transactions of this Society, vol. 18 (1917), and by Knopp, Mathematische Zeitschrift, vol. 15 (1922).

27. Professor H. W. Sibert: *A new type of singular solution of differential equations of the first order.*

The differential equation of the first order and nth degree $\Psi(x, y, p) = A_0p^n + A_1p^{n-1} + \cdots + A_{n-1}p + A_n = 0$, where the A's are integral functions in x and y, may be written $\Psi(x, y, p) = (p-\theta_i)^2 [A_0p^{n-2} + B_{3i}p^{n-3} + \cdots + B_{n-1,i}p + B_{ni}] + R_i$, where $\theta_i (i = 1, 2, \cdots, n-1)$ is any root of $\Psi_p'(x, y, p) = 0$. It may be shown that the p-discriminant $D_p(x, y) = CA_0^{n-1} \prod_{i=1}^{i=n-1} \Psi(x, y, \theta_i) = CA_0^{n-1} \prod_{i=1}^{i=n-1} R_i$, and hence any function of x and y which satisfies $\prod_{i=1}^{i=n-1} R_i = 0$ and $A_0p^{n-2} + B_{3i}p^{n-3} + \cdots + B_{ni} = 0$ is a singular solution of $\Psi(x, y, p) = 0$ and is *not* an envelope.

28. Professor H. M. Hosford: *On the summability of Fourier-Bessel and Dini expansions.*

The object of this paper is to investigate what conditions may be placed upon a function $\phi_n(\omega)$ such that the Fourier-Bessel series and the Dini series associated with $f(t)$ will be summable by the following method. If $\lim_{n\to\infty} \sum_{m=1}^{n} \phi_n(j_m) a_m J_\nu(j_m x)$ $[\lim_{n\to\infty} \sum_{m=1}^{n} \phi_n(\lambda_m) b_m J_\nu(\lambda_m x)]$ exists, the Fourier-Bessel [Dini] series will be said to be summable (ϕ), and this limit is taken as the sum (ϕ) of the series. A method of summability $(\overline{\phi})$ similar to the method (ϕ) is given where less stringent restrictions are placed on the function $\phi_n(\omega)$ and more stringent restrictions on $f(t)$. The summability theory of both methods is extended to the multiple Fourier-Bessel series and the multiple Dini series. The uniformity of summability by both methods is shown under the appropriate conditions

of continuity on the function $f(t)$. The method used is essentially a generalization to summability of a method due to Birkhoff for convergence, the tools being Green's function and contour integration.

29. Dr. D. V. Widder (National Research Fellow): *A generalization of Taylor's series.*

The series discussed in this paper arises from a particular phase of the problem of the representation of an arbitrary function $f(x)$ in terms of linear combinations of prescribed functions. The function of approximation $s_n(x)$ of order n is defined as that linear combination of the first $(n+1)$ of the prescribed functions which has closest contact with $f(x)$ at a given point $x=t$. The series $s_0(x)+\sum_{n=1}^{\infty}[s_n(x)-s_{n-1}(x)]$ is then a generalization of Taylor's series to which it reduces if the prescribed functions are $1, x, x^2, \cdots$. Conditions for the convergence of the series are discussed. It is found that under certain rather general conditions any analytic function can be expanded in such a series. Examples are given to show that sequences exist possessing the properties imposed.

30. Dr. D. V. Widder (National Research Fellow): *On the interpolatory properties of a linear combination of continuous functions.*

In his treatment of the mean-value theorems belonging to a linear differential equation, G. Pólya obtained a condition on a set of functions $u_0(x), u_1(x), \cdots, u_n(x)$ which insured that any linear combination of them should enjoy the most important interpolatory properties of a polynomial of degree n. The condition imposed presupposes that the functions $u_i(x)$ possess continuous derivatives of order n. The author obtains an analogous condition, which presupposes only the continuity of $u_i(x)$.

31. Dr. D. V. Widder (National Research Fellow): *Note on Tchebycheff approximation.*

Application is made of the results of the preceding paper to the problem of Tchebycheff approximation. It is found that the condition there imposed on the functions $u_0(x), u_1(x), \cdots, u_n(x)$ is necessary and sufficient that there exist a unique function of approximation in the sense of Tchebycheff to a given continuous function $f(x)$.

32. Dr. Jesse Douglas (National Research Fellow): *Reduction of the problem of Plateau to an integral equation.*

In this paper the problem of Plateau is formulated so as to require not only a regular simply connected segment M of a minimal surface, bounded by a prescribed contour Γ, but also a conformal representation of M on a half-plane. Γ being defined by the equations $x=x(t)$, $y=y(t)$, $z=z(t)$, the problem is made to depend on the integral equation (1): $\int_\Gamma (K(t,\tau)/(\phi(\tau)-\phi(t)))d\tau=0$, where $K(t,\tau)=Sx'(t)x'(\tau)$, S denoting summation over x, y, z. This equation is to be solved with a continuous function ϕ which takes each real value, including the (signless) infinite value, once and

only once on Γ, and which is in general differentiable. The integrand in (1) thus becomes infinite when and only when $\tau=t$. The integral is to be taken in its *principal value*. The function ϕ found, it remains simply to solve three Dirichlet problems for the half-plane, one for each of the coordinates x, y, z. This is effected by means of the definite integral corresponding to Poisson's integral for a circular domain. In the important special case where Γ is a skew polygon, (1) can be replaced by a certain linear (singular) integral equation.

33. Professor G. D. Birkhoff: *Stability and the equations of dynamics.*

After earlier researches by Laplace, Poisson, and others, on the stability of the solar system, Poincaré proved the complete formal stability of any system leading to equations of Hamiltonian canonical form. (See his *Méthodes Nouvelles de la Mécanique Céleste*, vol. 2.) The inverse question of the characterization of the differential equations for which such formal stability holds is taken up in the present paper. (See a note by G. D. Birkhoff, Comptes Rendus, August 30, 1926.) It is shown that such formal stability is possible if and only if the differential equation results from a Pfaffian variational principle $\delta\int_{t_0}^{t_1}\{P_1dx_1+\cdots+P_ndx_n+Q\,dt\}=0$, which can also be transformed into the Hamiltonian form $\delta\int_{t_0}^{t_1}\{\eta d\xi_1+\cdots+\eta_m d\xi_m-H\,dt\}=0$ $(n=2m)$, by a proper choice of formal variables ξ_i, η_i. It is shown that a necessary and sufficient condition that the formal series used in the transformation be convergent is that for a proper selection of the variables $x_1,\cdots,x_n$, the differential form $P_1dx_1+\cdots+P_ndx_n$ becomes an exact differential when half of the variables are annulled. To a certain extent, then, it appears that the property of formal stability alone ensures Pfaffian or Hamiltonian form, and thus constitutes their only characteristic property; also, the Pfaffian form has the advantage of being maintained under the most general point transformation of dependent variables. These results will appear in the author's Chicago Colloquium lectures on *Dynamical Systems*, soon to appear in book form.

34. Professor Elijah Swift: *Canonical forms for ordinary linear differential equations of the second order with periodic coefficients.*

This paper considers the ordinary linear differential equation of the second order $y''+p(x)y'+q(x)=0$, where p and q are periodic functions of x, and the two groups of transformations $y=\lambda(x)\bar{y}$, $x=\psi(\bar{x})$, where x and y are the original variables and $\bar{x}$ and $\bar{y}$ the transformed variables, and where λ and ψ are such functions that the coefficients of the transformed equation are periodic with the same period as those of the original equation. By using transformations of the two groups mentioned the equation may be reduced to one of a few canonical forms. If we classify differential equations of the above type using the roots of a certain (algebraic) quadratic equation, we find for the equations of one class that

the invariants are the ratio of these roots, the number of zeros of a certain solution of our differential equation in a certain interval, and, in some cases the sign of a Wronskian determinant. Any two differential equations with the same invariants may be transformed the one into the other.

35. Dr. C. C. Camp: *An expansion associated with a partial differential equation.*

The convergence proof for a development in terms of the most general solutions of $\sum_{i=1}^{p}(\partial u/\partial x_i)+(\lambda h+k)u=0$ and the boundary conditions $u]_{x_i=-\pi}=u]_{x_i=\pi}$, $i=1, 2, \cdots, p$, can be effected by an extension of Birkhoff's contour method provided the number of parameters be changed from 1 to p. (It is believed that in general this number of parameters is essential to the convergence proof when partial differential equations are involved.) These parameters enter naturally for solutions of the form $u_1=\prod_{i=1}^{p}X_i(x_i)$. The more general solution $u=e^{-\lambda H-K}u_1\phi$ can then be handled with similar Green's functions. Choose $h=DH+\sum_{i=1}^{p}a_i(x_i)$, $k=DK+\sum_{i=1}^{p}g_i(x_i)$, where $D\equiv\sum_{i=1}^{p}(\partial/\partial x_i)$, and take $a_i(x_i)$ either >0 or $\equiv 0$, but some $a_j>0$; then if H, K and ϕ satisfy the boundary conditions, the expansion of f of the usual sort in the form $\sum c^*u^*$ converges for any interior point to the mean value. The multiple series is to be summed in such a way as to make the transformed principal parameter values together with their sum become infinite.

36. Professor G. C. Evans: *Logarithmic potential and convergence in the mean of order less than one.*

The author proves that if $u(r, \theta)$ is given by a Poisson-Stieltjes integral it converges in the mean, as r approaches the boundary radius, of all orders less than one. The theorem rests on theorems on convergence in the mean, of order less than one, e.g., if $f_m(x)$, not negative, is convergent in the mean of order ν, a subsequence from $f_m(x)$ converges to a function $f(x)$ whose νth power is summable, and this $f(x)$ is the only function such that the integral of $|f-f_m|$ to the power ν has the limit zero. The converse statement is obvious. Finally, if $f_m(x)$, not negative, approaches zero almost everywhere, and fails to converge in the mean, of order ν, then there will be a subsequence of these functions the integral of whose κth powers will become infinite, if $\kappa>\nu$.

37. Dr. S. Mandelbrojt: *On the behavior of functions developable in Dirichlet series.*

The author considers the conditions on functions represented by Dirichlet series $f(s)=\sum a_n e^{-\lambda_n s}$, $s=\sigma+it$, necessary and sufficient for absolute convergence. Such conditions show that the functions so developable form a very special class, possessing, if the coefficients are real, a certain symmetry in the distribution of values for the line $\sigma=\sigma'$. Let R be the upper bound of $|f(s)|$ on $\sigma=\sigma'$. The measures of the sets of points on $\sigma=\sigma'$ where $f(s)$ takes on values near given values $e^{i\phi}R$ and $e^{i(\phi+\pi)}R$,

respectively, differ asymptotically little. More precise formulas are given, in the direction of earlier results stated in the Comptes Rendus, August 2, 1926, which hold only when λ_n takes the form $\log p_n$, p_n being the nth prime.

38. Professor Tomlinson Fort: *An elementary proof by mathematical induction of the equivalence of the Cesàro and Hölder sum formulas.*

The author gives an explicit formula for the Cesàro sum to n in terms of the Hölder sum to n, and vice versa. These formulas are readily proved by mathematical induction, and from them, in case of the existence of one limit, the existence and equality of the other follows almost by inspection.

39. Mr. Mark Kormes: *Concerning a property of nowhere dense, perfect sets.*

In this note the following theorem is proved: Given a perfect, nowhere dense, linear set $\mathfrak{P}$ and let P be its kernel of equal measure. The set of points R, where the density is undetermined, is of the second category with respect to P.

40. Professor R. G. Putnam: *On elements of accumulation and neighborhoods in the theory of abstract sets.*

This paper compares the neighborhood conditions resulting from the four conditions of F. Riesz and also a fifth condition with the Hausdorff neighborhood axioms. It is shown in particular, if the four Riesz conditions are satisfied, that corresponding to any pair of distinct elements of the class there exists at least one pair of disjoint neighborhoods.

41. Professor Henry Blumberg: *On the generalization of the theorem that a perfect set has the cardinal number c.*

An extension of this theorem concerning sets of points to sets of sets may be made by means of the notion of distance between sets (Hausdorff's "Entfernung") or, in case of components of a compact closed set, by means of the notion of ρ-connectedness. The latter method exhibits the theorem in question and R. L. Moore's extension of it (Proceedings of the National Academy, May, 1924), as derivable from a common source.

42. Professor Henry Blumberg: *A theorem on arbitrary real functions of two variables with an application to derivatives.*

Let $f(x, y)$ be any real function of two variables defined for the entire xy-plane, and l any straight line of this plane. By the interval of approach I_{Pd} we understand the interval (i, s) where i is lim inf $f(x, y)$ as (x, y) approaches the point P (of l) along the direction d, and s equals the corresponding lim sup. If now d_0 is any fixed direction, there exists an enumerable set E_{d_0} such that if P is any point of l not in E_{d_0}, then all the intervals of approach I_{Pd}, for all possible directions from the same side of l as that

of d_0, overlap, or at least abut, I_{Pd_0}. If $\{d_n\}$ is a sequence of directions of approach, all from one and the same side of l, there exists an enumerable set $E_{\{d_n\}}$ such that if P is any point of l not in $E_{\{d_n\}}$, then not only do the intervals I_{Pd_n} all mutually overlap or abut, but every I_{Pd} where d is arbitrary but from the same side of l as the d_n, overlaps or abuts every I_{Pd_n}. As a particular application to functions of one variable, we have the familiar result that $D^+ \geqq D_-$ except at most at $\aleph_0$ points.

43. Professor Henry Blumberg: *Note concerning closed sets.*

The proof that certain point sets, like the derived set, the set of condensation points, or the set of points of inexhaustibility, are closed is simple enough as usually presented, but a separate effort of thought is expected in every new case. It is true, however, that if a point set S is defined as the aggregate of points every neighborhood of which has a given neighborhood property, then S is closed; and conversely, every closed set is definable as the set of points every neighborhood of which has a certain neighborhood property. This fact makes obvious, from the definition of the above-mentioned or similar sets, that they are closed.

44. Professor R. L. Wilder: *Proof of the non-existence of a certain type of regular point set.*

The author has shown in another connection that a regular connected point set which contains more than one point and possesses the property of remaining connected upon the omission of any connected subset is a simple closed curve. Hence there does not exist an unbounded regular connected set which has this property. The present paper takes up the case of an unbounded regular connected point set which remains connected upon the omission of any bounded connected subset. It is shown that such a set does not exist. For the more special case of closed sets, this was proved by J. R. Kline (Fundamenta Mathematicae, vol. 5 (1924), pp. 3–10).

45. Dr. H. M. Gehman: *Note on Schoenflies' definition of a continuous curve.*

According to Schoenflies' definition, in order that a bounded continuum be a continuous curve, it is necessary and sufficient that the remainder of space have certain properties. R. L. Moore has shown that these properties are neither necessary nor sufficient if the space be of more than two dimensions. In this paper, it is shown that Schoenflies' properties are *necessary, but not sufficient*, if the space be itself a continuous curve in the plane.

46. Professor H. S. Vandiver: *Transformations of the Kummer criteria in connection with Fermat's last theorem.* Second paper.

This paper continues the investigations of the author's first paper with this title (Annals of Mathematics, (2), vol. 27 (1926), pp. 171–176), and

transformations are obtained which will be used later in the derivation of the criteria $m^{p-1} \equiv 1 \pmod{p^2}$ for small values of m, with respect to the first case of Fermat's last theorem.

47. Professor Olive C. Hazlett: *On the composition of polynomials.*

This paper considers the problem of the determination of all homogeneous polynomials $f(x)$ in n variables, $x_1, x_2, \cdots, x_n$, such that $f(x)f(\xi) = f(X)$, where the X's are bilinear in the x's and ξ's. If the field of definition is the field of complex numbers, then it appears that there is no such $f(x)$ which is irreducible of degree n. Denote by $\Delta(x)$ and $\Delta'(x)$ the first and second determinants of the general number $x_1e_1 + x_2e_2 + \cdots + x_ne_n$ of the linear algebra defined by the composition. If $f(x)$ cannot be expressed in less than n variables, then the irreducible factors of $f(x)$ coincide with the irreducible factors of $\Delta(x)$ and $\Delta'(x)$. From this theorem are proved some general properties of homogeneous polynomials that admit a composition.

48. Professor Olive C. Hazlett: *Note on formal modular invariants.*

This paper gives a second proof of the finiteness theorem for formal modular invariants and covariants of a system of binary forms with respect to the Galois field $GF[p^n]$, of order p^n.

49. Professor Olive C. Hazlett: *Ideals for any linear associative algebra.*

This paper defines an ideal for a general linear associative algebra A of order n over the field of rational numbers as a set of numbers belonging to an arithmetic of A which is closed under addition and subtraction and under multiplication (on right or left or both) by any integer of this arithmetic. Then it is proved that, for a semi-simple algebra, every ideal has a finite basis of order n and that if A is the direct sum of two algebras B and C then we know the ideals of A as soon as we know those of B and C and conversely. Moreover, a knowledge of the factorization of the ideals of A is equivalent to a knowledge of the factorization of the ideals of B and of C, and thus the study of ideals of a semi-simple algebra is reduced to that of simple algebras. Likewise, a study of ideals of a simple algebra is reduced to that for a division algebra. By using a generalization of Kronecker's fundamental theorem, it is proved that if $\alpha \neq 0$ is any ideal of a division algebra, then there exists an ideal β such that $\alpha\beta$ is a principal scalar ideal (q) and such that $\beta\alpha = (q)$, which we call the norm of α. From this theorem flow the theorems about the uniqueness of factorization into prime ideals, which are closely analogous to those for the ideals defined for an algebraic field.

50. Professor James Pierpont: *A generalization of the secular equation.*

This paper will appear in full in an early issue of this Bulletin.

51. Professor Harris Hancock: *An extremely simple proof of a fundamental theorem in ideals.*

In the Dedekind theory of moduls, it is shown that if $\mathfrak{a}$ and $\mathfrak{b}$ are any two integral moduls and if $\mathfrak{a}$ is contained in $\mathfrak{b}$ (written $\mathfrak{a}>\mathfrak{b}$), then $\mathfrak{a}/\mathfrak{b}=\mathfrak{i}$, where $\mathfrak{i}$ is an integral modul. If the discussion has to do with integral ideals, which are a special kind of modul, and if T is any algebraic integer that is divisible by the ideal $\mathfrak{a}$, then $\mathfrak{O}T/\mathfrak{a}=\mathfrak{b}$, where $\mathfrak{b}$ is an integral ideal, and where $\mathfrak{O}$ is the order-modul ("Ordnung" or "Art") as defined and used by Dedekind. It follows that $\mathfrak{a}\mathfrak{b}=\mathfrak{a}(\mathfrak{O}T/\mathfrak{a})=(\mathfrak{a}/\mathfrak{a})\mathfrak{O}T=\mathfrak{O}\mathfrak{O}T=\mathfrak{O}T$. That is, associated with every ideal $\mathfrak{a}$ there is another integral ideal $\mathfrak{b}$ of the same realm such that the product $\mathfrak{a}\mathfrak{b}$ is a principal ideal. With this it is proved (see Bachmann, *Allgemeine Arithmetik der Zahlenkörper*, p. 168) that if $\mathfrak{i}$ is contained in $\mathfrak{a}$, then $\mathfrak{i}=\mathfrak{a}\mathfrak{g}$, where $\mathfrak{g}$ is an integral ideal.

52. Professor Philip Franklin: *A theorem of Frobenius on quadratic forms.*

For a regularly arranged quadratic form, i.e., one in which no two consecutive principal minors vanish, the signature of the form may be determined from the signs of these minors. In certain cases where consecutive minors vanish, the result still holds, as Frobenius has shown. In this paper a simplified proof of this result is given.

53. Professor O. E. Glenn: *A list of differential combinants.*

Combinants of pairs of binary differential quantics are expressions in certain determinants whose elements are derivatives. In this paper, thirty cases for forms of low orders are tabulated in Maschke's symbolic notation, some being expressed also in the actual form.

54. Professor Harris Hancock: *Equivalent (quadratic) ideals have similar lattice points.*

The proof for the general realm $\Re(m^{1/2})$ is the same as that for a particular realm, which for convenience is taken as $\Re(i\,5^{1/2})$. The canonical form of ideals of this realm is $\mathfrak{i}=(a,\ b+i\,5^{1/2})$, where a and b are rational integers and $a>b$. (Sommer, *Vorlesungen über Zahlentheorie*, pp. 207 ff.) If $\mathfrak{i}$ is equivalent to $\mathfrak{i}_1=(A,\ B+i\,5^{1/2})$, $A,\ B<A$ rational integers, then there exist two algebraic integers $\alpha=c+i\,5^{1/2}\,d$ and $\alpha_1=C+i\,5^{1/2}\,D$, $a,\ b,\ C,\ D$ rational integers, such that $(\alpha)\mathfrak{i}=(\alpha_1)\mathfrak{i}_1$. This relation gives, if

$$\mathfrak{i}=ax+by+i\,5^{1/2}\,y, \quad \text{and} \quad \mathfrak{i}_1=AX+BY+i\,5^{1/2}\,Y,$$

the following equations:

$$cax+(cb-5d)y=CAX+(CB-5D)Y,\ dax+(bd+c)y=DAX+(BD+C)Y.$$

From these two equations, express x and y in terms of X and Y. Due to the fact that $N(x)N(\mathfrak{i})=N(x_1)N(\mathfrak{i}_1)$, it is seen that the determinant of the two expressions for x and y is unity. Reversing the process, and interchanging capitals and small letters, it is seen that the determinant for the corresponding expressions for X and Y is unity.

55. Professor I. J. Schwatt: *The sum of certain types of Fourier series.*

The author has developed methods by which the sum of types of Fourier series like $\sum_{n=1}^{\infty}(\pm 1)^{n-1}(\sin^p(b+n\theta))/(a+nh)$, and $\sum_{n=1}^{\infty}(\pm 1)^{n-1}\binom{m+n-1}{n-1}(\sin^p(b+n\theta))/(a+nh)$, can be obtained. The methods and the results are believed to be new.

56. Professor J. L. Walsh: *On the expansion of a harmonic function in terms of harmonic polynomials.*

A necessary and sufficient condition that a finite region R of the plane have the property that every function harmonic interior to R and continuous in the corresponding closed region be uniformly expansible in this closed region in a series of harmonic polynomials is that the boundary of R be likewise the boundary of an infinite region.

57. Professor C. N. Reynolds: *On the problem of coloring maps in four colors.* Second paper.

This paper is a continuation of the author's earlier paper on this subject (Annals of Mathematics, (2), vol. 28, pp. 1–15), in which he developed analytically some of the implications of the known theorems concerning irreducible maps, i.e., maps of simply connected closed surfaces which, while requiring five colors, have as few regions as any other map requiring five colors. In the present paper the author applies to the non-pentagons of an irreducible map the methods which he previously applied to the pentagons of such a map. Corresponding to the proposition that an irreducible map must contain at least one pentagon in contact with at least two regions of fewer than seven sides each (proved by Franklin) there follows, as a corollary of a theorem of this paper, the existence, in an irreducible map, of at least twelve pentagons of this type. Other known propositions are strengthened and new propositions established.

58. Professor Marston Morse: *The type number and rank of a closed extremal and a consequent theory in the large.*

In the ordinary regular form of the calculus of variations theory, let there be given a periodic extremal g. Let g be cut across by n arcs which, taken in circular order, are set nearer together than any two successive conjugate points on g. Let a closed broken extremal be formed with successive vertices on these successive arcs. The value of the given integral taken along this broken extremal will be a function, say f, of the n distances of the n vertices from a given side of g. The author shows that the matrix $\mathbf{a}$ of the symmetric quadratic form Q making up the terms of the second order of f will be of rank n, $n-1$, or $n-2$, respectively, according as the Jacobi differential equation set up for g has (i) no non-zero periodic solution, (ii) a one-parameter family of periodic solutions and no more, or (iii) only periodic solutions. The author means by the "type number of g" the "index of inertia" of $-Q$, and this number is determined in terms of the number of

successive conjugate points of g and another invariant. These results, in the small, taken with the author's "relations between critical points," lead to a theory in the large.

59. Dr. C. H. Langford: *Theorems on deducibility.* Second paper.

Questions of deducibility arise especially in connection with sets of defining properties for types of order. A set of properties $p, \cdots, s$ may be related to a property q in such a way that q follows from $p, \cdots, s$ jointly, or in such a way that q *is false* follows from $p, \cdots, s$ jointly, or in such a way that neither q nor q *is false* follows from $p, \cdots, s$; and similarly, a class of properties Q may be such that if q be any one of these properties, then either q or q *is false* follows from $p, \cdots, s$, or Q may be such that for at least one property neither that property nor its contradictory follows from $p, \cdots, s$. This paper is concerned with Q as the class of all first-order functions on the base K, R_2, and with $p, \cdots, s$ as a set of defining properties for discrete series with a first but no last element and with but one element having no immediate predecessor. For any first-order function q on K, R_2, q follows from the set or q *is false* follows from the set.

60. Professor Philip Franklin: *The canonical form of one-parameter groups.*

The author deals with a single Lie one-parameter group, and uses the canonical form to find the nth extended group, and the differential equation of the nth order invariant under the group. This method leads to the tables of known classes of solvable groups much more directly than the methods customarily used. Second-order equations invariant under two groups are next considered. It is shown that a pair of infinitesimal transformations can *not* always be reduced to canonical form by quadratures (contradicting Theorem 3, Annals of Mathematics, (2), vol. 25, p. 363) but that if we know a differential equation of the second order which is invariant under them, we can so reduce them by quadratures. This explains an apparent paradox in Lie-Scheffers, *Differentialgleichungen*, pp. 424–457.

61. Professor B. H. Camp: *Note on the correlation between two frequencies.*

A well known theorem in the theory of sampling gives the coefficient of correlation between the frequencies with which two given variables will occur in a sample as $r_{ij} = -(p_i p_j / q_i q_j)^{1/2}$, where $p_i = 1 - q_i$, and $p_j = 1 - q_j$ are the relative frequencies with which these variables are found in the infinite universe from which the sample is drawn. Pearson's proof (Biometrika, vol. 2 (1902–03), pp. 273–81) makes an explicit assumption which, though seemingly reasonable and actually correct, can as well be avoided, and then subsequently derived as a corollary if the method of proof is based on the equation of the frequency surface to which this coefficient of correlation applies. This surface has certain other interesting characteristics.

62. Professor James Pierpont: *Optics in hyperbolic space.*

The author treats some of the more elementary problems, assuming the rays of light to obey Fermat's law $\delta\int nds = 0$. Reflection and refraction on plane and spherical surfaces and in prisms are treated. It is shown that the theory of central optical collineation, so useful in euclidean space, does not hold. Finally, Bouguer's theorem is extended to this space. The method employed applies also to elliptic space. In a second paper some of the more advanced parts of optics will be discussed.

63. Professor H. S. Vandiver: *Applications of algebraic number theory to congruences involving binomial coefficients.*

In this paper the theory of power characters in an algebraic field is used to obtain new theorems concerning congruences involving binomial coefficients, including the following result: If p and l are odd primes; p belongs to the even exponent f modulo l; $p^f - 1 = cl$, then

$$\binom{c}{i} + \binom{c}{i+l} + \binom{c}{i+2l} + \cdots$$

is divisible by p for $i = 0, 1, \cdots, l-1$, with the exception of $i = k$, where k is the least positive residue of $c/2$, modulo l, and for $i = k$ is congruent to 1, modulo p.

64. Professor H. F. MacNeish: *Theorems concerning spheres associated with a tetrahedron generalized to space of n dimensions.*

For any tetrahedron $A_1A_2A_3A_4$ there is an inscribed sphere with center O, four escribed spheres with centers $O_1O_2O_3O_4$, and three *exscribed* spheres tangent to the four faces externally with centers O_{12}, O_{13}, O_{14}. We have the following theorems. Theorem I: The linear polar planes of O, O_{12}, O_{13}, and O_{14} as to the given tetrahedron and as to the tetrahedron $O_1O_2O_3O_4$ coincide, and the three points of the set O, O_{12}, O_{13}, O_{14} not chosen as the pole are the vertices of the diagonal triangle of the complete quadrilateral section of the two tetrahedrons in the linear polar plane. Theorem II: The three tetrahedrons $A_1A_2A_3A_4$, $O_1O_2O_3O_4$, and $OO_{12}O_{13}O_{14}$ are perspective in pairs in four ways from the vertices of the third tetrahedron. Theorem I is generalized to space of odd dimension $2n+1$.

65. Professor J. F. Ritt and Mr. Eli Gourin: *An assemblage-theoretic proof of the existence of transcendentally transcendental functions.*

This paper appears in full in the present issue of this Bulletin.

R. G. D. Richardson,
Secretary.

THE WESTERN CHRISTMAS MEETING OF THE SOCIETY AT CHICAGO

The twenty-sixth Western meeting of the Society was held at the University of Chicago on Friday, December 31, 1926. It throws an interesting light upon the growth of the Society that, in spite of the strong attractive power of the meeting in Philadelphia, the attendance at this meeting ran up to about 75, among whom were the following 54 members of the Society, a number which would have been counted large for a meeting ten or twelve years ago:

Frank Ayres, R. W. Babcock, Barnett, Ethelwynn R. Beckwith, Brenke, Clack, H. T. Davis, Dickson, Dresden, Edington, H. P. Evans, Feltges, Frink, Garver, Georges, G. H. Graves, L. M. Graves, Griffiths, V. G. Grove, Harshbarger, Hickson, Hildebrandt, Hillard, Hodge, Rosa L. Jackson, LaPaz, Logsdon, W. D. MacMillan, March, William Marshall, Marquis, Mirick, E. H. Moore, E. J. Moulton, A. L. Nelson, Newson, Nyswander, Pettit, Rainich, Roos, Roth, Schottenfels, Sheffer, Shohat, Simmons, Skinner, J. H. Taylor, E. L. Thompson, J. S. Turner, Van Vleck, Wait, G. T. Whyburn, John Williamson, F. E. Wood.

It was agreed to hold the spring meeting in Chicago on April 15 and 16, 1927. At that meeting Professor Chittenden of the University of Iowa will give the symposium lecture on *Some phases of general topology*.

The following thirty papers were read at the two sessions on Friday. The first session was presided over by Professor Dickson, relieved by Professor Hildebrandt; the afternoon session was presided over by Professor Van Vleck. Mr. Sheffer, Dr. Trjitzinsky, and Professor Uspensky were introduced to the Society by Professor Dresden. The fourth paper of G. T. Whyburn, and the papers by Trjitzinsky, W. M. Whyburn, Campbell, Coble, Dodd, Roos, Miller, and Manning were read by title.

1. Mr. G. T. Whyburn: *Cyclicly connected continuous curves*.

In this paper a new type of continuous curve is studied. A continuous curve M is said to be *cyclicly connected* if and only if every two points of

M lie together on some simple closed curve which is contained in M. A cyclicly connected continuous curve C which is a subset of a continuous curve M is a *maximal cyclic curve* of M provided C is not a proper subset of any other cyclicly connected continuous curve which is a subset of M. The most important theorems proved are as follows: (1) A continuous curve M is cyclicly connected if and only if M has no cut point. (2) If M is a continuous curve and J is any simple closed curve of M, then M contains a maximal cyclic curve which contains J. (3) If N denotes the point set obtained by adding together all the simple closed curves contained in a continuous curve M, then N is the sum of a countable number of maximal cyclic curves of M. (4) If H is a maximal connected subset of N, then $\overline{H}$ (H plus its limit points) is a two-way continuous curve.

2. Mr. G. T. Whyburn: *Some properties of continuous curves.*

This paper will appear in full in an early issue of this Bulletin.

3. Mr. G. T. Whyburn: *Concerning the structure of a continuous curve.*

The author shows that if the *cyclic elements* of a continuous curve M are defined as follows: E is a cyclic element of M provided either E is a maximal cyclic curve of M or E is a point of M not belonging to any maximal cyclic curve of M, then every continuous curve is acyclic with respect to its cyclic elements. The following familiar properties of acyclic continuous curves are established for any continuous curve M which is regarded as being composed of its cyclic elements: (1) every collection of whole elements of M whose sum is a connected point set is arcwise connected in the ordinary sense and is also *cyclic chainwise connected*; and every such collection whose sum is a continuum is a continuous curve; (2) an element of M is an end element if and only if it is a non-cut element; (3) if C is any *simple cyclic chain* of elements of M, there exists in M a maximal simple cyclic chain which contains C.

4. Mr. G. T. Whyburn: *On continuous curves.*

In this paper it is shown that if the continuous curve N is a proper subset of the continuous curve M, if R is a maximal connected subset of $M-N$, and B is the boundary of R with respect to M, then (1) $R+B$ is a continuous curve, and (2) every maximal connected subset of B is a continuous curve and at most a finite number of these curves are of diameter greater than any preassigned positive number. It is further shown that if the boundary B (with respect to M) of any S-domain R of a continuous curve M either (a) is connected im kleinen, or (b) has no continuum of condensation, then $R+B$ is a continuous curve. An example is given to show that this theorem does not remain true if B, instead of satisfying either (a) or (b), merely satisfies the conditions of (2) above.

5. Professor G. Y. Rainich: *On the definition of differential.*

If the usual definition of differential of a function of several variables or of a vector function is adopted, the existence of the differential does not imply that it is a linear function of the differentials of the independent variables, and that it is a first total differential in the sense of Thomae and Stolz. In this paper another definition of differential as a limit of an expression is proposed and it is proved that if the differential so defined exists, it is a linear function, and that if the approach to the limit is uniform, it is a first total differential (Stolz).

6. Mr. I. M. Sheffer: *On entire function interpolation.*

With every entire function $f(z)$ we associate a function $P(q)$, $q>0$, determined by the absolute values of the Taylor coefficients of $f(z)$. Some simple properties of $P(q)$ are proved, and it is shown that if $a_n (n=0, 1, \cdots)$ is a given set of numbers, there exist infinitely many entire functions $E(z)$ such that $E(n)=a_n$. This theorem, usually proved by means of a theorem due to Mittag-Leffler, follows here from the properties of $P(q)$ in a very elementary manner. It is shown by induction that the first k derivatives of $E(z)$ (k arbitrary) may also be assigned at $z=0, 1, \cdots$. The results are applied to give a new proof of a theorem due to Borel on the singularities of analytic functions whose Taylor coefficients satisfy certain conditions. This leads to an interesting set of polynomials of which some important properties are stated.

7. Dr. W. J. Trjitzinsky: *On integral transcendental functions.*

It is proved that there exists at least a set of entire functions $F(z)$, such that the expansion $F(z)=a_1F(\xi_1 z)+a_2F(\xi_2 z)+\cdots+a_nF(\xi_n z)+\cdots$ holds under certain restrictions, ξ_i being constant; explicit expressions for the coefficients a_i are found as infinite products. Absolute and uniform convergence of the expansion is established and also the equality of the expansion and the function. The expansion is obtained as the limit of a similar terminating expansion in polynomials. The validity of the process is established by the representation of double limits as the sums of terms of an absolutely convergent double array.

8. Professor W. M. Whyburn: *On differential equations and their related algebraic systems.*

The method of successive approximations is used to obtain the solution of a system of n difference equations of the first order. It is proved that this solution tends in the limit to the solution (obtained by the same method) of a system of differential equations whose coefficients are Lebesgue summable functions. The coefficients of the difference system are defined as sequences of horizontal functions having as their limit functions almost everywhere the summable coefficients of the differential system. The above result enables the author to complete the carrying over of

results previously obtained for difference systems to their related differential systems. See a paper presented to the American Mathematical Society at the summer meeting, September 8–9, 1926.

9. Professor J. A. Nyswander: *A direct method of solving systems of linear differential equations of the Fuchsian type.*

The only exhaustive treatment to be found in the literature of a general system of n linear differential equations of the first order in n dependent variables and one independent variable, the singular points of whose coefficients are poles of the first order, viz., that given by J. Horn about 1890, is based on the theory of transformations and elementary divisors. The present paper gives a new theory which arrives at the general solution, in each of the various cases, by a direct determination of the coefficients of the power series that enter into the solutions. Early in the theory arises the so-called characteristic determinant-equation $D(\lambda)=0$ whose n roots $\lambda_1, \lambda_2, \cdots, \lambda_n$ occur as powers of the independent variables in each of the n solutions respectively. The coefficients that enter into the solutions are obtained, in the more involved cases, from identities in λ; these identities are built up directly from the determinant $D(\lambda)$.

10. Professor J. H. Taylor: *Concerning an application of tensor analysis to the first variation of an integral.*

J. L. Synge, Proceedings of the London Mathematical Society, (2), vol. 25 (1925), has applied the absolute calculus to the development of the first and second variations of the length integral of a Riemannian space. In the present paper the first variation of an integral in parametric form for the "regular" problem of the calculus of variations is treated in an analogous fashion. The paper is an application of existing theory, as the extensions of the Ricci tensor analysis which are required have been previously developed. (J. H. Taylor, *A generalization of Levi-Civita's parallelism and the Frenet formulas*, Transactions of this Society, vol. 27 (1925), pp. 246–264.)

11. Professor V. G. Grove: *Transformations of nets.*

In this paper the notion of conjugate nets in the relation of a transformation F(Eisenhart, Jonas) is generalized to any net. Two nets are said to be C transforms, or in relation C, if the points of the curves of the nets are in one to one correspondence and the developables of the congruence of lines joining corresponding points cut the sustaining surfaces in the curves of the nets. Thus two conjugate nets in the relation of a transformation F are C transforms. With suitable restrictions on the nets, many properties of conjugate nets in relation F are enjoyed by non-conjugate nets. A generalization of transformation K(Koenig) is made, and, again with suitable restrictions, properties of conjugate nets in the relation of a transformation K are enjoyed by non-conjugate nets.

12. Professor A. D. Campbell: *Nets of conics in the Galois fields of order* 2^n.

In this paper the nets of conics $\lambda C_1+\mu C_2+\nu C_3=0$ in these finite domains are classified into 32 classes; typical nets are derived for each class. The results of previous papers on pencils of conics and on plane cubic curves in these Galois fields are made use of. In these fields $(ax+by+cz)^2=a^2x^2+b^2y^2+c^2z^2$, the discriminant of the general conic is $fgh+af^2+bg^2+ch^2$, every point is its own harmonic conjugate with respect to any pair of collinear points. Some cubics in λ, μ, ν have no associated nets of conics; some nets exist only in the Galois field of order 2, others only when $n>1$, still others only when $2^n=3k+1$; some nets have no degenerate conics in them. When the net has a non-degenerate cubic in λ, μ, ν this cubic is first reduced to a normal form and then the net is reduced. The nets with degenerate cubics (hence with degenerate pencils of conics in them) are first divided into sets according to the nature of the degenerate pencils they contain.

13. Professor A. B. Coble: *Modular manifolds determined by the binary* $(2p+2)$*-ic.*

The cross-ratio group of E. H. Moore is for the binary $(2p+2)$-*ic* a Cremona group of order $(2p+2)!$ in a linear space S_{2p-1}. This group has two important invariant linear systems which map the space S_{2p-1} upon two manifolds of dimension $2p-1$ in the same higher space. The object of the present paper is to develop some of the geometric properties of these two manifolds and to exhibit their relation to each other.

14. Professor G. Y. Rainich: *Periodic electromagnetic fields with non-periodic stresses.*

Together with an electromagnetic field certain second degree quantities are considered, viz., the stresses, the Poynting vector, and the energy. Given a set of quantities the conditions are established under which they may be considered as second degree quantities of an electromagnetic field, and the question how far the field is determined by these quantities is solved. An example is given in which the field of the second degree quantities is not periodic but determines a periodic electromagnetic field. The possibility of interpreting such fields as radiation is discussed.

15. Professor H. W. March: *The Heaviside operational calculus.*

This paper will appear in full in an early issue of this Bulletin.

16. Professor W. E. Edington: *The relation* $s_2^{-1}s_1^{\epsilon}s_2^{\delta}s_2=s_1^{-1}s_1^{\epsilon}s_2^{\delta}s_1$ *in the definition of non-abelian groups of finite order.*

Given $s_1^{\alpha} = s_2^{\beta} = 1$, $s_1 s_2 \neq s_2 s_1$, $s_2^{-1} s_1^{\epsilon} s_2^{\delta} s_2 = s_1^{-1} s_1^{\epsilon} s_2^{\delta} s_1$, $\epsilon < \alpha$, $\delta < \beta$ certain general relations exist among some of the operators of the group generated by s_1 and s_2 which suggest certain values different from 0 for ϵ and δ so as to impose conditions on α and β sufficient in some cases to determine infinite systems of non-abelian groups of finite order such as the dihedral and dicyclic systems and several other systems.

17. Professor J. V. Uspensky: *On the development of arbitrary functions in series of Hermite's and Laguerre's polynomials.*

The question concerning the development of arbitrary functions in series of Hermite's and Laguerre's polynomials has recently attracted the attention of several authors. Their results, however, are far less general than those published by the author ten years ago in the Proceedings of the Russian Academy of Sciences. As this paper, published in Russia during the World War did not attract the attention of mathematicians, the author reproduces its contents in a new and simplified form.

18. Professor J. V. Uspensky: *On Jacobi's arithmetical theorems contained in his paper: Ueber diejenigen unendlichen Reihen deren Exponenten zugleich in zwei verschiedenen quadratischen Formen enthalten sind.*

The author shows how the arithmetical theorems contained in Jacobi's paper mentioned above can be obtained by means of purely arithmetical considerations based upon the study of a certain kind of partitions of numbers.

19. Professor H. T. Davis: *A solution of Bourlet's generatrix equation.*

C. Bourlet has shown that, in general, the solution of a non-homogeneous linear functional equation $F(x, z) \rightarrow u = f(x)$, where z^n is the symbol of the nth derivative, can be reduced to that of finding a function $G(x, z)$ which satisfies the generatrix equation $[G \cdot F] = 1$. This paper deals with the solution of the equation when $F(x, z)$ is of the form $1 - \lambda g(x, z)$ and determines conditions for the existence of the solution regarded as a function of λ.

20. Professor J. A. Shohat: *On the asymptotic expressions of Tchebycheff polynomials and their derivatives.*

Consider Tchebycheff polynomials $\phi_n(p; x)$ corresponding to the finite interval (a, b) with $\log p(x)/[(x-a)(b-x)]^{1/2}$ integrable (L). The author derives the asymptotic expressions of $\phi_n(p; z)$ for $n \rightarrow \infty$, also of $\phi_n^{(i)}(p; z)$ and of $\sum_{k=0}^{n} \phi_k^{(i)}(p; z)\, \phi_k^{(l)}(p; z_1)$, $(i, l = 0, 1, \cdots$ finite), the real points z, z_1 being arbitrarily taken outside of (a, b). The method applied is very elementary. The first expression is derived *without the use of difference*

or *differential equations* but by means of some previously established properties of algebraic continued fractions. The second expression is obtained from a difference equation *of the first order*. Some of these results have been presented to the French Academy (Comptes Rendus, vol. 183, (1926) p.697).

21. Professor E. L. Dodd: *The convergence of general means of measurements, and the invariance of form of certain frequency functions.*

With continuous increasing functions $f_i(u)$, let

$$v = F(u) = \frac{1}{n}\sum_1^n f_i(u)\ ,\ u = G(v),\ M = G\left[\frac{1}{n}\sum_1^n f_i(m_i)\right],$$

where m_i are measurements. Conditions are given for a probability greater than $1-\eta$ that $|M-a|<\epsilon$, where a is the true value and ϵ and η are preassigned, small at pleasure. A certain invariance of form of the associated frequency functions is shown. Moreover, if the probability that an error be negative is 1/2, which does not require any symmetry in the frequency function, then the probability law for M can be expressed in simple form by the use of the common probability integral.

22. Dr. C. F. Roos: *Generalized Lagrange problems in the calculus of variations.*

In the theory of dynamical economics a problem arises which is a generalization of the Lagrange problem in the calculus of variations. It is desired to find in the space u_1, u_2, u_3, x a curve Γ which is a solution of a first order differential equation, which maximizes an integral π_1 when u_2 is kept fixed and which at the same time maximizes a second integral π_2 when u_1 is fixed. By employing the theory of Volterra integral equations, Eulerian necessary conditions are obtained without the use of multipliers. By allowing one end point to vary, conditions similar to the Weierstrass necessary condition are obtained. Sufficient conditions are then found by means of an analogue of the Hilbert integral. A similar treatment is given of the Lagrange problem with one integral but with several differential equations. It is shown that the analysis applies also to the problem in which the differential equations are replaced by functional equations.

23. Dr. C. F. Roos: *Dynamical economics.*

The hypotheses of Cournot, Walras and others concerning the equation of demand are modified so that the rate of change of price and production enter into the demand equation. Necessary conditions for a solution of the problems of competition, cooperation and monopoly are given for the case where the demand equation can be taken as a differential equation of the type $G(u_1, u_1', \cdots, u_n, u_n', p, p', t)=0$. In order to take account of the cumulative effects of price changes it is suggested that the differential equation be replaced by a functional equation. Some economic experiments which seem to support this theory are stated in brief. The mathematical

problem of competition is then discussed from the view point of Walras and Pareto, using functional equations of demand.

24. Professor G. A. Miller: *Possible orders of two generators of the alternating and of the symmetric group.*

If $l>3$ represents a prime number which divides the order of the symmetric group of degree $n\neq 2l-1$ then it is always possible to find two operators of orders 2 and l respectively which generate this symmetric group and also two such operators which generate the alternating group of this degree. When $n=2l-1$ it is possible to find two such generators of the alternating group of degree n but it is impossible to find two such generators of the symmetric group of this degree. Every symmetric group whose order is divisible by 4 except the symmetric group of degree 6 can be generated by two operators of orders 2 and 4 respectively, and every alternating group which involves operators of order 4 can be generated by two such operators. Whenever an alternating group involves a substitution of order $l_1>3$ then it contains two substitutions of orders 2 and l_1 respectively which generate the entire group, and the theorems noted above include all the cases when the symmetric group cannot be thus generated. The paper will be offered for publication to the Transactions of this Society.

25. Professor W. A. Manning: *Simply transitive primitive groups.*

Dr. E. R. Bennett (American Journal of Mathematics, vol. 34 (1912)) gave a series of useful theorems on a restricted class of simply transitive primitive groups. The present paper obtains the conclusion contained in Corollary II to Theorem V of this paper from much weaker hypotheses, by proving the following theorem: Let G_1, the subgroup that leaves fixed one letter of the simply transitive primitive group G of degree n and order g, have a transitive constituent M of degree m, in which the subgroup M_1 that fixes one letter is primitive. Let M be "paired with itself" in G_1 and let the order of M be less than g/n. Then G_1 contains an imprimitive constituent in which there is an invariant intransitive subgroup with m transitive constituents of $m-1$ letters each, permuted according to the substitutions of M. This pairing of two transitive constituents of G_1 (or the pairing of a transitive constituent of G_1 with itself) was introduced by Burnside in his paper on *Groups of odd order* (Proceedings of the London Mathematical Society, vol. 33 (1900)). The author also extends and simplifies Theorems I to VI of Bennett's paper.

26. Professor L. E. Dickson: *Integers represented by ternary quadratic forms.*

One part of this paper has appeared in the January-February issue of this Bulletin. The other part will appear in the Annals of Mathematics.

27. Professor L. E. Dickson: *Quaternary quadratic forms representing all integers.*

There is developed a method for finding all such quadratic forms, involving cross-products. The memoir will appear in the American Journal.

28. Professor L. E. Dickson: *Extensions of Waring's theorem on sums of powers.*

The part dealing with fourth powers will appear in an early issue of this Bulletin. The part concerning cubes will appear in the American Mathematical Monthly.

The form $ax^3+by^3+\cdots$ is denoted by $(a, b, \cdots)$. From it is derived $(r, s, b, \cdots)$ by partition of a into $r+s$. The latter form evidently represents every integer which can be represented by the first form. Call $a+b+\cdots$ the weight of the form. If a form of minimum weight 9 represents all positive integers, it is $f=(1, 1, 2, 2, 3)$ or one of the six forms derived from f by partition. It is shown that f represents all integers <1400. One of its partitions is $g=S+3v^3$, where S is a sum of six cubes; g represents all integers $<40{,}000$. Many further forms are discussed, including all of weights 10 and 11. In particular, $S+4v^3$ represents all integers $<40{,}000$.

29. Mr. John Williamson: *Conditions for associativity of division algebras corresponding to non-abelian groups.*

It has been shown by L. E. Dickson (Transactions of this Society, April, 1926) that with every solvable group G there is connected a system of division algebras. In this paper the conditions that the division algebras be associative are obtained when the group G is generated by three independent generators Θ_1, Θ_p and Θ_q, where Θ_p transforms Θ_1 into some power of Θ_1, and Θ_q transforms Θ_1 and Θ_p into some power of Θ_1 and Θ_p respectively.

30. Professor E. J. Moulton: *A note on attraction.*

In this note the author criticizes derivations of the formulas for the attraction of a continuous body on a particle under the Newtonian Law of gravitation. Various writers, in an effort to improve the usual methods of mechanics, have used some formulation of Duhamel's Theorem. The author calls attention to an error introduced by some of these writers (W. F. Osgood, Annals of Mathematics, (2), vol. 4 (1903), p. 161; R. L. Moore, Annals of Mathematics, (2), vol. 13 (1912), p. 161; G. A. Bliss, Annals of Mathematics, (2), vol. 16 (1914), p. 45). They assume that the attraction of an element of the body is greater than the attraction of a particle of equal mass which is placed at the maximum distance from the given particle to a point of the element. This assumption is invalid.

ARNOLD DRESDEN,
Assistant Secretary.

THE NINETEENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION

The nineteenth regular meeting of the Southwestern Section of the Society was held at the University of Nebraska on Saturday, November 27, 1926. Visiting members were entertained at the University Club on the Friday evening before the meeting. The morning and afternoon sessions on Saturday were presided over by Professor W. C. Brenke.

The total attendance at the meetings was twenty-five, among whom were the following eighteen members of the society.

Ashton, Brenke, Candy, Collins, J. T. Colpitts, Congdon, Doole, G. C. Evans, Gaba, Gouwens, F. S. Harper, Holl, Louis Ingold, J. V. McKelvey, Pierce, Runge, Sherer, J. S. Turner.

The morning session closed with an extended paper, *A survey of discontinuous boundary value problems for Laplace's equation in two dimensions*, by Professor Griffith C. Evans of Rice Institute. This paper was presented by invitation of the program committee.

At the close of the afternoon session, which was devoted to the reading of further papers, a resolution was passed thanking the Department of Mathematics of the University of Nebraska for their hospitality. The section also extended a vote of thanks to Professor Evans for his address.

Abstracts of the papers presented at this meeting, except the extended paper mentioned above, are given below. The papers by Anderson, Ettlinger, and Wahlin were read by title. Mr. Atanasoff was introduced by Professor E. S. Allen.

1. Professor D. L. Holl: *A solution of* $\nabla^4 = 0$ *occurring in the problem of viscous fluid motion.*

The problem of viscous incompressible fluid motion is the problem of determining a function ψ satisfying $\nabla^4\psi = 0$ in the region considered and subject to a given set of boundary conditions. In this paper a solution

is found by the use of conjugate cylindrical coordinates characterized by the transformation $\alpha+i\beta=\log(z+k)-\log(z-k)$ for a region between two eccentrically placed cylinders. The solution contains the special case of concentric cylinders, for which certain minimizing properties are evident immediately. Subsequent computations for all the relevant physical quantities concerned are found by which the stability of an apparatus, designed for experimental determinations, is predicted.

2. Professor G. E. Wahlin: *On a quadratic algebra over a field.*

This report contains the definition of an algebra of order 2^σ over a general field. The multiplication of the basal elements as defined is not associative. Every element of the algebra is a root of a quadratic equation with coefficients in the field.

3. Professor J. S. Turner: *Tchebycheff determinants.*

Tchebycheff gave forms $\pm(x^2-Dy^2)$ with small values of D, and explained their use in finding the factors of large integers (Journal de Mathématiques, vol. 16(1851), pp. 257–282). In the present paper 40 determinants are given which have the following properties: (a) D contains no square factor, and $T^2-DU^2=1$, $U\leqq 4$, (b) each genus of D contains only one class of reduced forms. The method of using reduced forms ax^2-by^2 to find the factors of large integers is explained. In certain cases this method is superior to that in which Euler determinants are used.

4. Miss Nola L. Anderson: *An extension of Maschke's symbolic method.*

Maschke represented the quadratic differential form $ds^2=\sum g_{ij}dx^i dx^j$ in the symbolic form $(df)^2=(\sum f_i dx^i)^2$ where the f_i were treated as partial derivatives of a symbolic function. In the present paper the coefficients of this differential form are represented symbolically by the equations $f_i f_j=g_{ij}$, but it is not assumed that $\partial f_i/\partial x^j$ is equal to $\partial f_j/\partial x^i$. It is found that differential parameters of the first order by the generalized method are represented in precisely the same manner as in Maschke's theory. Invariant expressions of higher order, however, are best regarded as invariants of the set of quantities g_{ij} and a certain other set analogous to Christoffel's triple-index symbols. A geometric interpretation is given for the two-dimensional case, and certain invariant vectors related to a two-dimensional surface are studied.

5. Professor M. G. Gaba: *A set of postulates for euclidean geometry in terms of point and inversion.*

The basis consists of a class of elements called points and a class of permutations among the points. Sufficient postulates or restrictions are placed upon the permutations to make them the circular inversions. Circles are defined and, by specializing a point, lines are defined. The usual

axioms of plane euclidean geometry are then derived as theorems or stated without using any other undefined notion.

6. Professor H. J. Ettlinger: *Note on Riemann-Stieltjes integrals.*

By means of the Duhamel-Moore theorem, a direct and simple proof is given of the following theorem established by J. M. Whittaker (*On integrals with a nucleus*, Proceedings of the London Society, vol. 25(1926), pp. 213–218): If $f(x)$ is bounded on (a, b) and $g(x)$ is bounded and R-integrable on (a, b), and $h(x)$ is the indefinite integral of $g(x)$ from a to x, then a necessary and sufficient condition that the Stieltjes integral $\int_a^b f(x)dh(x)$ exist is that the Riemann integral $\int_a^b f(x)g(x)dx$ exist.

7. Mr. J. V. Atanasoff: *On the dynamics of a new type of molecular model.*

A molecular model is chosen in which the distribution of charge approximates that of modern theories of atomic structure. Force distance relations are developed on the assumption of classical electrodynamics. An investigation is made of the law of state of a gas composed of such molecules.

8. Professor Louis Ingold: *Quadratic first integrals in affine and related geometries.*

In a previous paper the author has indicated a method of transforming a given affine geometry by means of an arbitrary non-singular matrix $\|a_i^i\|$ whose elements are functions of the coordinates. The present paper discusses conditions for the existence of quadratic first integrals for the differential equations defining the paths in the transformed geometries.

9. Professor Julia T. Colpitts: *The real zeros and other properties of a certain entire function of genus unity.*

The function considered is $f_a(x) = \sum_{n=0}^{\infty} x^n/(n+a)^n, a > -1$. There is an even number of negative zeros. There is not more than one greater than $-2e$. When $a \geqq 2$, there is no real zero. Approximations of the greatest zero for small values of a are obtained.

LOUIS INGOLD,
Secretary of the Section.

A MATHEMATICAL CRITIQUE OF SOME PHYSICAL THEORIES*

BY G. D. BIRKHOFF

1. *Introduction.* My purpose today is to review some of the mathematical-physical theories of the past and of the present, indicating briefly the nature of certain concepts upon which these theories rest as well as the attendant logical difficulties, and even proposing modifications which have occurred to me. Of course the subject is too large for an address of this kind, but interest in mathematical-physical ideas is very widespread, and their importance for both mathematicians and physicists is profound. I feel more justified in choosing this subject because it has occupied so much of my attention recently.

2. *Geometry.* It goes without saying that geometry is the first and simplest of such theories, unless arithmetic be regarded as of this type. But the only assumption of a physical type underlying arithmetic is that material objects possess a permanent identity; and this can hardly be considered as an assumption if we are to do any thinking at all.

Now ordinary three-dimensional geometry of space is extremely simple in essence. Some day, when the field of knowledge has extended so far that simplification becomes necessary, ordinary geometry may be approached somewhat as follows:

(1) Geometry treats of elements called *points* and the relation called *distance* between pairs of points.

(2) The complete tabulation of distances between pairs of points may be arranged as follows:

* Presidential Address, delivered before a joint meeting of this Society and Section A of the American Association for the Advancement of Science, December 30, 1926. This paper was awarded the Prize of the American Association for the Advancement of Science, for the 1926 meeting.

(a) the points P correspond to real number triples (x, y, z);

(b) the squared distance between P_1 and P_2 is $(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2$.

All of geometry follows very readily from these agreements. To begin with, the line segment P_1P_2 may be defined as the set of points P such that the sum of the distance P_1P and the distance PP_2 is the distance P_1P_2. The definition of a line segment makes the definition of a complete line possible of course, and then of a plane, while the corresponding simple algebra yields the analytic equations of lines and planes. Similarly, the perpendicularity of two lines l_1 and l_2 with the common point P can be defined as the relation existing between l_1 and l_2 when the sum of the squares of the two distances from any point on either line to the common point is equal to the square of the distance between these two points themselves. In this way one may successively define line segment, line, plane, perpendicularity, rectangular coordinate systems, etc. The whole body of geometrical fact with corresponding analytic framework is thus easily deducible, and yet one may stop at the fundamental principles, without taking up beautiful but less vital geometrical studies. More generally, an entirely analogous direct development of general differential geometry of n dimensions, based on the well known differential formula for the squared element of distance, can be made. Indeed much of specialized geometry as well as of analysis is likely to give way some day, for as my predecessor, Professor Veblen, said in his address: "The main current of mathematics will flow by carrying with it only the more important facts." One thing is plain, however: it will remain the first duty of the mathematician to develop interesting and beautiful theories of all types, without being much concerned with the question of ultimate importance.

The theories of relativity of Einstein have made us appreciate that geometry arises directly out of the comparison of rigid bodies, and that its meaning becomes more and more precise as uniformity of pressure and temperature

of the bodies, freedom from gravitational, rotational and other stresses are secured; but such precision is limited, because of the atomic structure of matter. In its origin the geometric concept of space is always to be associated with that of a corresponding body of reference.

3. *Classical Dynamics.* Classical dynamics arises in the attempt to use euclidean space and absolute time as the means for expressing the laws of nature. Natural laws are then interpreted as describing the interaction of various particles, rigid and elastic bodies in a selected space of reference.

There lie at the very basis of this attempt to make space the container of matter, certain fundamental difficulties, to some of which I wish to direct your attention. The simplest illustration of them arises in dealing with a collection of "equal rigid elastic spheres," which move in straight lines

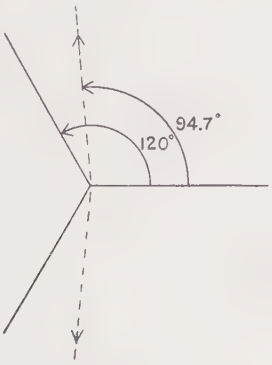

FIG. 1

with uniform velocity except in so far as they may collide. The spheres will be taken to be non-rotating and smooth. This is the model on which the statistical treatment of the properties of a perfect gas is often based. When only two spheres collide, the assumed laws of contact action determine uniquely their directions and velocities under collision; but

when more than two spheres collide, the situation is entirely different.

Suppose that three equal spheres approach a point with equal velocities, the lines of motion being 120° apart and in the same plane. If all three spheres collide at the same instant, considerations of symmetry alone demand that the spheres must rebound back along the same lines with a velocity equal to that of approach. But it is easily verified that if two of them collide ever so little before they collide with the third, the resultant motion will be decidedly different in character, as indicated by the dotted lines of the figure. Such a result seems to contradict the fundamental physical requirement of continuity. In fact the laws of action suffice to determine the behavior of two spheres which collide, but as more and more complicated simultaneous collisions between three or more spheres are considered, it is not possible to infer their behavior by an argument based on continuity or even on symmetry. Hence these laws need to be supplemented by indefinitely many others of arbitrary type, if the mathematical theory is to be determinate.

FIG. 2.

The situation is similar with n mass particles, attracting one another according to the Newtonian law. The laws of motion are then incorporated in $3n$ differential equations of the second order. In the case of only three particles, if certain three area integral constants are not all zero, Sundman* has shown that triple collision is impossible, and that the differential equations themselves suffice to define the motion after double collision; in fact, at double collision the bodies approach each other in a certain direction and may

* Acta Mathematica, 1912.

be taken to recede with nearly the same velocity in the opposite direction after collision; this is proper because for nearly a double collision the behavior is almost the same, as indicated by the dotted lines in the figure below. Hence considerations of continuity determine the behavior at collision.

But, if there are further particles, three particles may happen to collide simultaneously, and then there arises the same kind of indeterminateness as in the case when three rigid elastic spheres collide. Hence, here too, other supplementary conditions are required in addition to those furnished by the differential equations.

Finally, the concept of elastic bodies so fundamental in classical dynamics presents even more formidable logical objections. The whole of that theory is based on the notion of continuously distributed matter subject to certain strains and stresses obeying Hooke's law. It is a characteristic feature of the isotropic elastic body that the effects of any disturbance are propagated at a definite velocity, namely, the disturbance velocity. Bearing this property in mind, let us ask what happens when two equal elastic spheres under no pressure approach along their line of centers with equal velocities which exceed the disturbance velocity. The parts of the spheres approaching the plane of collision have no possibility of reacting to the disturbance of collision since that plane is approaching the center of either sphere at a velocity greater than the disturbance velocity. On the other hand, the parts of the spheres which collide at the plane cannot rebound without interpenetration. Thus it appears as if the spheres are converted into a kind of lamina of infinite density moving radially outward in the plane of symmetry. But this yields a total change of state, which the theory of elasticity does not contemplate.

Another but lesser stumbling block is contained in the following situation: Consider two equal elastic hemispheres at rest with the two circular boundaries in contact and com-

pare these with a single sphere of the same size and also at rest. The differential equations, initial densities and pressures are identical and yet the reactions under the same force may be different, for in the first case the two hemispheres can be separated.

These illustrations show that the classical theory of particles, rigid and elastic bodies, needs to be supplemented by a variety of conditions, some of which do not appear to have been formulated, and that such formulation will of necessity be artificial in character.

There are other facts to be considered also. Hertz pointed out the peculiarities of such cases as an elastic sphere resting on a plane surface. Furthermore, if elastic spheres collide successively with one another, the internal kinetic energy will increase, while the relative velocities of the spheres becomes smaller and smaller. Thus a collection of small elastic spheres does not really furnish a model for a perfect gas. But these are questions of complication rather than of principle.

The central difficulty of indeterminateness disappears in the case of a single elastic body, but then the situation is no longer analogous to the case of a set of particles in empty space, which formed our starting point. The question now confronts us: Is it possible to conceive of simple laws of motion for systems of particles and for continuous bodies in empty space which will be unified and determinate?

To secure such a system of particles, it suffices to subject the particles to forces acting directly between them and of the form

$$kmM\left(\frac{1}{r^2} - \frac{a}{r^3}\right),$$

where m and M are the masses of the two particles, and r is the distance between them; in other words, in addition to the ordinary Newtonian force of attraction there is a repulsive force inversely proportional to the cube of the distance. Since the potential energy of the system then

increases indefinitely when any two particles approach collision, it follows that collision can never take place. Furthermore, if there are dissipative forces, there will be obviously a position of stable equilibrium for such a set of particles.

In order to deal with a continuous distribution of matter, we need merely to set up an appropriate potential energy function

$$U = k \int \left(\frac{1}{r} - \frac{a}{2r^2} \right) dm ,$$

where dm is the element of mass and the integration is taken over space. This means that the law of force for the continuous distribution is the same as that for the system of particles just considered; the acceleration in any direction of a point of the body is given by the corresponding gradient of the potential function U. It is obvious that such a body cannot contract indefinitely since then its potential energy would exceed the total initial energy; nor can it expand indefinitely unless sufficient kinetic energy is available. Moreover, there will clearly be a single spherical state of equilibrium in case dissipative forces are present.

This type of body is instructive in another way. It illustrates how necessary it is to take the differential equations of motion as fundamental rather than to rely upon what appear to be simple physical intuitions. For, consider what happens if two such bodies collide. If we let ordinary intuition speak, we might declare that they will either rebound from one another or continue indefinitely in contact. But the nature of the differential equations of motion excludes both of these possibilities. In fact, when the bodies first touch, the points of contact have differing velocities, whereas the accelerations will be the same according to the formula. Consequently, the formula shows that they must interpenetrate rather than remain in contact or rebound from one another. Evidently such a fluid is entirely different in character from the elastic body under pressure, but it has

at least the theoretical advantage of being free from the fundamental defect of indeterminateness. As perhaps of some interest, it may be remarked that two colliding bodies of this description will in general separate after a transitional period of interpenetration.

The chief mathematical instruments used by the physicists in dealing with space, time and matter according to the conception of classical physics have been the Lagrangian and Hamiltonian equations. I wish to digress somewhat from my main theme in order to make a remark concerning the nature and significance of these equations.

A dynamical system representable approximately by means of a set of "mass particles" subject to "rigid constraints" and "conservative forces" of gravitational or elastic character, has its potential energy given by a function of n spatial coordinates, while its kinetic energy is expressed in terms of these coordinates and their velocities in the usual way. It is readily demonstrated that the equations of motion may be written in either of the above mentioned forms. The mathematician has thus been led to ask: What are the characteristics of equations of these types? In the particular case of the solar system, it was demonstrated by Laplace and Poisson that there would be partial stability. Poincaré proved that it was a general characteristic of equations of this type to possess complete formal stability. This means that the disturbances from periodic motion are essentially periodic in type and representable to all orders by means of trigonometric series with a certain degree of approximation.

I have recently succeeded in establishing a kind of converse result: If a dynamical system is such that its state is determined by $2n$ coordinates, and if the perturbations from a periodic motion can be thus represented by trigonometric series, then the equations may be given Hamiltonian form.* Thus perhaps the *only* significance of the Hamiltonian form of equations in classical dynamics is to insure automatically that periodic motions are completely stable. A more general

* Comptes Rendus, September 20, 1926.

Pfaffian type of equation seems better adapted to be useful to the physicist, for it possesses greater ease and flexibility of transformation of the variables. I believe that the whole apparatus of transformation which is used in connection with the Hamiltonian equations has been overrated.

4. *The Electromagnetic Theory.* The equations of Maxwell, giving the interplay between the electric and magnetic forces in space, have never been modified, although it has been shown that these admit of very elegant formulation when presented in the Maxwell-Lorentz form. The space-time background corresponding to this form is that of the special theory of relativity, in which space and time are taken relative to some reference body and in which the velocity of light appears as a characteristic limiting disturbance velocity.

Fundamental in this theory is the notion of the minute test charge of electricity in the field, which is always supposed to be associated with ordinary matter. It is the acceleration of this electrified mass particle which measures the electric and magnetic forces in the field. In recent years statements have been made by physicists conjecturing that all mass is electromagnetic in origin. Now I have never been able to understand such a point of view, for in that case the ultimate particle of electricity would be "free" and would constantly move in directions determined by the electric and magnetic forces with a velocity equal to that of light.

A first question which arises is this: Shall the forces arising from the charge of the particle itself be considered as forming part of the total force? If we admit that this can be possible, we are confronted with the difficulty that the forces acting on the point particle will be infinite; but if we consider the resultant electromagnetic forces acting on a hypothetical small electrified sphere with the electrified particle at its center, the limiting forces can be determined. However, for a system of such particles there will be an indefinite radiation outward of electromagnetic energy as oppositely electrified particles fall into one another, while those similarly electrified

tend to separate indefinitely under the mutual forces of repulsion. Evidently such a system of electrified particles is of little physical interest.

The use of an elastic fluid as the carrier of electricity seems at first sight to offer greater prospects of success. Not long ago I attempted to develop a theory based upon an adiabatic fluid having the property that the tension increased as the square of the density of the electrical charge. This tension was introduced in order to offset the electrical forces of repulsion within the proton and electron. It turned out that there was a unique sphere of equilibrium, of definite radius but of unstable type. This sphere contracted under slight displacement, and I tried to develop some of the properties of what I thought might be the final stable form.*

Further examination of the problem has made it appear impossible to secure stability of the kind desired, no matter what type of relation between pressure and density is assumed to hold. For, if we imagine the bits of the fluid to be widely distributed but under the same pressures and densities as at the outset, it is clear that the elastic potential energy is the same as before, while the electromagnetic energy of the field has been diminished. From this fact and some more purely mathematical considerations, it follows that there will always be a tendency toward breaking up of the fluid into smaller and smaller nuclei so that no final stable condition is possible.

But, aside from this instability which arises from the fact that electricity of one sign exerts strong forces of repulsion upon itself, there is another difficulty which arises even in the consideration of neutral matter not carrying any electrical charge. In fact, we have two types of disturbance velocities: firstly, that of light, and secondly, that of the elastic wave of the fluid. Now if the velocity of the latter wave is less than that of light, we can imagine two bodies approaching each other from opposite directions with veloc-

* *The Origin, Nature and Influence of Relativity*, New York, The Macmillan Company, 1925. Chapters 6 and 7.

ities greater than the disturbance velocity but less than that of light. In this case the same paradox would arise as in the analogous classical case considered above.

Therefore the only possible elastic fluid would seem to be one with a disturbance velocity equal to that of light at all densities. The fluid of this type I shall term the "perfect fluid," by analogy with the ordinary perfect gas. The pressure in such a fluid is readily determined to be one half the density in absolute units, and so enormously great. Evidently the perfect fluid would tend to expand indefinitely with velocities approaching that of light. Of course such a perfect fluid would afford a much more unstable carrier of electricity than the one which I had proposed (loc. cit.). It should be noted in passing that because of the relativistic character of this perfect fluid, the mass of a small part of it is not invariable but changes as the square of the density of the attached charge.

Consequently, attempts to make use of an elastic fluid as the carrier of electricity seem to fail, and we are led to inquire whether it is not in the nature of the case that the elementary bodies such as the protons and electrons must be allowed to have some sort of autonomous existence in the same sense that empty space has. I shall make a suggestion of this sort in the "atomic potentials" used below. For the moment, however, I wish to emphasize the difficulty in maintaining the point of view of the "field" consistently. Consider a stretched elastic string which is vibrating longitudinally. No matter how the linear density or elastic coefficient varies, any local disturbance is propagated in both directions along the string with the disturbance velocity. Consequently the particle cannot be treated as a moving local disturbance. However, we might compare the particle to a bead on a string and moving with a velocity of its own, whose motion is affected by and affects that of the string. The bead would not appear then as a part of the string but would have autonomous existence. Here the strings correspond to the "field" and the bead to the body.

The idea of atomic potentials may be explained as follows. It is well known that the kinetic and elastic energy of an adiabatic fluid at low velocities can be defined just as in classical dynamics, in such wise that the principle of conservation of energy holds. Let us suppose in addition that there is an individual atomic potential energy of positive volume density, ψ, where ψ has a value fixed for all time for each point of the fluid. This leads to a supplementary body force, proportional to the gradient of ψ in space. In this way indefinite expansion of the corresponding proton or electron is prevented, for it would involve an indefinite increase in the atomic potential energy.

At first sight this seems to insure a stable spherical form of equilibrium. However, further consideration of small motions near the equilibrium state shows that the radial forces are so small as to make the nucleus amorphous under radial displacement, with a tendency to spread over space in wisp like form.

But now suppose that the protons are made of very small parts of the fluid with charge $+e$, while the electrons are made of parts of the perfect fluid with the charge $-e$, and with suitable atomic potentials. Let us suppose furthermore that such an electron can be penetrated freely by the proton.

Under these circumstances there will be a stable spherical form of equilibrium in which the proton coincides with the electron; for the tendency towards amorphous shape of the electrons and protons will be destroyed by the attractive forces between them.

The fact ought not to pass unnoted that all of the forces due to the atomic potential will have a resultant zero if the density of potential ψ is constant at the boundary. Hence the electron as well as the proton will respond to external forces in the required way.

Here perhaps is a kind of two substance theory of matter and electricity which will be found to meet the fundamental mathematical requirements of determinateness and stability.

5. *General Relativity.* The essential logical structure of the gravitational theory of relativity of Einstein is extremely simple. Let us suppose that the universe of events is four-dimensional. Imagine a very slight disturbance of the existing physical condition to be made at an event. This disturbance will spread and modify physical phenomena in a certain region of space-time, in accordance with the principle of local causation. This region is analogous to a three-dimensional cone with vertex at the given event. Now the simplest mathematical form of definition for such a cone is that obtained from the vanishing of a differential quadratic form ds^2. Hence it seems probable that such a form ds^2 will lie at the base of any physical theory; in consequence, the symbolic language of tensor analysis becomes available. If we assume also that a mass particle moves in the space-time of general relativity according to the same underlying law as in the special theory of relativity (principle of equivalence) and if we determine the coefficients of ds^2 in such wise as to be as nearly as possible the same in character as the coefficients in the special theory, they become to a large degree determined.

Now it seems obvious that the space-time in the vicinity of the sun possesses spherical symmetry, and the law of motion obtained on this assumption is essentially that of Newton. In this way the fundamental experimental tests of Einstein are obtained.

Thus the general theory of relativity appears as a theory of empty space and throws no light upon the structure of matter. It is only the spherical symmetry of the space-time about the sun which allows us to come to a conclusion.

The logical situation is analogous to that involved in the following example. Suppose that a uniform square plate is maintained with two opposite sides at temperature 0°C, and the other two sides at temperature 100°C. I say that it is possible to prove that the temperature along the diagonals is exactly 50°C, without making any other assumption than that states of temperature are additive. In fact, imagine

the square rotated about one diagonal. In the new position the sides at temperature 0° take the position of the sides previously at temperature 100°, and *vice versa*. By addition of the first and second states, a state is obtained at temperature 100°C around the four sides, and therefore 100°C within. But evidently along the invariant diagonal the temperatures of the two states are identical. In consequence the temperature along this diagonal, and similarly along the other, must be 50°C. A like treatment of a rectangular plate would not be possible, because of the lesser symmetry.

In attempting to extend the general theory of relativity to space occupied by matter, Einstein was led to "field equations"

$$R_{ij}-\tfrac{1}{2}Rg_{ij}=XT_{ij}, \qquad (X=-8\pi),$$

where T_{ij} is the "energy tensor" of matter and electricity. These equations are supposed to be supplemented by various "constitutive equations" such as the Maxwell-Lorentz equations. But it should be noted that there are in reality two types of energy tensors, namely, one for empty space and another for the interior of matter; for instance, it is only within matter that the velocity tensor is defined. Thus these are not really field equations as they stand.

The space-time framework of general relativity is adapted to our concept of atomic potential; for this purpose we define T_{ij} as consisting of the usual elastic and electromagnetic energy tensor of an atomic potential term ψg_{ij}. The mathematical fact that the divergence of the energy tensor vanishes will yield the observed law of motion of the protons and electrons.

6. *Atomic Theory.* It is evident today that all earlier ideas in physics have been statistical in character, and that the fundamental laws are those of the atomic domain. The kinetic theory of heat illustrates this method of explanation of physical fact in the simplest possible way. Naturally the attention of physicists everywhere is fixed upon the discovery

of the laws of the atom. The main theoretical attack is based upon electrical phenomena in rarefied gases and the properties of spectra.

If we grant the four-dimensional nature of space-time, it is natural to assume to argue by analogy about the atomic domain. This argument of continuity seems to make it imperative that the atom is an oscillating electromagnetic system, which radiates or absorbs electromagnetic energy.

Now the central facts about the atomic oscillator are essentially two: first, it acts like a combination of simple resonators of perfectly definite frequencies, such as those given by the Balmer formula in the case of the hydrogen atom; and secondly, these frequencies are excited only by means of certain quanta of energy (Planck's hypothesis).

Now in my opinion there need be no essential difficulty in accounting for this second fact. Imagine a pendulum to swing in a viscous medium whose viscosity diminishes rapidly as the distance from the position of equilibrium increases. With less than a certain initial velocity, the pendulum would swing back slowly towards its position of equilibrium, never passing it. On the other hand, if the initial velocity is sufficient to carry the pendulum beyond the zone of considerable viscosity, it will oscillate back and forth, traversing the viscous region in damped harmonic motion. Consequently we may conceive of the so-called "energy levels" as defining the amount of energy necessary to carry the oscillators so far from equilibrium that they will move back and forth past the position of equilibrium. Thus the first and most fundamental task appears to be to find an oscillator possessing the desired frequencies. Afterwards one may investigate in detail the rate of electromagnetic radiation, which will correspond to viscosity.

It seems premature to abandon the attempt to explain the facts upon some such basis. Computations of the frequencies should be made for some simple conjectural theories of matter and electricity which meet the elementary mathematical demands of determinateness and stability. As far as

I know this has not been done for a single case. The tendency of the quantum theory as developed by N. Bohr, Heisenberg, Born and others, has been altogether in the direction of obtaining a discrete theory of atomic structure. No doubt the discovery of a consistent theory of this kind would be of the utmost interest for mathematics as well as physics. But from my limited mathematical point of view, I can discern no kernel of thought in their work which tends toward the construction of a logically satisfactory discrete model, although this work is obviously of the highest value for physics.

Very recently Schrödinger* has obtained a highly suggestive "wave equation," so that spectroscopic frequencies are treated by him as somewhat analogous to the frequencies of vibration of an electrodynamic system. He determines a sequence of frequencies and conjectures that the Balmer formula, which is obtained by taking the difference of two such frequencies, may be a difference frequency in the usual physical sense.

It will be of interest to compare his wave equation with that for the small oscillations of the fluid proton and electron in the theory outlined above. Now we observe that in this theory there figures a "substance coefficient" ϕ whose square gives the ratio of the density of matter to the square of the electrical density at any point; there is also the "atomic potential" ψ. Thus there are two functions ϕ and ψ which may be given arbitrarily at the outset, and which need to be specified before definite conclusions can be made. I have not as yet had the opportunity of determining the boundary conditions and the actual frequencies.† The three wave equations are of the same general type as the Schrödinger equation.

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* Annalen der Physik, 1926.

† Further development of the theory outlined above will appear shortly in the Proceedings of the National Academy. The theory leads to a formula of the Balmer type for the frequencies.

There is another remark about atomic dynamics that seems to me of importance. It is usual to approximate to an atomic problem by an ordinary dynamical problem such as the two-body problem, and the equations are then of Hamiltonian form. It is well known that any state of such a system tends to recur. Thus in general the motion will not be limited to certain periodic motions as demanded, or at least suggested, by the quantum theory. But, for a non-Hamiltonian system I have shown that there will exist a restricted set of "central motions" near which any motion will in general be found.* Consequently if the central motions are periodic, we may expect a corresponding set of sharp frequencies. Thus an approximating set of differential equations may yield quantum orbits without any quantum conditions, provided that the equations are not of Hamiltonian type.

If I have convinced you of the necessity for a careful critique of the logical questions inherent in the laws ordinarily given for matter and electricity, I shall feel very well satisfied. The mathematician can be of great service in analyzing these basic ideas, in forming physical models of as simple type as possible which meet the most pressing physical requirements, and in making the necessary calculations of oscillation frequencies, etc. The possibilities of such models based upon an underlying four-dimensional space-time continuum have by no means been exhausted.

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* Göttinger Nachrichten, 1926.

AN ASSEMBLAGE-THEORETIC PROOF OF THE EXISTENCE OF TRANSCENDENTALLY TRANSCENDENTAL FUNCTIONS*

BY J. F. RITT AND ELI GOURIN

1. *Introduction.* The first example of a transcendentally transcendental function,† that is, of an analytic function not a solution of any algebraic differential equation, was given by Hölder,‡ who, in 1887, showed that the gamma function does not satisfy any such equation. Other investigations have followed Hölder's. In some of these, functional equations of different types are studied, to find which of their solutions satisfy algebraic differential equations. In others, e.g., in that of Hurwitz, on power series with rational coefficients§ and in that of Ostrowsky on Dirichlet series,‖ certain analytic expressions are examined and conditions found under which they represent transcendentally transcendental functions.

In the present note, the existence of transcendentally transcendental functions is shown on an a priori basis, by a proof resembling that proof of the existence of transcendental numbers which is based on the countability of the algebraic numbers. The totality of algebraic differential equations has the same power as the totality of analytic functions, but there exists a countable set of the equations, namely, those with integral coefficients, whose solutions form the totality of the solutions of all algebraic differential equations. It follows that the solutions of the countable set of equations do not exhaust the analytic functions.

* Presented to the Society, December 29, 1926.

† This term is due to E. H. Moore, Mathematische Annalen, vol. 48 (1896), p. 49.

‡ Mathematische Annalen, vol. 28 (1887), p. 1.

§ Annales de l'Ecole Normale Supérieure, vol. 6 (1889), p. 327.

‖ Mathematische Zeitschrift, vol. 8 (1920), p. 241.

Our result throws some light on the distribution of transcendentally transcendental functions among the analytic functions. We prove that *it is possible to choose the coefficients in the series*

$$a_0 + a_1 x + \cdots + a_n x^n + \cdots$$

so that the series does not satisfy formally any algebraic differential equation, with the following degree of freedom:

a_0 *may be given any value not on a certain countable set*;

a_1 *may then be given any value not on a certain countable set which depends on the choice of* a_0;

. . . . . .

a_n *may be given any value not on a certain countable set which depends on the choices of* $a_0, \cdots, a_{n-1}$;

. . . . . .

Any set of a's which do not increase too rapidly give a transcendentally transcendental function.

2. *Algebraic Differential Equations.* Let $y(x)$ be any function satisfying an algebraic differential equation. Representing by y_p the pth derivative of y, we write the equation for y in the form

$$\sum c_{(i)} x^{i} y^{i_0} y_1^{i_1} \cdots y_n^{i_n} = 0. \tag{1}$$

Here each c is a constant not zero. It is understood that the expressions $x^i \cdots y_n{}^{i_n}$ are distinct from one another.

Now (1) states that, for the given function $y(x)$, the expressions $x^i \cdots y_n{}^{i_n}$ are linearly dependent. Thus, if we set the wronskian of these expressions equal to zero, we shall have an algebraic differential equation for $y(x)$, usually of order greater than n, *with integral coefficients*. All that it is necessary to show is that the wronskian does not vanish identically in $x \cdots y_n$. If it did, every function with n derivatives would satisfy an equation like (1), that is, an equation with the same expressions $x^i \cdots y_n{}^{i_n}$ as appear in (1), with constants c not all zero. But because we can construct a function with any given values for itself and for its first n derivatives at any number of points, it

would follow easily that an equation like (1) exists, with coefficients c not all zero, which is an identity for $x, \cdots, y_n$ arbitrary. This is absurd. We know thus that $y(x)$ satisfies an algebraic differential equation with integral coefficients.*

3. *Transcendentally Transcendental Functions.* The algebraic differential equations with integral coefficients are countable. Let them be arranged in a sequence

$$e_1 = 0, \quad e_2 = 0, \cdots, \quad e_n = 0, \cdots. \tag{2}$$

If the polynomial e_i vanishes when $x=h$, for all values of the variables other than x which appear in e_i, then e_i is divisible by $x-h$. As each e_i has only a finite number of factors of the form $x-h$, the polynomials of (2) have, collectively, only a countable set of such factors.

If we replace x in (2) by any value distinct from the h's of the countable set just mentioned, the equations of (2) become non-identical equations which do not involve x,

$$e_1' = 0, \quad e_2' = 0, \cdots, \quad e_n' = 0, \cdots. \tag{3}$$

Of course, an e_i' may be a constant. This simply means that $e_i=0$ has no solution analytic for the chosen value of h.

We can now choose y anywhere, except on a countable set, in such a way as to reduce the equations of (3) to non-identical equations involving neither x nor y, and can continue in the same fashion for y_1, etc. We find thus a set of values of our variables which satisfy no equation (2).

If the values chosen for x, y, y_1, etc., are respectively h, k_0, k_1, etc., the power series $\sum k_n(x-h)^n/n!$ will not satisfy formally any algebraic differential equation. The series $\sum k_n x^n/n!$ cannot satisfy such an equation either. This proves the result stated in the introduction.

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* Hurwitz, loc. cit., states without proof that a power series with rational coefficients which satisfies an algebraic differential equation, satisfies such an equation with integral coefficients. Very probably he had the more general result just proved in mind. In any case, the countability of the equations with integral coefficients plays no role in Hurwitz's work.

THE MOST GENERAL CLOSED POINT SET OVER WHICH CONTINUOUS FUNCTION MAY BE DEFINED BY CERTAIN PROPERTIES*

BY G. T. WHYBURN

In a recent paper† C. H. Rowe states the following theorem.

THEOREM A. *In order that a function $f(x, y, \cdots)$, defined in a finite open region R, may be continuous in R, it is necessary and sufficient that, as we move along any continuous curve whose points belong to R, the function should never pass from one value to another without taking every intermediate value, and that, for every value of α, the set E of all points of R for which $f=\alpha$ should contain all those of its limit points that belong to R.*

Rowe points out difficulties in extending his theorem to functions defined over a closed region‡ instead of an open one. He states that his theorem is valid for functions defined over a closed region R provided R satisfies the condition that for every one of its boundary points P there are values of δ as small as we please such that the points of R whose distances from P do not exceed δ form a set having the property that every two of its points can be joined by an arc of a continuous curve which lies wholly in this set. The present paper has two main purposes: (1) to show that the condition imposed by Rowe upon the closed region R above is unnecessarily strong and (2) to give a condition on a closed point set M, whether M is a closed region or not, which is both necessary and sufficient in order that Theorem A should be valid for functions defined over M.

* Presented to the Society, September 9, 1926.

† *Note on a pair of properties which characterize continuous functions*, this Bulletin, vol. 32 (1926), pp. 285–287.

‡ For definitions of the terms *open region* and *closed region* see Rowe, loc. cit.

In view of Theorem I below, the following example of a simple closed curve plus its interior which does not satisfy Rowe's condition shows that his condition is much too strong. Let t_1 denote the simple continuous arc composed of the point (0, 0) together with the broken line joining, in the order here named, the points (4, 0), (4, 1), (−4, 1), (−4, −1), (2, −1), (2, $\frac{1}{2}$), (−2, $\frac{1}{2}$), (−2, −$\frac{1}{2}$), (1, −$\frac{1}{2}$), (1, $\frac{1}{4}$), (−1, $\frac{1}{4}$), (−1, −$\frac{1}{4}$), $\cdots$, indefinitely. Let t_2 denote the simple continuous arc consisting of the point (0, 0) plus the broken line joining, in the order here named, the points (3, 0), (3, $\frac{3}{4}$), (−3, $\frac{3}{4}$), (−3, −$\frac{3}{4}$), ($\frac{3}{2}$, −$\frac{3}{4}$), ($\frac{3}{2}$, $\frac{3}{8}$), (−$\frac{3}{2}$, $\frac{3}{8}$), (−$\frac{3}{2}$, −$\frac{3}{8}$), ($\frac{3}{4}$, −$\frac{3}{8}$), ($\frac{3}{4}$, $\frac{3}{16}$), $\cdots$, indefinitely. Let I denote the straight line interval from (3, 0) to (4, 0). It is easy to see that if J denotes the point set t_1+t_2+I, then J is a simple closed curve. Let R denote J plus its interior. Then R is a closed region. But if P denotes the point (0, 0), then no matter what positive number δ we take which is less than 1, the set K_δ of all those points of R whose distances from P do not exceed δ is not connected. Hence R does not satisfy Rowe's condition, even though it belongs to one of the simplest types of closed regions.

THEOREM I. *In order that Theorem* A *should be valid for functions defined over a closed point set M it is necessary and sufficient that M should be connected im kleinen.**

PROOF. The condition is sufficient. Let $f(x, y, \cdots)$ be any functon which is defined over a closed and connected im kleinen point set M and which satisfies the conditions of Theorem A. . Let P denote any definite point of M, and ϵ any positive number. Let E_1 and E_2 respectively denote the

* A point set M is connected im kleinen provided that for each point P of M and for every positive number ϵ there exists a positive number $\delta_{\epsilon P}$ such that every point X of M whose distance from P is less than $\delta_{\epsilon P}$ lies, together with P, in a connected subset of M of diameter less than ϵ. By the diameter of a point set K is meant the upper limit of the aggregate of numbers $\delta(X, Y)$, where X and Y are any pair of points of K and $\delta(X, Y)$ is the distance between X and Y. The notation $\delta(X, Y)$, to represent the distance between the points or point sets X and Y, will be used in this paper. See Hahn, *Mengentheoretische Charakterisierung der stetigen Kurve*, Wiener Akademie Sitzungsberichte, vol. 123, pp. 2433–2489.

set of all those points X of M such that $f(X)=f(P)+\epsilon$ and $f(X)=f(P)-\epsilon$, respectively. By hypothesis both E_1 and E_2 are closed or vacuous point sets, and neither contains the point P. Hence, there exists a positive number e such that no point of M whose distance from P is $\leqq e$ belongs to either E_1 or E_2. Since M is connected im kleinen and closed, there exists a positive number d such that every point Q of M such that $\delta(Q, P)<d$ can be joined to P by a simple continuous arc* every point of which belongs to M and is at a distance less than e from P. Let K denote the set of all those points of M whose distance from P is less than d. Then for every point Q of $K, f(P)-\epsilon<f(Q)<f(P)+\epsilon$. For if for some point A of K, this statement is not true, then since A can be joined to P by an arc t which lies in M but which has no point in common with either of the sets E_1, E_2, f does not take on all values between $f(A)$ and $f(P)$ as we move along t from A to P, contrary to hypothesis. Hence, $f(x, y, \cdots)$ is continuous at every point P of M.

The condition is also necessary. Let M denote any closed point set which is not connected im kleinen. I will show that a function can be defined over M which will satisfy the conditions of Theorem A but which will not be continuous at every point of M. Since M is not connected im kleinen, there exists a point P of M and a positive number ϵ such that no matter what positive number δ may be, there exists some point Q of M such that $\delta(Q, P)<\delta$ but such that Q and P do not lie together in any connected subset of M of diameter less than ϵ. It easily follows that M contains a sequence of points $P_1, P_2, P_3, \cdots$, having P as its sequential limit point and such that for no positive integer n does P_n lie together with P in a connected subset of M of diameter $\leqq\epsilon/2$. Let K denote the set of all those points of

* See R. L. Moore, *A theorem concerning continuous curves*, this Bulletin, vol. 23 (1917), pp. 233–236; J. R. Kline, *Concerning approachability of simple closed and open curves*, Transactions of this Society, vol. 21 (1920), footnote to p. 453, and S. Mazurkiewicz, *Sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 1 (1920), pp. 166–209.

M which can be joined to P by a connected subset of M of diameter $\leqq \epsilon/2$. Since M is closed, it easily follows that K is closed. Let C denote the set of all points whose distances from P are equal to $\epsilon/2$. Now let us define a function $f(x, y, \cdots)$ over M as follows. For every point X of M which is on K let $f(X) = \delta(X, C)$, and for every point X of M which is not on K let $f(X) = \delta(X, K)$.

It is easy to see that $f(x, y, \cdots)$, thus defined over M satisfies both conditions of Theorem A. But since no point of the sequence $P_1, P_2, P_3, \cdots$, belongs to K, and since $\delta(P_n, P) \to 0$ as $n \to \infty$, then $f(P_n) \to 0$ as $n \to \infty$. But $f(P) \neq 0$. Hence $f(x, y, \cdots)$ is not continuous at P. It follows that the condition of Theorem I is necessary.

THEOREM II. *In order that Theorem A should be valid for functions defined over a point set M, (closed or not), it is sufficient that M should be arcwise connected im kleinen.**

Theorem II can be proved by an argument almost identical with the first part of the proof of Theorem I.

It is interesting to note that Theorem A is a special case of Theorem II, since every open region is arcwise connected im kleinen.

It can easily be shown by means of an example similar to the one given in the second part of the proof of Theorem I that in order that Theorem A should be valid for functions defined over a point set M, (closed or not), it is *necessary* that M should be connected im kleinen. It would be interesting to determine a necessary and sufficient condition on a point set M, which is otherwise unrestricted, in order that Theorem A should be valid for functions defined over M.

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* A point set M is arcwise connected im kleinen provided that for every point P of M and every positive number ϵ there exists a positive number $\delta_{\epsilon P}$ such that every point X of M such that $\delta(X, P) < \delta_{\epsilon P}$ can be joined to P by a simple continuous arc which is a subset of M and is of diameter less than ϵ. The properties "connected im kleinen" and "arcwise connected im kleinen" are equivalent for closed point sets.

INVARIANT RELATIONS

BY F. H. MURRAY

1. *Introduction.* In a preceding paper* a discussion was given of the systems of invariant relations as defined by Poincaré.† The methods employed there required that all the functions appearing in the differential system or in the invariant relations should be analytic. This restriction is not essential, however, and it is the object of the present note to give a corresponding discussion in which the functions appearing are required to possess only a finite number of derivatives. The integration of the differential system by the method of successive approximations of Picard, after a suitable transformation, gives at once the justification for the name "invariant relation."

2. *Transformation of the Differential System.* Suppose given a differential system

$$\frac{dx_i}{dt} = X_i(x_1, \cdots, x_n), \quad (i = 1, \cdots, n), \tag{1}$$

and a system of equations

$$\phi_k(x_1, \cdots, x_n) = 0, \quad (k = 1, \cdots, r). \tag{2}$$

The functions X_i, ϕ_k are assumed to possess continuous partial derivatives of the first and second orders with respect to all their arguments, $|x_i - x_i^0| < C$. Assume that the equations

$$\sum_{i=1}^{n} X_i \frac{\partial \phi_k}{\partial x_i} = 0, \qquad (k = 1, \cdots, r), \tag{3}$$

* *On certain families of orbits with arbitrary masses in the problem of three bodies*, Transactions of this Society, vol. 28, pp. 74–77.

† *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, pp. 45–47.

are a consequence of the equations (2); the equations (2) are then said to form a system of invariant relations. If the functions ϕ_k are not independent, but equations (2) are a consequence of a sub-set of these equations, then (3) is also a consequence of the equations of the sub-set, and the discussion can be restricted to the smaller set.

Without loss of generality, therefore, it may be assumed that the equations (2) are independent. It will be assumed also that the determinant

$$\begin{vmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_r} \\ \cdots & \cdots & \cdots \\ \frac{\partial \phi_r}{\partial x_1} & \cdots & \frac{\partial \phi_r}{\partial x_r} \end{vmatrix}$$

does not vanish for $x_i = x_i^0$, $i = 1, \cdots, n$. Under these assumptions the equations

$$(4) \qquad \begin{aligned} y_1 = \phi_1, \cdots, y_r = \phi_r, \\ y_{r+1} = x_{r+1} - x_{r+1}^0, \cdots, y_n = x_n - x_n^0 \end{aligned}$$

can be solved for $x_1, \cdots, x_n$, if $|y_i| \leqq a$, $(i = 1, \cdots, n)$, if a is sufficiently small. If transformed functions are indicated by brackets, (1) becomes

$$(1') \qquad \begin{cases} \dfrac{dy_i}{dt} = \left[\sum_{k=1}^{n} X_k \dfrac{\partial \phi_i}{\partial x_k} \right] = Y_i(y_1, \cdots, y_n), (i = 1, \cdots, r), \\ \dfrac{dy_k}{dt} = [X_k] = Y_k(y_1, \cdots, y_n), \qquad (k = r+1 \cdots n). \end{cases}$$

Now if $y_1 = y_2 = \cdots = y_r = 0$, equations (2) are satisfied, hence (3), and consequently $Y_1 = Y_2 = \cdots = Y_r = 0$, for all values of y_{r+k} such that $|y_{r+k}| < a$. Or,

$$(5) \qquad Y_i(0, \cdots, 0, y_{r+1}, \cdots, y_n) = 0, \ (i = 1, \cdots, r).$$

3. *Integration of the Equations.* Consider a differential system

$$(6) \qquad \frac{dy_i}{dt} = Y_i(y_1, \cdots, y_n), \qquad (i = 1, \cdots, n),$$

in which the functions Y_i satisfy the conditions

$$Y_i(0, \cdots, 0; y_{r+1}, \cdots, y_n) = 0, \quad (i = 1, \cdots, r),$$

$$|Y_k(0, \cdots, 0)| \leqq M, \quad \left|\frac{\partial Y_k}{\partial y_i}\right| \leqq A_k, \ (i, k = 1, \cdots, n),$$

in the region

$$|y_i| \leqq b. \tag{7}$$

Suppose $K = \sum_{i=1}^{n} A_i$; the system (6) can now be integrated by the method of successive approximations of Picard:

$$y_i' = \int_0^t Y_i(0, \cdots, 0)dt,$$

and in general

$$y_i^{(k+1)} = \int_0^t Y_i(y_1^{(k)}, \cdots, y_n^{(k)})dt,$$
$$(i = 1, \cdots, n; k = 1, 2, 3, \cdots).$$

The series

$$y_i'(t) + (y_i'' - y_i') + \cdots + (y_i^{(k+1)} - y_i^{(k)}) + \cdots$$

converges uniformly in t if $t \leqq T$,

$$T < \frac{1}{K} \log\left(1 + \frac{bK}{M}\right).$$

By hypothesis,

$$Y_i(0, \cdots, 0) = 0, \qquad (i = 1, \cdots, r);$$

consequently

$$y_i'(t) = 0, \qquad (i = 1, \cdots, r).$$

Also

$$Y_i(0, \cdots, 0; y_{r+1}', \cdots, y_n') = 0, \qquad (i = 1, \cdots, r),$$

whence

$$y_i''(t) \equiv 0, \qquad 0 < t \leqq T, \qquad (i = 1, \cdots, r).$$

By the method of induction it is seen at once that every one of the functions $y_i^{(k)} (i = 1, \cdots, r)$ vanishes identically in t, and consequently the sum of the series y_i must likewise vanish identically. *If the conditions* $y_i^0 = 0$, $(i = 1, \cdots, r)$, *are satisfied, then* y_i *vanishes identically for* $t \leqq T$.

Suppose that after the transformation of §1, the differential system satisfies the hypotheses stated at the beginning of this paragraph. The equations $[\phi_i]_{t=0}=0$ imply the equations $y_i^0=0$, $(i=1, \cdots, r)$, which imply that

$$y_i \equiv \phi_i(x_1, \cdots, x_n) = 0, \qquad 0 \leqq t \leqq T, \qquad (i = 1, \cdots, r).$$

The equations (2) are satisfied, therefore, for all values of $t \leqq T$, and the name "invariant relations", applied to (2), is completely justified.

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DETERMINATION OF THE NUMBER OF SUBGROUPS OF AN ABELIAN GROUP

BY G. A. MILLER

The present note aims to exhibit a simple method for finding the total number of the subgroups contained in an arbitrary abelian group G of order g. It will be shown that this problem can be reduced to a determination of the number of ways in which the first l independent generators of highest order of a given prime power abelian group can be selected. Hence we shall first consider this question. Therefore we shall assume for the present that g is of the form p^m, p being a prime number, and that G has k largest invariants which are separately equal to p^α. It is well known that G contains a fundamental characteristic subgroup of order p^k and of type $(1, 1, 1, \cdots)$.* Each operator of order p contained in this subgroup is the $p^{\alpha-1}$ power of p^{m-k} operators of G, and no other operator of order p contained in G has this property. After selecting $l \leqq k$ independent generators of highest order in a set of such generators the next independent generator of this order can therefore be selected in

$$(p^k - p^l)p^{m-k}$$

* G. A. Miller, American Journal of Mathematics, vol. 27 (1905), p. 15.

different ways, since every operator of order p^α which does not generate an operator of order p contained in the group generated by the given l independent generators can be used as such an additional generator. Hence the first l independent generators of order p^α can be selected in

$$p^{m-k} \prod_{x=0}^{l-1} (p^k - p^x)$$

different ways.

From this formula it results directly that after l_1 independent generators of order p^α in a set of such generators of G have been selected, l_2 additional such generators can be selected in

$$p^{m-k} \prod_{x=l_1}^{l_1+l_2-1} (p^k - p^x)$$

different ways whenever $l_1+l_2 \leqq k$. To select a set of independent generators of a subgroup H of G, when g is of the form p^m, we may obviously proceed as follows: Consider the characteristic subgroup of G which is generated by its operators whose order is equal to the largest invariant of H. The number k_1 of independent generators of highest order in a set of reduced independent generators of this characteristic subgroup of order p^m, is evidently equal to the number of the invariants of G which are not less than the largest invariant of H. The number of different ways in which the l' largest independent generators of H in a set of such reduced independent generators can be selected from the operators of G is given by the formula noted at the close of the preceding paragraph, when m, k, and l are replaced by m_1, k_1, and l' respectively. After these l' independent generators have been selected we may consider the characteristic subgroup generated by all the operators of G whose order is equal to the second largest invariant of H, in case H has different invariants. The number of different ways in which the l_2' next to the largest independent generators in a set of reduced generators of H can be selected from the

operators of this second characteristic subgroup is given by the formula of the present paragraph, since these l_2' generators have been preceded by the l' generators of highest order, in a set of reduced independent generators of H.

When the invariants of H have more than two different values we obviously repeat this process by considering next the characteristic subgroup of G which is generated by all its operators whose order is equal to the third largest invariant of H and letting l_2'' be the number of operators of this order in a set of reduced independent generators of H while l_1'' represents the number of all the larger operators in such a set. Hence it is clear that the given formulas suffice to determine the number of ways in which a set of reduced independent generators of H can be selected from the operators of G, and if this number is divided by the number of ways in which such a set can be selected from the operators of H itself, which is given by the same formulas, the quotient is equal to the number of the different subgroups contained in G which are separately of the same type as H is. This, therefore, solves the problem of determining the number of the subgroups of G whenever g is a power of a prime number. When g does not have this property G is the direct product of its Sylow subgroups and the subgroups of G are the direct products of their Sylow constituents whenever they are not prime power subgroups. The number of such subgroups having given invariants is therefore equal to the product of the numbers of the possible ways in which the Sylow subgroups involved therein can be selected from the corresponding subgroups of G. Hence the problem of determining the number of the subgroups involved in G is completely solved by the formulas noted above.

The University of Illinois

THE ASYMPTOTIC OSCULATING QUADRICS OF A CURVE ON A SURFACE

BY E. P. LANE

1. *Definition.* Let us consider a surface S, a curve C on S, and three neighboring points P, P_1, P_2, on C. The three tangents at these points to the asymptotic curves of one family determine a quadric whose limit, as P_1, P_2 independently approach P along C, is a quadric called* by Bompiani an *asymptotic osculating quadric* of C at P. A second asymptotic osculating quadric is obtained by using the other family of asymptotics. We shall now derive the equations of these quadrics, using Wilczynski's notation, and shall deduce some of their fundamental properties.

2. *Equations.* Let the four homogeneous coordinates y of a point P on a non-degenerate non-developable surface S be given as analytic functions of two independent variables u, v; and let the curves $u=\text{const.}$, $v=\text{const.}$ be the asymptotics. Then the functions y, when multiplied by a suitably chosen proportionality factor, are solutions of Wilczynski's canonical system of differential equations,

$$(1) \qquad y_{uu} + 2by_v + fy = 0, \quad y_{vv} + 2a'y_u + gy = 0.$$

The one-parameter family of curves on S represented by the equation

$$(2) \qquad dv - \lambda du = 0$$

contains one curve C through P. The coordinates Y of any point P_1 on C near P are given by an expansion of the form

$$Y = y + \frac{dy}{du}\Delta u + \frac{1}{2}\frac{d^2y}{du^2}\Delta u^2 + \cdots.$$

* Bompiani, *Geometria delle superficie considerate nello spazio rigato*, Rendiconti dei Lincei, 1926.

If the points y, y_u, y_v, y_{uv} are used as the vertices of a local tetrahedron of reference with a suitably chosen unit point, the local coordinates x_i of P_1 are represented by the series

$$(3)\quad \begin{cases} x_1 = 1 - \frac{1}{2}(f + g\lambda^2)\Delta u^2 + \Big[-\frac{1}{6}f_u + \frac{1}{2}(2bg - f_v)\lambda \\ \quad + \frac{1}{2}(2a'f - g_u)\lambda^2 - \frac{1}{6}g_v\lambda^3 - \frac{1}{2}g\lambda\lambda' \Big]\Delta u^3 + \cdots, \\ x_2 = \Delta u - a'\lambda^2\Delta u^2 + \Big[-\frac{1}{6}f + 2a'b\lambda \\ \quad - \frac{1}{2}(g + 2a_u')\lambda^2 - \frac{1}{3}a_v'\lambda^3 - a_u'\lambda\lambda' \Big]\Delta u^3 + \cdots, \\ x_3 = \lambda\Delta u - \Big(b - \frac{1}{2}\lambda'\Big)\Delta u^2 + \Big[-\frac{1}{3}b_u \\ \quad - \frac{1}{2}(f + 2b_v)\lambda + 2a'b\lambda^2 - \frac{1}{6}g\lambda^3 + \frac{1}{6}\lambda'' \Big]\Delta u^3 + \cdots, \\ x_4 = \lambda\Delta u^2 - \frac{1}{3}\Big(a'\lambda^3 + b - \frac{3}{2}\lambda'\Big)\Delta u^3 + \cdots, \end{cases}$$

where λ', λ'' are total derivatives of λ. The derivatives of these series with respect to Δu are the coordinates x_{iu} of a point on the tangent of the asymptotic $v = \text{const.}$ through P_1. And any point on this tangent is given by a linear combination of the form

$$\eta_i = hx_i + kx_{iu}.$$

If now the algebraic equation of a quadric surface is subjected to the condition that it be satisfied by the functions η identically in h, k and in Δu up to and including terms in Δu^2, the result is

$$(4)\quad \begin{cases} (2a'b\lambda^3 - 2b_v\lambda^2 - b_u\lambda + 2b^2 + b\lambda')x_4^2 + 2b\lambda x_3x_4 \\ \quad + \lambda^3(x_1x_4 - x_2x_3) - 2b\lambda^2x_2x_4 = 0. \end{cases}$$

This is the equation of the first asymptotic osculating quadric $Q^{(u)}$ of C at P. The equation of the second asymptotic osculating quadric $Q^{(v)}$ of C at P is

$$(5)\quad \begin{cases} (2a'b - 2a_u'\lambda - a_v'\lambda^2 + 2a'^2\lambda^3 - a'\lambda')x_4^2 - 2a'\lambda x_3x_4 \\ \quad + (x_1x_4 - x_2x_3) + 2a'\lambda^2x_2x_4 = 0. \end{cases}$$

3. *Properties.* Some simple properties of the quadrics $Q^{(u)}$ and $Q^{(v)}$ will now be deduced. First of all, it is clear that the tangent plane $x_4=0$ of S at P cuts each of these quadrics in the asymptotic tangents $x_2x_3=0$. And $Q^{(u)}$ becomes the quadric Q of Lie, whose equation is

$$(6)\qquad x_1x_4 - x_2x_3 + 2a'bx_4^2 = 0,$$

in case $\lambda\to\infty$, while $Q^{(v)}$ becomes the quadric of Lie in case $\lambda=0$. The quadrics $Q^{(u)}$ and $Q^{(v)}$ coincide only for a curve C which is tangent to a curve of Darboux, $a'\lambda^3+b=0$, on a surface for which

$$a'\left[\frac{\partial}{\partial u}\log a'^2b\right]^3 = b\left[\frac{\partial}{\partial v}\log a'b^2\right]^3.$$

We shall suppose from now on that Q, $Q^{(u)}$, $Q^{(v)}$ are distinct, and that S is unrestricted.

The result of eliminating λ' from equations (4) and (5) is

$$(7)\quad \begin{cases} (a'\lambda^3 + b)(x_1x_4 - x_2x_3 + 2a'bx_4^2) \\ + a'b\left[2(a'\lambda^3 + b) - \lambda\dfrac{\partial}{\partial u}\log a'^2b\right. \\ \left. - \lambda^2\dfrac{\partial}{\partial v}\log a'b^2\right]x_4^2 = 0. \end{cases}$$

This is a quadric of Darboux through the intersection of $Q^{(u)}$ and $Q^{(v)}$. For a curve C which passes through P tangent to a curve of Darboux this quadric becomes the tangent plane counted twice, so that the intersection of $Q^{(u)}$ and $Q^{(v)}$ is the asymptotic tangents each counted twice; in this case $Q^{(u)}$ and $Q^{(v)}$ are tangent to each other at every point of each asymptotic tangent.

We have found here a new characterization of the directions of Darboux: *the directions of Darboux are the di-*

*rections of curves whose asymptotic osculating quadrics intersect only in the asymptotic tangents.**

The quadrics $Q^{(u)}$ and $Q^{(v)}$ intersect, besides in the asymptotic tangents, also in a residual conic which lies in the plane whose equation is

$$(8)\quad \begin{cases} 2\lambda(a'\lambda^3+b)(x_3-\lambda x_2)+[\lambda'(a'\lambda^3+b)-2(a'^2\lambda^6-b^2) \\ \quad +\lambda(2a_u'\lambda^3-b_u)+\lambda^2(a_v'\lambda^3-2b_v)]x_4=0, \end{cases}$$

obtained by eliminating x_1 from equations (4) and (5). This plane cuts the tangent plane $x_4=0$ in the line $x_3-\lambda x_2=0$, which is tangent to C at P; and the residual conic has this line for tangent at P.

The equation of the osculating plane of C at P is

$$(9)\qquad 2\lambda(x_3-\lambda x_2)-(\lambda'+2a'\lambda^3-2b)x_4=0.$$

This plane coincides with the plane (8) in case C is such that

$$(10)\quad 2(a'\lambda^3+b)\lambda'+\lambda(2a_u'\lambda^3-b_u)+\lambda^2(a_v'\lambda^3-2b_v)=0.$$

But this is the differential equation† of the pangeodesics. Thus we have found a new characterization of the pangeodesics: *a curve is a pangeodesic in case its osculating plane contains the residual conic of intersection of its asymptotic osculating quadrics.*

The quadric of Lie intersects $Q^{(u)}$ in the asymptotic tangents and in a conic which lies in the plane

$$(11)\quad 2b\lambda(x_3-\lambda x_2)+(b\lambda'+2b^2-b_u\lambda-2b_v\lambda^2)x_4=0.$$

This plane coincides with the osculating plane of C in case C is such that

$$(12)\qquad 2b\lambda'+2a'b\lambda^3-2b_v\lambda^2-b_u\lambda=0.$$

The importance of the pole-polar correspondence with respect to the quadric of Lie in the projective differential geometry of surfaces suggests that the polar relation with respect to the quadrics $Q^{(u)}$ and $Q^{(v)}$ would be of interest.

* This theorem was also found by Bompiani. See Fubini and Čech, *Geometria Proiettivo Differentiale*, vol. 2, Appendix 2.

† Fubini and Čech, *Geometria Proiettiva Differenziale*, p. 147.

We shall merely mention here that a curve C defines by means of these quadrics a collineation in the tangent plane such that to any point x corresponds that point x' which has the same polar plane with respect to $Q^{(v)}$ as the point x has with respect to $Q^{(u)}$. The equations of this collineation are

$$(13)\quad \begin{cases} \rho x_1' = \lambda^2 x_1 + 2(a'\lambda_3 + b)(x_3 - \lambda x_2), \\ \rho x_2' = \lambda^2 x_2, \qquad \rho x_3' = \lambda^2 x_3. \end{cases}$$

This collineation is an *elation*, with the tangent of C for axis and with the point P for center. It is the identity if C is tangent to a curve of Darboux.

4. *Conjugate Asymptotic Osculating Quadrics.* The family of curves

$$(14)\qquad dv + \lambda du = 0$$

is conjugate to the family (2). The curve $C_{-\lambda}$ of this family through P is conjugate to the curve C_λ of the family (2) through P, so that the corresponding quadrics $Q_\lambda^{(u)}$, $Q_{-\lambda}^{(u)}$ and $Q_\lambda^{(v)}$, $Q_{-\lambda}^{(v)}$ may be called conjugate. The equations of conjugate quadrics differ only in the sign of λ.

It is easy to verify that $Q_\lambda^{(u)}$ and $Q_{-\lambda}^{(u)}$, intersecting in the asymptotic tangents, are tangent to each other along the line $x_2 = x_4 = 0$. The remainder of their intersection is the straight line

$$(15)\quad \begin{cases} x_2 - \left(\frac{1}{2}\frac{\lambda_v}{\lambda} - \frac{b_v}{b} + \frac{b}{\lambda^2}\right)x_4 = 0, \\ x_1 - \left(\frac{1}{2}\frac{\lambda_v}{\lambda} - \frac{b_v}{b} - \frac{b}{\lambda^2}\right)x_3 \\ \qquad\qquad + \left(2a'b - \frac{b_u}{\lambda^2} + b\frac{\lambda_u}{\lambda^3}\right)x_4 = 0, \end{cases}$$

which intersects the asymptotic tangent $x_2 = x_4 = 0$ in the point

$$\left(-\frac{b}{\lambda^2} + \frac{1}{2}\frac{\lambda_v}{\lambda} - \frac{b_v}{b}, 0, 1, 0\right).$$

Similarly, the quadrics $Q_\lambda^{(v)}$ and $Q_{-\lambda}^{(v)}$ define the point

$$\left(-a'\lambda^2 - \frac{1}{2}\frac{\lambda_u}{\lambda} - \frac{a_u'}{a'}, 1, 0, 0\right)$$

on the other asymptotic tangent. The line joining these two points coincides with the ray of the conjugate net λ, which joins the points

$$\left(-\frac{1}{2}\frac{\lambda_v}{\lambda} - \frac{b}{\lambda^2}, 0, 1, 0\right), \qquad \left(\frac{1}{2}\frac{\lambda_u}{\lambda} - a'\lambda^2, 1, 0, 0\right),$$

in case

$$\frac{\lambda_u}{\lambda} + \frac{a_u'}{a'} = 0, \qquad \frac{\lambda_v}{\lambda} - \frac{b_v}{b} = 0.$$

Then the surface S has the property

$$\frac{\partial^2}{\partial u \partial v} \log a'b = 0,$$

so that, after a change of parameters, $a'b = 1$, $\lambda = \text{const. } b$. For such surfaces Wilczynski's canonical form (1) of the differential equations coincides with Fubini's canonical form.* The line y, y_{uv} is therefore the projective normal. The curvature of Fubini's fundamental form $\phi_2 \equiv 8a'b\, du\, dv$ is zero, and the mean projective curvature† is -2.

We reach thus the following conclusion. *A surface has mean projective curvature equal to -2 if, and only if, there exists on it a conjugate net whose ray, for each surface point, coincides with the line joining the points on the asymptotic tangents where each of these tangents is met by the residual line of intersection of a pair of conjugate asymptotic osculating quadrics of the curves of the conjugate net through the surface point.*

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* Lane, *Wilczynski's and Fubini's canonical systems of differential equations*, this Bulletin, vol. 32 (1926), p. 365.

† Fubini and Čech, loc. cit., p. 146.

A NEW CHARACTERIZATION OF PLANE CONTINUOUS CURVES*

BY W. L. AYRES

A number of authors† have given necessary and sufficient conditions that a bounded continuum be a continuous curve. However new conditions are always of interest as no one characterization applies without difficulty to all problems. It is the purpose of this paper to give a new necessary and sufficient condition that a bounded plane continuum be a continuous curve. Also this gives a condition under which a subcontinuum of a continuous curve is itself a continuous curve. Finally we prove a new property of continuous curves.

THEOREM I. *In order that a continuum N, which is a subset of a plane continuous curve M and such that $M-N$ consists of a finite number of maximal connected subsets‡, be a continuous curve, it is necessary and sufficient that if $P_1, P_2, P_3, \cdots$ is any sequence of distinct points of a maximal connected subset of $M-N$ which has a sequential limit point P, then there exists an increasing sequence of positive integers $n_1, n_2, n_3, \cdots$*

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* Presented to the Society, October 30, 1926.

† For definitions relating to and characterizations of continuous curves, see R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, this Bulletin, vol. 29 (1923), pp. 289–302. Hereafter we shall refer to this paper as *Report*. See also R. L. Wilder, *A property which characterizes continuous curves*, Proceedings of the National Academy, vol. 11 (1925), pp. 725–728; R. L. Moore, *A characterization of a continuous curve*, Fundamenta Mathematicae, vol. 7 (1925), pp. 302–7; H. M. Gehman, *Some conditions under which a continuum is a continuous curve*, Annals of Mathematics, vol. 27 (1926), pp. 381–4; R. L. Wilder, *A characterization of continuous curves by a property of their open subsets*, this Bulletin, vol. 32 (1926), p. 217.

‡ A point set K which is a subset of a point set M is said to be a *proper subset* of M if $M-K$ is not vacuous. A connected subset K of a point set M is said to be a *maximal connected subset* of M if K is not a proper subset of any connected subset of M.

and a set of arcs of $M-N$, $P_{n_1}P_{n_2}$, $P_{n_2}P_{n_3}$, $\cdots$, *such that the set* $P+\sum_{i=1}^{\infty} P_n P_{n_{i+1}}$ *is closed.*

PROOF. A. The condition is necessary. Let $P_1, P_2, P_3, \cdots$ be any sequence of points of a maximal connected subset D of $M-N$ which has a sequential limit point P. There are two cases to consider.

(a). If P is a point of $M-N$, D contains P and there exists a circle C_1 with center at P which encloses no point of N. We may suppose that for every i, $P_i \neq P$, for if any P_i were P we could drop this point from the sequence and consider the remainder. Since M is connected im kleinen, there exists a circle C_2 with center at P such that $r_2 \leqq r_1/2$, where r_i denotes the radius of C_i, and such that every point of M in the interior of C_2 can be joined to P by an arc* of M which lies wholly in the interior of C_1. Let n_1 be the smallest integer so that P_{n_1} is interior to C_2. In general there exists a circle C_{i+1} with center at P such that $r_{i+1} \leqq r_i/2$ and $P_{n_{i-1}}$ lies in the exterior of C_{i+1} and such that every point of M in the interior of C_{i+1} can be joined to P by an arc of M which lies wholly in the interior of C_i. Let n_i be the smallest integer such that P_{n_i} lies in the interior of C_{i+1} and let $P_{n_i}P$ denote the arc of M (actually of $M-N$) whose existence is shown above. For every i, the set $P_{n_i}P+P_{n_{i+1}}P$ contains an arc $P_{n_i}P_{n_{i+1}}$ from P_{n_i} to $P_{n_{i+1}}$. Since every arc $P_{n_i}P_{n_{i+1}}$ lies in the interior of the circle C_i and the numbers r_i approach 0 as i increases, the set $P+\sum_{i=1}^{\infty} P_{n_i}P_{n_{i+1}}$ is closed.

(b). If P is a point of N, let C_1 be a circle with center at P and radius r so small that N and D contain points exterior to C_1. This is possible unless N is identical with M and in this case our theorem is obvious. Let $D_{11}, D_{12}, D_{13}, \cdots$ be the maximal connected subsets of $D \cdot I(C_1)$.†

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* That this can be done by an *arc*, see J. R. Kline, *Concerning the approachability of simple closed and open curves*, Transactions of this Society, vol. 21 (1920), page 453 and footnote.

† If C is a circle, $I(C)$ denotes the interior of C. If A and B are point sets, $A \cdot B$ denotes the set of points common to A and B.

We shall show that one of these sets, which we will denote by D_1, contains infinitely many of the points P_i. If this is not true, then if C_2 denotes the circle with center at P radius $r/2$, for infinitely many values of i, D_{1i} has a point within C_2 and C_1 contains a limit point of D_{1i}. Thus infinitely many of the sets D_{1i} are of diameter greater than $r/4$. But this contradicts the theorem that if $M+C_1$ and $N+C_1$ are continuous curves and $N+C_1$ is a subset of $M+C_1$ then $M+C_1-(N+C_1)$ cannot contain more than a finite number of maximal connected subsets of diameter greater than $r_1/4$.*

Let n_1 be the smallest integer such that D_1 contains P_{n_1}. Similarly one, D_2, of the maximal connected subsets of $D_1 \cdot I(C_2)$ contains infinitely many of the points P_i. Let n_2 be the smallest integer greater than n_1 such that D_2 contains P_{n_2}. In general let $C_j(j=1, 2, 3, \cdots)$ be a circle with center at P and radius r/j and let D_j be a maximal connected subset of $D_{j-1} \cdot I(C_j)$ which contains infinitely many points of the sequence $[P_i]$. Let n_j be the smallest integer greater than n_{j-1} such that P_{n_j} lies in D_j. For every j, D_j contains an arc $P_{n_j}P_{n+j_1}$.† Since for every j, the arc $P_{n_j}P_{n_{j+1}}$ lies interior to C_j we see easily that the set $P+\sum_{j=1}^{\infty} P_{n_j}P_{n_{j+1}}$ is closed.

B. The condition is sufficient. If N is not a continuous curve there exist‡ two concentric circles K_1 and K_2 and a countable infinity of continua $\overline{N}$, N_1, N_2, N_3, $\cdots$, such that (1) each of these continua belongs to N, contains a point on K_1 and a point on K_2 and is a subset of the set H which is composed of K_1+K_2+I, I denoting the annular domain between K_1 and K_2, (2) no two of these continua have a point in common and, indeed, no one of them except possibly $\overline{N}$ is a proper subset of any connected subset of $N \cdot H$,

* See the abstract of my paper, *Concerning the arcs and domains of a continuous curve*, this Bulletin, vol. 32 (1926), p. 37.

† See R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260.

‡ See *Report*, p. 296.

(3) the set $\overline{N}$ is the sequential limiting set of the sequence of sets $N_1, N_2, N_3, \cdots$. For each i, let A_i and B_i be points of $K_1 \cdot N_i$ and $K_2 \cdot N_i$ respectively. There exist arcs X_1Y_1A and X_2Y_2B of K_1 and K_2 and an increasing sequence of integers $n_1, n_2, n_3, \cdots$, such that X_1Y_1A contains A_{n_i} for every i and in the order $X_1Y_1A_{n_1}A_{n_2}\cdots A$ and X_2Y_2B contains B_{n_i} for every i and in the order $X_2Y_2B_{n_1}B_{n_2}\cdots B$.

Let P denote a point of $\overline{N}$ which lies in I. There exists a circle C_1 with center at P such that C_1, together with its interior, lies in I. Let r_1 be the radius of C_1. Since M is connected im kleinen at P there exists in any circle C_i a concentric circle $\overline{C}_i$ such that every point of M within $\overline{C}_i$ can be joined to P by an arc of M lying wholly within C_i. Let $N_{11} \equiv N_{n_j}$, where j has the smallest value such that N_{n_j} contains a point Q_1 within $\overline{C}_1$. There exists an arc PQ_1 of M lying wholly in C_1. The arc PQ_1 from P to Q_1 contains a first point E_1 in common with N_{11} and the arc E_1P, a subset of Q_1P, has a first point F_1 in common with $\overline{N}$. The set $\{E_1F_1\}$* contains a point P_1 of $M-N$. Let C_2 be a circle with center at P and radius $r_2 \leqq r_1/2$ such that P_1 and N_{11} lie in the exterior of C_2. Let $N_{12} \equiv N_{n_j}$, where j has the smallest value such that N_{n_j} contains a point Q_2 within $\overline{C}_2$. Let us determine a point P_2 of $M-N$ as above. Continue this process indefinitely each time taking C_i with center at P and radius $r_i \leqq r_{i-1}/2$ and such that P_{i-1} and N_{1i-1} lie outside C_i. Thus we obtain an infinite sequence of points $P_1, P_2, P_3, \cdots$, and continua $N_{11}, N_{12}, N_{13}, \cdots$, such that (1) P_i belongs to $M-N$ and lies interior to C_i and thus P is the sequential limit point of the sequence $[P_i]$, (2) $\{E_iF_i\}$ contains P_i, where C_i encloses E_iF_i, and $\{E_iF_i\}$ contains no point of $N_{1i}+\overline{N}$.

Since $M-N$ consists of only a finite number of maximal connected subsets one of these must contain infinitely many of the points $[P_i]$ say $\overline{P}_1, \overline{P}_2, \overline{P}_3, \cdots$. For each i, let D_i be the maximal connected subset of $M+K_1+K_2-(\overline{N}+\overline{N}_{1i}$

* If AB is an arc from A to B then $\{AB\}$ denotes $AB-(A+B)$.

$+K_1+K_2$)* which contains $\overline{P}_i$. We see easily that there exists an integer t_2 such that $\overline{P}_1$ does not lie in D_{t_2}. Then any arc of $M-N$ from $\overline{P}_{t_1}$ ($t_1=1$) to $\overline{P}_{t_2}$† must contain a point of either K_1 or K_2. There exists an integer $t_3>t_2$ such that D_{t_3} does not contain $\overline{P}_{t_2}$. In general there exists an integer $t_i>t_{i-1}$ such that D_{t_i} does not contain $\overline{P}_{t_{i-1}}$ and thus any arc of $M-N$ from $\overline{P}_{t_{i-1}}$ to $\overline{P}_{t_i}$ must contain a point of K_1 or K_2. Let $p_i=\overline{P}_{t_i}$. Then if $k_1, k_2, \cdots$ is any increasing sequence of positive integers, the set $\overline{N}$ must contain a limit point of the set $P+\sum_{i=1}^{\infty} p_{k_i} p_{k_{i+1}}$ which lies on K_1 or K_2 and thus the set cannot be closed. But this set is closed by hypothesis. Thus the condition is sufficient.

THEOREM II. *In order that a bounded plane continuum M be a continuous curve, it is necessary and sufficient that* (1) *for any given positive number ϵ there are not more than a finite number of complementary domains of M of diameter greater than ϵ;* (2) *if $P_1, P_2, P_3, \cdots$ is any sequence of distinct points of a complementary domain D of M which has a sequential limit point P, then there exists an increasing sequence of positive integers, $n_1, n_2, n_3, \cdots$, and a sequence of arcs of D, $P_{n_1}P_{n_2}, P_{n_2}P_{n_3}, P_{n_3}P_{n_4}, \cdots$, such that the set $P+\sum_{i=1}^{\infty} P_{n_i} P_{n_{i+1}}$ is closed.*

PROOF. The necessity of condition (1) has been proved by Schoenflies.‡ The necessity of condition (2) can be proved exactly as in Theorem I since no property of the continuous curve M was used that is not also a property of the entire space. The sufficiency of the conditions is proved as in Theorem I except that the fact that some one complementary domain of M contains infinitely many of the points $P_1, P_2, P_3, \cdots$, which are chosen in the course of the argument, follows from condition (1) rather than the condition *$M-N$ consists of a finite number of maximal connected subsets.*

* If $\overline{P}_i=P_j$, then $\overline{N}_{1i}$ denotes N_{1j}.

† For the proof that such an arc exists, see R. L. Moore, *Concerning continuous curves in the plane*, loc. cit.

‡ See *Report*, pp. 290, 291.

THEOREM III. *If M is a plane continuous curve then M cannot contain, for any positive number ϵ, an infinite number of mutually exclusive continua $M_1, M_2, M_3, \cdots$, such that (1) the diameter of each set M_i is greater than ϵ, (2) $M-M_i$ is closed except for a set K_i and if η is any positive number there exists an integer n_η so that if $i>n_\eta$ then K_i can be enclosed in two circles each of radius less than η.*

PROOF. Suppose that there exists a positive number ϵ and a continuous curve M such that M contains an infinite number of continua $M_1, M_2, M_3, \cdots$, which satisfy restrictions (1) and (2) of the theorem. From condition (2) it follows that we may divide each set K_i into two subsets K_{1i} and K_{2i} such that

$$\lim_{i\to\infty} d(K_{1i})=0 \text{ and } \lim_{i\to\infty} d(K_{2i})=0.^*$$

For each i and j ($i=1, 2, 3, \cdots, j=1, 2$) let A_{ji} be a point of K_{ji}, unless K_{ji} is vacuous. For no value of i can both K_{1i} and K_{2i} be vacuous. Several cases arise here according to the existence or non-existence of the various points A_{ji} but we can see easily that there exist a point A or two points A and B and an increasing sequence of integers $n_1, n_2, n_3, \cdots$, such that either (1) K_{1n_i} is vacuous for each i, and A is the sequential limit point of $[A_{2n_i}]$, (2) K_{2n_i} is vacuous for each i and A is the sequential limit point of $[A_{1n_i}]$, (3) all of the points of the sequences $[A_{1n_i}]$ and $[A_{2n_i}]$ exist and A is the sequential limit point of each sequence, or (4) all of the points of the sequences $[A_{1n_i}]$ and $[A_{2n_i}]$ exist and A and B are the sequential limit points of the sequences $[A_{1n_i}]$ and $[A_{2n_i}]$ respectively ($A\neq B$). For cases (1), (2) and (3), let $t=\epsilon$; for case (4) let $t=d(A, B)$. By condition (2) of the hypothesis of the theorem, there exists an integer k_1 so that if $i>k_1$ then

$$d(K_{1n_i})<t/12 \quad \text{and} \quad d(K_{2n_i})<t/12.$$

* If K is a set of points the notation $d(K)$ denotes the diameter of K. If A and B are two points the notation $d(A, B)$ denotes the distance from A to B.

Also there exists an integer k_2 so that if $i>k_2$ then either

Case (1) $d(A_{2n_i}, A)<t/12$,

or

Case (2) $d(A_{1n_i}, A)<t/12$,

or

Case (3) $d(A_{1n_i}, A)<t/12$ and $d(A_{2n_i}, A)<t/12$,

or

Case (4) $d(A_{1n_i}, A)<t/12$ and $d(A_{2n_i}, B)<t/12$.

In any case if $k_3=k_1+k_2$ and $i>k_3$ then the circle C_1 with center at A and radius $t/6$, or the circles C_1 and C_2 with centers at A and B and radii $t/6$, enclose every point of K_{n_i}. For every $i>k_3$, M_{n_i} contains a point p_i such that $d(A, p_i)>t/3$ and $d(B, p_i)>t/3$ (if B exists). The sequence $M_{n_1}, M_{n_2}, M_{n_3}, \cdots$, contains a subsequence $\overline{M}_1, \overline{M}_2, \overline{M}_3, \cdots$, such that (1) for every i, if $\overline{M}_i=M_{n_j}$ then $j>k_3$, (2) for every i, if $\overline{M}_i=M_{n_j}$ and $\overline{M}_{i+1}=M_{n_m}$ then $j<m$, (3) the points $\bar{p}_1, \bar{p}_2, \bar{p}_3, \cdots$ * have a sequential limit point P. It follows that M contains P, that $d(P, A)\geqq t/3$ and $d(P, B)\geqq t/3$ (if B exists) and that if C_3 is a circle of radius $t/6$ with P as a center then no point of any set $\overline{K}_i$ is within C_3. As M is connected im kleinen at P the circle C_3 encloses a concentric circle C_4 such that every point of M within C_4 can be joined to P by an arc of M which lies entirely in C_3. Let $\bar{p}_s$ be the first point of the sequence $[\bar{p}_i]$ within the circle C_4. There exists an arc α from $\bar{p}_s$ to P which lies wholly in C_3. Let $\alpha_1=\overline{M}_s\cdot\alpha$ and $\alpha_2=\alpha-\alpha_1$. As α is connected one of these sets must contain a limit point of the other. The set $\overline{M}_s$, and consequently α_1, is closed. Then α_1 must contain a limit point q of α_2. As $\overline{M}_s$ contains α_1 and $M-\overline{M}_s$ contains α_2, by definition q must belong to $\overline{K}_s$. But no point of $\overline{K}_s$ is within C_3 while α is entirely within C_3. Thus the supposition that M contains an infinite set of this type has led to a contradiction.

The preceding theorem implies as an immediate corollary the following rather useful result.

* If $\overline{M}_i=M_{n_j}$, then $\overline{P}_i$ denotes p_{n_j}.

THEOREM IV.* *If M is a plane continuous curve then M cannot contain, for any positive number ϵ, an infinite number of arcs of diameter greater than ϵ which are mutually exclusive except possibly for common end-points and such that if α is any one of this set of arcs then $M-\{\alpha\}$ is closed.*

That Theorem I no longer remains true when the condition that "$M-N$ consists of a finite number of maximal connected subsets" is removed, even with the addition of the condition that "for any positive number ϵ, $M-N$ contains only a finite number of maximal connected subsets of diameter greater than ϵ," is shown by the following example. The modified conditions are necessary but not sufficient.

Let N denote the set of points consisting of the intervals from $(1, 0)$ to $(0, 0)$ and from $(0, 1)$ to $(0, 0)$ together with the intervals from $(1, 1/i)$ to $(0, 1/i)$ for every positive integer i. Let M be the set of points consisting of N together with the intervals from $(j/i, 1/i)$ to $(j/i, 0)$ for every positive integer $j<i$ and for every positive integer i. The modified conditions are then satisfied, but N is not a continuous curve.

Theorem III gives a necessary condition that a bounded continuum be a continuous curve. The following example shows that this condition is not sufficient:† Let M be an indecomposable continuum of diameter $\geqq 2\epsilon$ and let $\eta<\epsilon/10$. Now suppose that $M_1, M_2, M_3, \cdots$ is any sequence of mutually exclusive subcontinua of M of diameter greater than ϵ. As each set M_i is a proper subcontinuum of an indecomposable continuum it is a continuum of condensation of M.‡ Thus for each i, $K_i \equiv M_i$. Then no matter how large i is, K_i cannot be enclosed in two circles each of radius less than η. Thus M satisfies the condition but is not a continuous curve.

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* This theorem was presented to the Society October 31, 1925. I am indebted to Dr. H. M. Gehman for the suggestion that this theorem might be generalized. The resulting study led to Theorem III of this paper.

† This example is due to Professor J. R. Kline.

‡ Cf. Z. Janiszewski and C. Kuratowski, *Sur les continus indécomposables*, Fundamenta Mathematicae, vol. 1 (1920), pp. 210–222.

A METHOD FOR ACCELERATING THE CONVERGENCE IN THE PROCESS OF ITERATION*

BY C. C. CAMP

1. *Introduction.* The simplicity and directness of the process of iteration coupled with the fact that errors committed along the way do not vitiate the final result have won for the method a certain degree of widespread popularity. This has become imperiled, however, by the extreme slowness of the convergence in many cases. The object of the present paper is to overcome the difficulty by furnishing a powerful method based on the same kind of analysis as Newton's but better adapted to the process of iteration. The common practice of taking half the sum of two successive approximations, one of which is too large and the other too small, is seen to be inadequate by the following example: $x=2+\pi \sin x$. Here one iterates using the formula $\sin x_2=(x_1-2)/\pi$. For x_1 close to the true root $m_1\equiv dx_2/dx_1=1/(\pi \cos x_2)=-.331$ nearly. The half sum is no improvement here and one finds the same value for the ninth application of the formula as by unmodified iteration, provided one starts with x_1 as the equivalent of 164° in radians. This value is obtained by two applications of the new method. Its equivalent in degrees is 164.05131, and is correct to five decimals.

By ordinary iteration it requires about a score of approximations to solve $6k+10e^{-k}=10$† correct to six decimals. The result, $k=1.126261$, is found by two applications of the new method by starting with $k_1=1.1$, obtained graphically. Here $m_1=.548$. One can use the half sum only when m is negative; and if m is near $-1/3$, no improvement results. The present method consists of one iteration and one application of formula (8), which may be regarded as a formula of interpolation

* Presented to the Society, April 2, 1926.

† See Phillips, *Differential Equations*, p. 15, ex. 5.

or of extrapolation, according to the sign of m. In §2 the method is derived analytically and interpreted geometrically. The rapidity of the convergence is considered in §3, and the last three sections give modifications and extensions of the new method. It is obvious that one may also extend it so as to apply to functions which are merely tabulated by replacing m by a properly corrected first difference.

2. *Derivation and Interpretation of the New Method.* Consider the equation

$$f(x) = 0, \tag{1}$$

in which f is a real single-valued function of a real variable x, continuous together with its first and second derivatives in a properly chosen neighborhood of each root of (1). Let f be further restricted so that $f'(x_0) \neq 0$, $f''(x_0) \neq 0$, when $f(x_0) = 0$. This implies that f possesses only a finite number of maxima and minima in the neighborhood of a zero. Let f, or a modification of f possessing the same desired zero, be decomposed into $f_1(x) - f_2(x)$, so that

$$|f_1'(x_i)| > |f_2'(x_j)| \tag{2}$$

in one of these neighborhoods

$$x_0 - \delta \leqq x \leqq x_0 + \delta, \tag{3}$$

where δ is sufficiently small. That this is always possible is obvious from the following choice

$$f_1(x) \equiv x, \qquad f_2(x) \equiv x - \frac{f(x)}{M(x)}, \tag{4}$$

where $M(x)$ is a function which equals $f'(x)$ at a root of (1) and possesses a continuous second derivative. By continuity $M(x)$ will not vanish in some neighborhood (3). If $|f_1'(x_0)| > |f_2'(x_0)|$, then by continuity there will be a neighborhood about x_0 in which the minimum of the first is greater than the maximum of the second. Condition (2) will therefore be satisfied if it holds for the one point $x_0 = x_i = x_j$. This is evidently true for the choice (4).

By Maclaurin's Theorem we have

$$(5)\qquad \begin{cases} f_1(x) = f_1(x_2) + f_1'(x_2)\Delta x_2 + f_1''(x_2 + \theta_2\Delta x_2)\dfrac{\Delta x_2^2}{2}, \\ f_2(x) = f_2(x_1) + f_2'(x_1)\Delta x_1 + f_2''(x_1 + \theta_1\Delta x_1)\dfrac{\Delta x_1^2}{2}, \end{cases}$$

where $\Delta x_1 = x - x_1$, $\Delta x_2 = x - x_2$, $0 < \theta_1 < 1$, $0 < \theta_2 < 1$.

If we neglect infinitesimals of the first order we obtain

$$(6)\qquad f_1(x_2) = f_2(x_1),$$

the approximate relation used in the process of iteration. If instead of (5) we use the Law of the Mean in connection with (6), we get

$$(7)\qquad x - x_2 = \frac{f_2'(x_1 + \theta_1'\Delta x_1)}{f_1'(x_2 + \theta_2'\Delta x_2)}(x - x_1),$$

where $0 < \theta_1' < 1$, $0 < \theta_2' < 1$. By assumption (2) this shows that the error in the second approximation is less than that in the first, provided x_1 lies in the neighborhood (3). Clearly x_2 will then lie in the same interval.

If we neglect only infinitesimals of order two we have from (5) and (6), upon solving for x and denoting this new approximation by $\bar{x}_2$,

$$(8)\qquad \bar{x}_2 = x_1 + \frac{x_2 - x_1}{1 - m},$$

where

$$(9)\qquad m = f_2'(x_1)/f_1'(x_2).$$

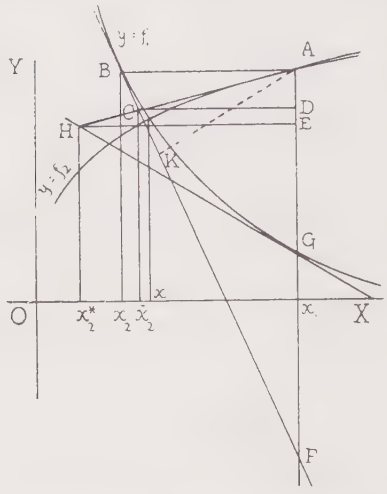
If we use (5) unchanged in connection with (6) and (8) we get

$$(10)\qquad x - \bar{x}_2 = \frac{\frac{1}{2}f_1''(x_2 + \theta_2\Delta x_2)\Delta x_2^2 - \frac{1}{2}f_2''(x_1 + \theta_1\Delta x_1)\Delta x_1^2}{f_1'(x_2) - f_2'(x_1)}.$$

By calling $\bar{m}$ the coefficient of $x - x_1$ in (7) one may put this in the form

$$(11)\qquad x - \bar{x}_2 = \tfrac{1}{2}\Delta x_1^2\left[\frac{\bar{m}^2 f_1''(x_2') - f_2''(x_1')}{f_1'(x_2) - f_2'(x_1)}\right],$$

where x_2' and x_1' are arguments in the numerator of (10). Since the quantities in brackets are bounded above and below, it is clear that the approximation given by formula (8) will be closer than both x_1 and x_2 when Δx_1 is sufficiently small.



If we clear (8) of fractions and simplify we obtain

$$(x_1 - \bar{x}_2)\left[f_1'(x_2) - f_2'(x_1)\right] = f_1'(x_2)(x_1 - x_2),$$

or, in the figure, $DF - DA = AF$.

Newton's method may be illustrated by using the tangents at A and G. His form gives

$$(x_1 - x_2^{+})\left[f_1'(x_1) - f_2'(x_1)\right] = f_1(x_1) - f_2(x_1),$$

or $EG - EA = AG$.

The new method may be modified by drawing at A a secant line with slope equal to $f_2'(x_2)$ intersecting BF at K. The value of m is thus replaced by $f_2'(x_2)/f_1'(x_2)$ and the interpretation gives $(x_1 - \underline{x})[f_{21}'(x_2) - f_2'(x_2)] = f_1'(x_2)(x_1 - x_2)$ or $NF - NA = AF$, where $\underline{x}_2$ is the abscissa of K, and N is the intersection of AF and a horizontal line through K.

3. *Rapidity of the Convergence of Ordinary Iteration and of the New Method.* For ordinary iteration one has from (7), $\Delta x_2 = \bar{m}\Delta x_1$. An upper bound for $|\bar{m}|$ may be found by replacing the numerator by an upper bound $|\bar{f}_2'|$ and the denominator by a lower bound $|\underline{f}_1'|$ in the interval $x_0 - |\Delta x_1| \leqq x \leqq x_0 + |\Delta x_1|$. If the resulting quotient is called M, one has a method of determining an upper bound for the number of successive approximations required for attaining a given degree of accuracy by pure iteration. For instance, if the error allowed is 10^{-k}, since obviously $|\Delta x_3| \leqq M^2 |\Delta x_1|$ and in general $|\Delta x_i| \leqq M^{i-1} |\Delta x_1|$, we need simply to determine $i - 1$

from the relation $M^{i-1}\,|\,\Delta x_1\,|=10^{-k}$. Hence the number sought is

$$n = i - 1 = (-k - \log_{10}|\Delta x_1|)/\log_{10} M. \tag{12}$$

Likewise one may obtain a lower bound for the number by replacing M by a lower bound for $|\bar{m}|$. $|\Delta x_1|$ may be replaced by a lower bound in the latter case and by an upper bound in (12).

From (11) one can estimate the errors made by using (8) in place of (5). Since the quantities in brackets are bounded above and below, evidently

$$|x - \bar{x}_2| < k\Delta x_1^2, \tag{13}$$

where k is a constant.

By repeated applications the sequence of values obtained from

$$\begin{cases} \bar{x}_k = x_{k-1} + \dfrac{x_k - x_{k-1}}{1 - m}, \\ f_1(x_k) = f_2(x_{k-1}), \qquad (k = 2,3,4, \cdots) \end{cases} \tag{14}$$

where $m = f_2'(x_{k-1})/f_1'(x_k)$, will have errors which are not greater than

$$\rho\epsilon^2,\ \rho\epsilon^4,\ \rho\epsilon^8, \cdots, \quad \text{where} \qquad \rho\epsilon = |\Delta x_1|, \qquad \rho = \frac{1}{k}.$$

In Newton's method one has similarly

$$x - x_2' = \tfrac{1}{2}\Delta x_1^2\left[\frac{f_1''(x_1 + \theta_3\Delta x_1) - f_2''(x_1 + \theta_4\Delta x_1)}{f_1'(x_1) - f_2'(x_1)}\right], \tag{15}$$

$$0 < \theta_3 < 1, \qquad 0 < \theta_4 < 1.$$

The two methods coincide when $f_1(x)$ is a linear function. But the new method is more powerful than Newton's in the following extensive cases:

Case I. When $f_1(x)$ is non-linear, $f_1''(x)$ is of constant sign and $f_2(x)$ is linear in the interval

$$x_0 - |\Delta x_1| \leqq x \leqq x_0 + |\Delta x_1|\ ; \tag{16}$$

Case II. When $f_1''(x)$, $f_2''(x)$ are of opposite sign and $f_1'(x)$, $f_2'(x)$ are each of constant sign in some interval (16).

We choose Δx_1 small enough so that an application of Newton's method will give x_2^+ a better value than x_1. It is sufficient* if x_1 is so chosen that

$$\left| \frac{f_1(x_1) - f_2(x_1)}{f_1'(x_1) - f_2'(x_1)} \right| < \frac{A}{2B},$$

where A is the minimum of $|f_1'(x) - f_2'(x)|$ and B is the maximum of $|f_1''(x) - f_2''(x)|$ in the interval between x_1 and

$$x_1 - \frac{2[f_1(x_1) - f_2(x_1)]}{f_1'(x_1) - f_2'(x_1)}.$$

Both cases may be proved geometrically. In Case I, since $|f_1'(x)| > |f_2'(x)|$, evidently f_1 will be monotonic and have a continuously turning tangent in view of the fact that f_1'' is of constant sign. The same is true for $f_1(x)$ and $f_2(x)$ in Case II. In either case let these curves intersect at C whose abscissa is x_0 in the restricted interval. Draw tangents at A on f_1 and at B on f_2 for $x = x_1$. Call the intersection of the tangents D. A horizontal line through B cuts f_1 at E, at which a tangent is drawn intersecting BD at F. The tangent at any point of arc AC will cut BD in a point of the line segment DG where G is the point on BD with the same abscissa as C. The abscissa $\bar{x}_2$ of F will therefore be between x^+ and x_0 the true value of the root.

As an illustration of Case I let us take the famous Wallis equation, namely $x^3 = 2x + 5$. Here if $x_1 = 2$, $m_1 = .1541$. By Barlow's Tables $1/(1 - m_1) = 1.182$, $x_2 = 2.08$, $\bar{x}_2 = 2.09456$. By iteration, $x_3 = 2.094552776$, and $\bar{x}_3 = 2.094551483$, where $1/(1 - m_2) = 1.179$. The result of two applications of the new method thus furnishes nine digits correct.† The same number is furnished by Newton's method with three applications. Iteration requires six operations to get six digits correct.

* Cf. Cauchy, Œuvres Complètes, (2), vol 4 (1899), p. 576, Theorem III.

† Cf. Whittaker and Robinson, *The Calculus of Observations*, p. 86.

4. *Modifications of the Method.* Just as one may modify Newton's method by various substitutions for the derivative, so it is expedient at times to change the value of m. One may determine approximately by sketching the curves whether a secant line drawn at $x=x_1$ on f_2 with a slope $f_2'(x_2)$ will tend to accelerate the convergence still further in the early terms of the sequence. With certain combinations of f_1 and f_2 this also simplifies the form of m and is very effective in diminishing the arithmetic. Formula (9) then becomes

$$m = f_2'(x_2)/f_1'(x_2), \tag{17}$$

and both advantages are shown in the following example: $4x^2=x^3+5$. Here $m=3x_2/8$. If we take $x_1=1.4$ then $x_2=1.391402$, and by the modified form $\underset{\sim}{x}_2=1.38202$ whereas $\bar{x}_2=1.3817745$ by (8), (9). Then by iteration from $\underset{\sim}{x}_2$ we have $x_3=1.381994$ and by (17) $\underset{\sim}{x}_3=1.38196603$. This is correct to eight figures.

One may make small errors either accidentally, by choosing simple values for $1/(1-m)$ or by using the same value of m more than once. The final result is correct as illustrated by the problem, $y=.5-\log_{10} y$. By inspection the initial value found from a table was $y_1=.6675$; m was taken as $-.65$ and by error y_2 was found to be .6756. Accordingly $\bar{y}_2=y_1+.606\times.0081=.6724$. By (6) $y_3=.6723723$ and by using .606 again $\bar{y}_3=.6723832$ which checks to seven digits.

If the f's have simple derivatives up to the third order, it may be desirable to use a formula which neglects infinitesimals of only the third order or higher. One needs merely to extend equations (5) to one more term each and then keep terms of the second order. It is easy to derive the new approximation by using (6) and solving a quadratic in x. If we designate the new approximation by $\tilde{x}_2$, we have

$$\begin{aligned}\tilde{x}_2(f_1'' - f_2'') = {} & f_2' - f_1' + x_2 f_1'' - x_1 f_2'' \\ & + [(f_1' - f_2')^2 + 2(x_2 - x_1)Q]^{1/2},\end{aligned} \tag{18}$$

where $Q\equiv(f_2'f_1''-f_1'f_2'')+\frac{1}{2}(x_2-x_1)f_1''f_2''$, where f_1', f_1'' have the argument x_2, and where f_2', f_2'' have the argument x_1. One

application each solves $4x^2=x^3+5$ and the Wallis equation with errors beyond the sixth decimal place, provided the first approximations are 1.4 and 2, respectively.

5. *Extensions to Systems of Equations.* The method may be extended to a system of n equations in n unknowns. Let us consider for simplicity the case $n=2$, and let the equations to be solved be

$$f(x,y) = 0, \qquad g(x,y) = 0. \tag{19}$$

Assume that the slopes of these curves at a common point (x_0, y_0) are different and that in the neighborhood of each intersection the ordinate of each curve represents a single-valued function of x possessing continuous derivatives up to the second order. At a solution (x_0, y_0), the Jacobian does not vanish, i. e.,

$$J = \begin{vmatrix} f_x & g_x \\ f_y & g_y \end{vmatrix} \neq 0. \tag{20}$$

Let f be broken up into $f_1(x, y)-f_2(x, y)$ and g similarly into $g_1(x, y)-g_2(x, y)$ so that at the solution sought

$$\left|\frac{\partial f_1}{\partial y}\right| > \left|\frac{\partial f_2}{\partial y}\right| \quad \text{and} \quad \left|\frac{\partial g_1}{\partial x}\right| > \left|\frac{\partial g_2}{\partial x}\right|, \tag{21}$$

also, if possible so that

$$\begin{cases} |f_x| + |f_{2y}| < r|f_{1y}| \\ |g_y| + |g_{2x}| < r|g_{1x}| \end{cases}, \tag{22}$$

where $0<r<1$, and the subscripts x and y denote partial differentiations. Condition (22) is sufficient to insure the convergence of the sequence of iterated values obtained from the system

$$f_1(x_1, y_2) = f_2(x_1, y_1), \tag{23}$$

$$g_1(x_2, y_1) = g_2(x_1, y_1), \tag{24}$$

since on account of continuity these inequalities will still hold if we keep x and y in sufficiently small intervals about x_0 and

y_0, respectively. To prove this it is sufficient to expand the f's and g's of x and y by Taylor's theorem about the points indicated by the arguments in (23), (24) and then employ these equations. The errors in the sequences of values for x and y will then have zero for their limit.

By extending the expansions to one more term each and then retaining only infinitesimals of order two, one obtains readily the analog of (8), namely

$$(25)\qquad \begin{cases} \bar{x}_2 = x_1 + \dfrac{1}{D}\begin{vmatrix} (y_2 - y_1)f_{1y}, & f_{1y} - f_{2y} \\ (x_2 - x_1)g_{1x}, & g_{1y} - g_{2y} \end{vmatrix} \\ \bar{y}_2 = y_1 + \dfrac{1}{D}\begin{vmatrix} f_{1x} - f_{2x}, & (y_2 - y_1)f_{1y} \\ g_{1x} - g_{2x}, & (x_2 - x_1)g_{1x} \end{vmatrix} \end{cases}$$

where D is the form taken by J when f, g are separated into parts and the arguments are changed to agree with (23), (24). These same arguments are to be understood in the partial derivatives of (25). It is clear that D will also be different from zero for a sufficiently small neighborhood of (x_0, y_0). It can be seen that if (x_1, y_1) is in such a neighborhood then (x_2, y_2) will be also, on account of (22). When $|x_0 - x_1|$, $|y_0 - y_1|$ are sufficiently small, $(\bar{x}_2, \bar{y}_2)$ will also be inside this neighborhood and the sequence of values (x_k, y_k), $(\bar{x}_k, \bar{y}_k)$ found by repeating the iterations

$$(26)\qquad \begin{cases} f_1(x_k, y_{k+1}) = f_2(x_k, y_k), \\ g_1(x_{k+1}, y_k) = g_2(x_k, y_k), \end{cases}$$

and applying (25) successively will converge to the solution (x_0, y_0).

It is obvious how the formulas would be written for the case of a greater number of variables n.

In case the inequalities (22) do not hold for a given system it is easy to show that (19) can always be replaced by a new system

$$(19')\qquad \begin{cases} F(x, y) = 0, \\ G(x, y) = 0, \end{cases}$$

which will satisfy (21), (22), when rewritten in the form $F \equiv F_1 - F_2$, $G \equiv G_1 - G_2$. The new system will have the same solution as (19) if (20) is satisfied, provided (x, y) is confined to a sufficiently small region about (x_0, y_0). It is sufficient to choose

$$G_1 \equiv x, \quad G_2 \equiv x - \frac{1}{J}\begin{vmatrix} f(x,y), & g(x,y) \\ f_y(x,y), & g_y(x,y) \end{vmatrix}, \tag{27}$$

$$F_1 \equiv y, \quad F_2 \equiv y - \frac{1}{J}\begin{vmatrix} f_x(x,y), & g_x(x,y) \\ f(x,y), & g(x,y) \end{vmatrix}. \tag{28}$$

The amount of arithmetic work may be shorter for this change in the equations to be solved than for more direct methods by which (19) can still be solved without (22) being satisfied.

Geometrically one may solve a set (19) by the theory of the method for one variable, provided $J \neq 0$, i. e., the curves (19) intersect without being tangent to each other. We assume as before that the functions defined by the graphs of (19) possess continuous second derivatives. By a proper choice of f's and g's suppose (21) is satisfied and that at (x_0, y_0) the slope of f is numerically less than that of the curve g. One may proceed as follows:

First Method. After solving by a table, by interpolation, or graphically for an approximate solution (x_1, y_1) of (19), one tests the value of y_1 by trial in (23) and then solves (24) for x_2. Next comes an application of the formula

$$\bar{x}_2 = x_1 + \frac{x_2 - x_1}{1 - m_x}, \tag{29}$$

where $m_x \equiv g_{2x}(x_1, y_1)/g_{1x}(x_2, y_1)$. Equation (23) is then solved in which x_1 is replaced by $\bar{x}_2$ and y_1 by y_1 as corrected above. One then employs the formula

$$\bar{y}_2 = y_1 + \frac{y_2 - y_1}{1 - m_y}, \tag{30}$$

where $m_y \equiv f_{2y}(\bar{x}_2, y_1)/f_{1y}(\bar{x}_2, y_2)$. The process is repeated until a set (x_k, y_k) is the same as (x_{k-1}, y_{k-1}).

Second Method. This is the same before the application of (29), in place of which one uses

$$(31)\quad \bar{x}_2 = x_1 + \frac{x_2 - x_1}{1 - m}, \quad \text{where } m \equiv \frac{\text{slope of } f \text{ at } (x_1, y_1)}{\text{slope of } g \text{ at } (x_2, y_1)};$$

$$(32)\quad m = \frac{(f_{1x} - f_{2x})(g_{1y} - g_{2y})}{(f_{1y} - f_{2y})(g_{1x} - g_{2x})},$$

wherein the arguments are the same as in (23), (24). The process is repeated as in the first method until the required accuracy is reached. The following example illustrates the method:

$$f(x, y) \equiv 5y^3 + x^2 - 2xy - 4 = 0,$$
$$g(x, y) \equiv x^3 + 2y^2 - 1 = 0.$$

Take $f_1 \equiv 5y_2^3$, $f_2 \equiv 2x_1y_1 - x_1^2 + 4$, $g_1 \equiv 1 - x_2^3$, $g_2 \equiv 2y_1^2$. For a first approximation start with $x_1 = -.65$, $y_1 = .8$. The second of conditions (22) is not satisfied and it is easily seen that (25) will give a diverging sequence. Nevertheless we can still obtain a solution as follows:

The value y_1 improved by trial gives .7977; x_2 from (24) $= -.64844$. Now $m = -.675$, hence from (31) $\bar{x}_2 = -.6491$. From (23) $y_2 = .798235$. In (30) $m_y = -.1358$ and $\bar{y}_2 = .79817$; $x_3 = -.649658$ from (24); and from (31) if we use $m = -.6725$ then $\bar{x}_3 = -.649434$; $y_3 = .798070$, $\bar{y}_3 = .798082$ if we take $m_y = -.136$; $x_4 = -.6494036$, $\bar{x}_4 = -.649416$ where m is not recalculated. Finally $y_4 = .7980875$ and $\bar{y}_4 = .798087$, where m_y is kept as $-.136$. The results are correct to six decimals.

6. *Extension to the Determination of Complex Roots.* Just as Newton's method remains valid, so the method of iteration is applicable to the computation of complex roots. Moreover, the latter method is simpler and in case the modulus of m is sufficiently small it has the usual advantages. However, when the convergence is slow it is imperative to have a method which leads more rapidly to the root. That the present method will accelerate the convergence as in the case of real roots is evident since Taylor's Theorem goes over with a slight change to

expansions in complex numbers. The arithmetic of iteration is easier than for Newton's method and the application of formula (8) is easier after one has found a few approximations than a change to Newton's method at that stage of the calculation.

The method will be illustrated by an example already solved by Cauchy* although he has erroneously stated that his final result, (163), p. 490, is correct up to the seventh decimal. If we write his equation, $e^x - x = 0$, in the form $x_2 = \log x_1$ then near the root sought $|m|$ is near .7275. Hence pure iteration gives a slowly converging sequence, which will approach the root spirally from $x_1 = i$, as follows:

i, $1.5708i$, $.45158+1.5708i$, $.49129+1.29086i$, $.32295+1.20713i$, $.22281+1.3094i$, $.28384+1.40225i$, $.35807+1.3715i$, $\cdots$.

If after the third approximation is computed formula (8) is employed, the fourth application of the new method gives the root $\bar{x}_7$ as $.3181315+1.3372357i$, which is less than Cauchy's result by 2 in the seventh decimal of the real part. That the value here given is correct and that Cauchy's is incorrect has been verified by the use of Peter's Zehnstellige Logarithmen, by which the calculation has also been extended to nine decimals. Cauchy used five applications of Newton's method starting with $x_1 = 0$, whereas the new method by the same number of applications furnishes nine decimals as follows:

$x_1 = i$; $x_2 = 1.5708i$, $\bar{x}_2 = .2854 + 1.2854i$;
$x_3 = .2751 + 1.3523i$, $\bar{x}_3 = .3310 + 1.3376i$;
$x_4 = .3206 + 1.3282i$, $\bar{x}_4 = .318165 + 1.337276i$;
$x_5 = .31816565 + 1.3371892i$, $\bar{x}_5 = .3181315 + .3372358i$;
$x_6 = .318131574 + 1.3372357215i$, $\bar{x}_6 = .318131505 + 1.337235701i$.

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* Œuvres, IIe Série, Tome IV, p. 485.

A QUADRATIC ALGEBRA AND ITS APPLICATION TO A PROBLEM IN DIOPHANTINE ANALYSIS*

BY G. E. WAHLIN

1. *Introduction.* Dickson† has studied the problem of the application of algebras in the determination of the solutions of certain Diophantine equations. In the following pages I shall define a system which is a generalization of those defined by Clifford‡ and Lipschitz.§

The algebra as here defined is sufficient for the application which I wish to make. By further restrictions on the basal elements we can define an algebra which is intimately connected with the theory of quadratic forms. This I hope to show in a later paper.

2. *Quadratic Extensions of a Field.* We shall assume a certain fundamental field K on which we shall construct an algebra by the adjunction of certain new elements. Let us first consider a quadratic equation

$$x^2 - ax + b = 0, \tag{1}$$

whose coefficients are numbers of K. We shall denote by r a root of this equation. This root r is so defined that $r^2 - ar + b = 0$ but it does not satisfy a linear equation with coefficients in K, even in the case when the equation (1) is reducible in this field. If $b = 0$, a simple linear transformation will transform (1) into an equation in which the last term is not zero and we shall therefore assume that all the quadratic equations with which we are concerned are such that the term which does not involve x is different from zero.

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* Presented to the Society, April 10, 1925.

† *Algebras and their Arithmetics*, University of Chicago Press.

‡ American Journal, vol. 1, p. 350.

§ Comptes Rendus, vol. 91. pp. 619–660.
Bulletin de la Société de France, (2), vol. 11, p. 115.

The multiplication of r by any number of K shall be commutative and in any product of r and numbers of K multiplication shall be associative. We shall also assume the distributive law.

By a quadratic extension of K we mean the set of all elements of the form $mr+n$ where m and n are numbers of K.

Since $r^2=ar-b$ any power of r is equal to a linear function of r and hence it is easily seen that the sum, difference or product of any two elements of the quadratic extension of K belong to the same. Moreover, since r does not satisfy a linear equation in K, $mr+n$ is zero when and only when m and n are both zero.

In the case when the equation (1) is irreducible the quadratic extension of K is closed also with respect to division and is a field which is isomorphic with the ordinary algebraic number field defined by the equation. We shall call this a quadratic extension of the first type.

When the equation (1) is reducible in K we may write $x^2-ax+b=(x-\rho_1)(x-\rho_2)$. Due to the assumption of the commutative and distributive laws we see that this equality is true when we replace x by r and hence

$$(r-\rho_1)(r-\rho_2) = r^2 - ar + b = 0.$$

But $r-\rho_1 \neq 0$ and $r-\rho_2 \neq 0$ since r does not satisfy an equation of the first degree. Hence $r-\rho_1$ and $r-\rho_2$ are proper divisors of zero and a unique and universal division process is not possible. We shall call this a quadratic extension of the second type.

It is easily seen that in either of the two types of quadratic extensions the multiplication is associative and commutative.

The element r as above defined cannot satisfy more than one quadratic equation in K, for if we have $r^2-ar+b=0$, and also $r^2-a'r+b'=0$, we find $(a'-a)r-(b'-b)=0$, which is impossible unless $a'=a$ and $b'=b$.

3. *Conjugate Roots.* Let us next consider the element $a-r$ and substitute it for x in (1). This gives

$$(a - r)^2 - a(a - r) + b = a^2 - 2ar + r^2 - a^2 + ar + b$$
$$= r^2 - ar + b = 0.$$

Hence $a-r$ is also a solution of (1). We shall write $r'=a-r$ and speak of r and r' as a pair of conjugate roots of (1).

From the definition of r' it follows that $r+r'=a$ and $rr'=r'r=(a-r)r=ar-r^2=b$. The sum $S(r)=r+r'$ is called the *trace* of r, and the product $rr'=N(r)$ is called the *norm* of r.

4. *An Algebra of Order Four.* Let us consider a ternary quadratic form

$$Q(x, y_1, y_2) = x^2 + b_1y_1^2 + b_2y_2^2 + a_1xy_1 + a_2xy_2 + cy_1y_2,$$

and let us suppose that r_1 and r_2 are a pair of elements such that with another, associated, pair of elements r_1' and r_2' and a properly defined multiplication

$$(2) \quad \begin{cases} Q(x, y_1, y_2) = (x + r_1y_1 + r_2y_2)(x + r_1' y_1 + r_2' y_2) \\ \qquad = x^2 + r_1r_1' y_1^2 + r_2r_2' y_2^2 + (r_1 + r_1')xy_1 \\ + (r_2+r_2')xy_2 + (r_1r_2' + r_2r_1')y_1y_2. \end{cases}$$

Then

$$(3) \quad r_1 + r_1' = a_1, \ r_2 + r_2' = a_2, \ r_1r_1' = b_1, \ r_2r_2' = b_2,$$
$$r_1r_2' + r_2r_1' = c.$$

We shall assume that the coefficients a_1, a_2, b_1, b_2, c belong to the field K and that each of the elements r_1 and r_2 has a multiplication with the elements of K subject to the assumptions of the preceding sections.

From (2) we have

$$Q(x, -1, 0) = x^2 - a_1x + b_1 = (x - r_1)(x - r_1')$$

and hence r_1 and r_1' are a pair of conjugate roots of the equation $Q(x, -1, 0)=0$. In the same way, r_2 and r_2' are a pair of conjugate roots of $Q(x, 0, -1)=0$, and r_1+r_2 and $r_1'+r_2'$

a pair of conjugate roots of $Q(x, -1, -1)=0$. Hence we have the following relations:

$$r_1^2 = a_1 r_1 - b_1, \tag{4}$$

$$r_2^2 = a_2 r_2 - b_2, \tag{5}$$

$$(r_1 + r_2)^2 = (a_1 + a_2)(r_1 + r_2) - b_1 - c - b_2, \tag{6}$$

and (6) can be written in the form

$$r_1^2 + r_1 r_2 + r_2 r_1 + r_2^2 = a_1 r_1 + a_2 r_1 - a_1 r_2 + a_2 r_2 - b_1 - c - b_2,$$

which by (4) and (5) reduces to

$$r_1 r_2 + r_2 r_1 = a_2 r_1 + a_1 r_2 - c. \tag{7}$$

Let us now consider two elements $u_0+u_1 r_1+u_2 r_2$ and $u_0+u_1 r_1' +u_2 r_2'$. Their sum is $2u_0+u_1 a_1+u_2 a_2$ and their product is $Q(u_0, u_1, u_2)$ and hence the u_0, u_1, u_2 being numbers of K we see that the sum and product are numbers of K and hence the two elements as given are a pair of conjugate roots of a quadratic equation in K.

We shall now form a new extension of K by the adjunction of the two elements r_1 and r_2 subject to the above specified conditions and further assume that $r_1 r_2$ and $r_2 r_1$ are also roots of quadratic equations. We also suppose that

$$S(r_1 r_2 + r_2 r_1) = S(r_1 r_2) + S(r_2 r_1). \tag{8}$$

Let $q(x)=0$ be the quadratic equation of which $r_1 r_2$ is a root. Using the associative law, which we shall also assume, we have

$$\frac{1}{r_1}(r_1 r_2) r_1 = r_2 r_1,$$

$$\frac{1}{r_1}(r_1 r_2)^2 r_1 = \frac{1}{r_1}(r_1 r_2) r_1 \frac{1}{r_1}(r_1 r_2) r_1 = (r_2 r_1)(r_2 r_1) = (r_2 r_1)^2;$$

and hence

$$\frac{1}{r_1} q(r_1 r_2) r_1 = q(r_2 r_1).$$

Since $q(r_1 r_2) = 0$, it follows that $q(r_2 r_1) = 0$ and $r_1 r_2$ and $r_2 r_1$ are roots of the same equation, and $S(r_1 r_2) = S(r_2 r_1)$. As above, we here assume that b_1 and b_2 are not zero. It would suffice to make this assumption regarding only one of them. Using (7), and (8), we then have

$$S(r_1 r_2) + S(r_2 r_1) = 2S(r_1 r_2) = a_2 S(r_1) + a_1 S(r_2) - 2c$$
$$= 2a_1 a_2 - 2c.$$

Thus

$$S(r_1 r_2) = a_1 a_2 - c = a_{12}, \tag{9}$$

where for the present we use a_{12} to denote the trace of $r_1 r_2$. The conjugate root $(r_1 r_2)'$ of $r_1 r_2$ is then $a_{12} - r_1 r_2$.

If we now use the equations (3), we have

$$a_{12} = a_1 a_2 - r_1 r_2' - r_2 r_1' = a_1 a_2 - r_1(a_2 - r_2) - (a_2 - r_2')r_1'$$
$$= a_1 a_2 - a_2(r_1 + r_1') + r_1 r_2 + r_2' r_1'$$
$$= a_1 a_2 - a_1 a_2 + r_1 r_2 + r_2' r_1'.$$

Therefore $r_2' r_1' = a_{12} - r_1 r_2$, and we conclude that $r_2' r_1'$ is the conjugate of $r_1 r_2$. We shall denote $r_1 r_2$ by r_{12} and $r_2' r_1'$ by r_{12}'. We now have the following relations:

$$r_1 r_{12} = r_1 r_1 r_2 = r_1^2 r_2 = a_1 r_1 r_2 - b_1 r_2 = a_1 r_{12} - b_1 r_2, \tag{10}$$

$$r_{12} r_2 = r_1 r_2 r_2 = r_1 r_2^2 = a_2 r_1 r_2 - b_2 r_1 = a_2 r_{12} - b_2 r_1, \tag{11}$$

$$r_{12} r_1 = r_1(r_2 r_1) = r_1(a_2 r_1 + a_1 r_2 - c - r_1 r_2) \tag{12}$$
$$= a_2(a_1 r_1 - b_1) + a_1 r_{12} - c r_1 - a_1 r_{12} + b_1 r_2$$
$$= - a_2 b_1 + a_{12} r_1 + b_1 r_2,$$

$$r_2 r_{12} = (r_2 r_1) r_2 = (a_2 r_1 + a_1 r_2 - c - r_1 r_2) r_2 \tag{13}$$
$$= a_2 r_{12} + a_1 a_2 r_2 - a_1 b_2 - c r_2 - a_2 r_{12} + b_2 r_1$$
$$= - a_1 b_2 + b_2 r_1 + a_{12} r_2.$$

The equations (4), (5), (10), (11), (12), (13) can be summed up in the following multiplication table.

	1	r_1	r_2	r_{12}
1	1	r_1	r_2	r_{12}
r_1	r_1	$a_1r_1 - b_1$	r_{12}	$-b_1r_2 + a_1r_{12}$
r_2	r_2	$-c + a_2r_1 + a_1r_2 - r_{12}$	$a_2r_2 - b_2$	$-a_1b_2 + b_2r_1 + a_{12}r_2$
r_{12}	r_{12}	$-a_2b_1 + a_{12}r_1 + b_1r_2$	$-b_2r_1 + a_2r_{12}$	$a_{12}r_{12} - b_1b_2$

Let us now consider the extension of K consisting of all elements of the form $x_0+x_1r_1+x_2r_2+x_{12}r_{12}$, where x_0, x_1, x_2, and x_{12} are numbers of K. We have seen that r_1r_2 and $r_2'r_1'$ are a pair of conjugate roots of a quadratic equation. Let us next consider r_1r_{12} and $r_{12}'r_1'$. Their product is $b_1^2b_2$, a number of K. From the multiplication table

$$r_1r_{12} = -b_1r_2 + a_1r_{12},$$

$$\begin{aligned} r_{12}'r_1' &= (a_{12} - r_{12})(a_1 - r_1) = a_1a_{12} - a_{12}r_1 \\ &\qquad - a_1r_{12} + r_{12}r_1 \\ &= a_1a_{12} - a_{12}r_1 - a_1r_{12} - a_2b_1 + a_{12}r_1 + b_1r_2 \\ &= a_1a_{12} - a_2b_1 + b_1r_2 - a_1r_{12}, \end{aligned}$$

whence

$$r_{12}'r_1' = -b_1r_2' + a_1r_{12}',$$

and

$$r_1r_{12} + r_{12}'r_1' = a_1a_{12} - a_2b_1,$$

a number of K. Hence as before r_1r_{12} and $r_{12}'r_1'$ are a pair of conjugate roots of a quadratic equation with coefficients in K, and $S(r_1r_{12}) = a_1a_{12} - a_2b_1$. In the same way we can show that $S(r_{12}r_1) = a_1a_{12} - a_2b_1$ and $S(r_2r_{12}) = S(r_{12}r_2) = a_2a_{12} - b_2a_1$ and in all cases the conjugate of the product is the product of the conjugates in reversed order. Hence

$$\begin{aligned} -a_2b_1 &= S(r_{12}r_1) - S(r_1)S(r_{12}), \\ -a_1b_2 &= S(r_{12}r_2) - S(r_2)S(r_{12}). \end{aligned} \tag{14}$$

From the multiplication table

$$r_1r_{12} + r_{12}r_1 = -a_2b_1 + a_{12}r_1 + a_1r_{12}, \tag{15}$$

$$r_2r_{12} + r_{12}r_2 = -a_1b_2 + a_{12}r_2 + a_2r_{12}. \tag{16}$$

By (14) and the equation $-c=a_{12}-a_1a_2$, we can then write a common form for (7), (15), (16) as follows:

$$r_ir_j + r_jr_i = S(r_ir_j) - S(r_i)S(r_j) + S(r_j)r_i + S(r_i)r_j. \tag{17}$$

From this we get

$$\begin{aligned} r_ir_j' + r_jr_i' &= r_i(a_j - r_j) + r_j(a_i - r_i) \\ &= S(r_j)r_i + S(r_i)r_j - r_ir_j - r_jr_i \\ &= S(r_i)S(r_j) - S(r_ir_j), \end{aligned} \tag{18}$$

which is a number of K. Let us now consider two elements

$$\xi = x_0 + x_1r_1 + x_2r_2 + x_{12}r_{12},$$
$$\xi' = x_0 + x_1r_1' + x_2r_2' + x_{12}r_{12}'.$$

Then $\xi+\xi'=2x_0+a_1x_1+a_2x_2+a_{12}x_{12}$ a member of K. Also

$$\begin{aligned} \xi\xi' = {} & x_0^2 + b_1x_1^2 + b_2x_2^2 + b_1b_2x_{12}^2 + a_1x_0x_1 + a_2x_0x_2 \\ & + a_{12}x_0x_{12} + (r_1r_2' + r_2r_1')x_1x_2 \\ & + (x_1x_{12}' + r_{12}r_1')x_1x_{12} + (r_2r_{12}' + r_{12}r_2')x_2x_{12}, \end{aligned} \tag{19}$$

and it may be shown that $\xi\xi'=\xi'\xi$. By (18) and (19) we see that $\xi\xi'$ is also a number of K and hence ξ and ξ' are a pair of conjugate roots of a quadratic equation with coefficients in K.

Consider next the two elements

$$\alpha = u_0 + u_1r_1 + u_2r_2 + u_{12}r_{12}, \quad \beta = v_0 + v_1r_1 + v_2r_2 + v_{12}r_{12}.$$

Then

$$\alpha \pm \beta = u_0 \pm v_0 + (u_1 \pm v_1)r_1 + (u_2 \pm v_2)r_2 + (u_{12} \pm v_{12})r_{12}$$

and by the multiplication table we can also express the product in the same form. Hence the set of elements is closed with respect to addition, subtraction, and multiplication.

Since the product r_ir_j is a linear function of r_1, r_2, r_{12} and its conjugate $r_j'r_i'$ is the same linear function of r_1', r_2', r_{12}' we

see that $\alpha\beta$ and $\beta'\alpha'$ are the same linear function of r_1, r_2, r_{12} and r_1', r_2', r_{12}', respectively. Hence $\alpha\beta+\beta'\alpha'$ is a number of K and $\alpha\beta\cdot\beta'\alpha'=N(\alpha)N(\beta)$ is also a number of K. Therefore, in general for all products the conjugate of a product is the product of the conjugates in the reversed order.

If we let $r_0=r_0'=1$, we may write

$$\alpha\beta = \sum_{i,j} u_i v_j r_i r_j,$$

$$\beta\alpha = \sum_{i,j} u_i v_j r_j r_i,$$

whence we have

$$\begin{aligned}\alpha\beta+\beta\alpha &= \sum_{i,j} u_i v_j (r_i r_j + r_j r_i)\\ &= \sum_{i,j} u_i v_j [S(r_i r_j) - S(r_i)S(r_j) + S(r_i)r_j + S(r_j)r_i]\\ &= \sum_{i,j} u_i v_j S(r_i r_j) - \sum_i u_i S(r_i) \sum_j v_j S(r_j)\\ &\quad + \sum_i u_i S(r_i) \sum_j v_j r_j + \sum_j v_j S(r_j) \sum_i u_i r_i\\ &= S(\alpha\beta) - S(\alpha)S(\beta) + S(\alpha)\beta + S(\beta)\alpha.\end{aligned}$$

This shows that the equation (17) is true for any pair of elements α and β of the algebra.

5. *Extension of the Algebra.* Let us next consider a general quadratic form in $n+1$ variables

$$Q(x_0, x_1, x_2, \cdots, x_n) = \sum_{i,j} a_{ij} x_i x_j$$

having $a_{00}=1$ and $a_{ij}=a_{ji}$. From this we shall form n equations, each one being obtained by setting x for x_0 and one of the remaining variables equal to -1 and the rest equal to zero. The n equations are then

$$x^2 - a_{0i}x + a_{ii} = 0, \qquad (i = 1,2,3, \cdots, n).$$

The conjugate roots of these equations we shall denote by r_i and r_i' $(i=1, 2, \cdots, n)$. We shall also assume that r_i+r_j and $r_i'+r_j'$ are a pair of conjugate roots of the equation obtained from the equation

$$Q(x_0, x_1, x_2, \cdots, x_n) = 0$$

by putting $x_i = x_j = -1$, and all the remaining x_k, except x_0, equal to zero. We shall further assume that for any pair r_i, r_j all the conditions of the preceding sections are fulfilled so that 1, r_i, r_j, $r_i r_j$ constitute the basal elements of an algebra such as we have already discussed. The product $r_i r_j$ we shall denote by r_{ij}.

All polynomials in $r_1, r_2, \cdots, r_n$ having coefficients in K constitute an algebra over K. In a later paper I expect to show some consequences of further restrictions on products involving three or more of the elements $r_1, r_2, \cdots, r_n$. For the present the algebra as defined is sufficient.

6. *A Linear Set.** Having thus defined the algebra let us consider all elements of the form $x_0 + x_1 r_1 + x_2 r_2 + \cdots + x_n r_n$. We shall write

$$\alpha = u_0 + u_1 r_1 + \cdots + u_n r_n,$$

$$\alpha' = u_0 + u_1 r_1' + \cdots + u_n r_n'.$$

Then

$$S(\alpha) = \alpha + \alpha' = 2u_0 + a_{01}u_1 + a_{02}u_2 + \cdots + a_{0n}u_n,$$

$$N(\alpha) = \alpha\alpha' = u_0^2 + \sum_{i=1}^{n} a_{ii}u_i^2 + \sum_{i=1}^{n} a_{0i}u_0u_i + \sum_{i<j} (r_i r_j' + r_j r_i')u_i u_j.$$

But according to our assumptions regarding the $r_1, r_2, \cdots, r_n$, by equations (18) and (9), we have

$$r_i r_j' + r_j r_i' = S(r_i)S(r_j) - S(r_i r_j) = a_{ij}.$$

We note that a_{ij} has a different meaning here from what it had in the preceding section where $a_{12} = S(r_1 r_2)$. Here a_{ij} has taken the place of the c.

Hence we see that $S(\alpha)$ and $N(\alpha)$ are both numbers of K, and any element α of the linear set is a root of a quadratic equation with coefficients in K. We also note that

$$N(\alpha) = Q(u_0, u_1, u_2, \cdots, u_n).$$

It is not difficult to show that $\alpha\,\alpha' = \alpha'\,\alpha$.

* Dickson, loc. cit.

If α is some particular element of the linear set, the set also contains $m\alpha+n$ and hence with $u_0+u_1r_1+\cdots+u_nr_n$ is also found in the set $u_1r_1+\cdots+u_nr_n$ and all its elements are obtained by the addition of a number of K to an element of the last form. But $u_1r_1+u_2r_2+\cdots+u_nr_n$ is a root of

$$Q(x, -u_1, -u_2, \cdots, -u_n) = 0.$$

We therefore see that we may consider the linear set as the totality of all quadratic extensions to K formed by the adjunction of the roots of all equations obtained from the equation $Q(x, x_1, x_2, \cdots, x_n) = 0$ by assigning to the x_1, $x_2, \cdots, x_n$ all possible combinations of values from K.

7. *Application to a Diophantine Equation.* We shall next consider the equation

$$\sum_{i,j=0}^{n} a_{ij}x_ix_j = v_1v_2\cdots v_k, \qquad (a_{ij} = a_{ji}), \tag{20}$$

in which the coefficients a_{ii} and $2a_{ij}$ are rational integers. The application of the foregoing algebra will give a method for finding all rational, integral solutions of the equation. The unknowns are the $x_0, x_1, \cdots, x_n, v_1, v_2, \cdots, v_k$. The field K used in the preceding pages is, from now on, the field of rational numbers. If we multiply both members of (20) by a_{00} and put $a_{00}x_0=y_0$ we can write the left hand member of the equation as a quadratic form in $y_0, x_1, x_2, \cdots, x_n$. This form we shall denote by $Q(y_0, x_1, x_2, \cdots, x_n)$, and the equation (20) is equivalent to

$$Q(y_0, x_1, x_2, \cdots, x_n) = a_{00}v_1v_2\cdots v_k. \tag{21}$$

The quadratic form $Q(y_0, x_1, \cdots, x_n)$ has rational integral coefficients with that of y_0^2 equal to 1 and hence is of the form considered in the preceding sections. The left hand member of (21) is therefore the norm of the general element of a linear set in an algebra such as has been defined above. It is therefore possible to write the equation (20) in a new form

$$N(a_{00}x_0 + r_1x_1 + r_2x_2 + \cdots + r_nx_n) = a_{00}v_1v_2\cdots v_k. \tag{22}$$

For each set of rational values of $x_1, x_2, \cdots, x_n$ a root $\theta(x)$ of the equation

$$Q(x, x_1, x_2, \cdots, x_n) = 0 \tag{23}$$

is a generator of a quadratic extension of K. This may be either of the first or second kind.

Let us consider first the case when (23) is irreducible so that the quadratic extension is of the first kind. Then θ generates a quadratic number field and $1, \theta$ is a base of a ring in this field and (a_{00}, θ) is the base of an ideal in the ring. The problem of obtaining the solutions of

$$N(a_{00}x_0 + \theta t) = a_{00}v_1v_2 \cdots v_k \tag{24}$$

has been solved by the author,* and from this we see that for those rational integral values of $x_1, x_2, \cdots, x_n$ which cause (23) to be irreducible we can obtain the solutions of (22) by writing it in the form

$$N(a_{00}x_0 + r_1x_1t + r_2x_2t + \cdots + r_nx_nt) = a_{00}v_1v_2 \cdots v_k \tag{25}$$

and assigning values to $x_1, x_2, \cdots, x_n$ and solving for x_0 and t by the method for solving (24).

In the case when the values $x_1, x_2, \cdots, x_n$ cause (23) to be reducible the left-hand member of (25) is the product of two linear factors which are homogeneous in x_0 and t. Then elementary methods suffice for the determination of x_0 and t, and the v_i.

It is easily seen that in assigning the values to $x_1, x_2, \cdots, x_n$ only such as are relatively prime need be used as all others will be included among the ones so found.

Hence to obtain all solutions of (20) replace $x_1, x_2, \cdots, x_n$ by $x_1t, x_2t, \cdots, x_nt$; assign to $x_1, x_2, \cdots, x_n$ arbitrary rational integral values thus reducing the left hand member of (20) to a binary form and hence solve by means of known methods.

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* This Bulletin, vol. 31, p. 430.

FUNCTIONS EXPANSIBLE IN SERIES*

BY L. E. WARD

In the Transactions of this Society† Hopkins has stated and proved the following theorem.

THEOREM I. *If $f(x)$ is a function analytic in the interior and on the boundary of a circle centered at $x=0$ and of radius x_0, $0<x_0<\pi$, which involves in its power series expansion only powers of x of indices congruent to* 2 (mod 3), *and which has a continuous second derivative for real values of x in the interval $0\leqq x\leqq\pi$, then the formal development of $f(x)$ in a series whose terms are the characteristic functions of the differential system*

$$\frac{d^3u}{dx^3}+\rho^3u=0, \qquad u(0)=u'(0)=u(\pi)=0,$$

converges uniformly to $f(x)$ in the interval $0\leqq x\leqq x_0$.

Hopkins proved further that the development converges uniformly to $f(x)$ in the interior of an equilateral triangle centered at $x=0$ and having one vertex at $x=x_0$. The following theorem in which the adjoint differential system appears is obtained from Theorem I by the change of variable $x'=\pi-x$.

THEOREM II. *If $f(x)$ is a function analytic in the interior and on the boundary of a circle centered at $x=\pi$ and of radius x_1, $0<x_1<\pi$, which involves in its power series expansion only powers of $(\pi-x)$ of indices congruent to* 2 (mod 3), *and which has a continuous second derivative for real values of x in the interval $0\leqq x\leqq\pi$, then the formal development of $f(x)$ in a series*

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* Presented to the Society, April 2, 1926.

† J. W. Hopkins, Transactions of this Society, 1919, pp. 245, et seq. Published by D. Jackson.

whose terms are the characteristic functions of the differential system

$$\frac{d^3v}{dx^3} - \rho^3 v = 0, \qquad v(\pi) = v'(\pi) = v(0) = 0,$$

converges uniformly to $f(x)$ *in the interval* $x_1 \leqq x \leqq \pi$.

It can be shown further that the latter development converges uniformly to $f(x)$ in the interior of an equilateral triangle centered at $x=\pi$ and having one vertex at $x=x_1$.

The purpose of this note is to determine whether there are functions satisfying the conditions of both the above theorems; and if there are, to see whether the corresponding series may have a common range of convergence.

We derive first conditions which must be satisfied by a function $f(x)$ of the type demanded by both theorems. We must have $f(x)=x^2\phi(x^3)$ and $f(x)=(\pi-x)^2\Phi[(\pi-x)^3]$, where $\phi(x^3)$ and $\Phi[(\pi-x)^3]$ are analytic in x^3 at $x=0$ and in $(\pi-x)^3$ at $x=\pi$ respectively. Such a function $f(x)$ must be the derivative with respect to x of a function $\psi_1(x^3)$ and also of $\psi_2[(\pi-x)^3]$. Neglecting a constant of integration, which may be included in either function, we must have $\psi_1(x^3)=\psi_2[(\pi-x)^3]$. Can we find a function single valued and analytic in x^3 at $x=0$ and also single valued and analytic in $(\pi-x)^3$ at $x=\pi$? If there is such a function, it will be invariant under the transformations $x'=\omega x$ and $\pi-x'=\omega(\pi-x)$, where $\omega=e^{2\pi i/3}$; i. e., under the transformations

$$\left\{\begin{array}{l} x' = \omega x \\ x' = \omega x + (1-\omega)\pi \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l} x' = \omega x \\ x' = x + (1-\omega)\pi \end{array}\right\}.$$

Repetition of the first of these gives $x'=\omega^m x$, and of the second $x'=x+n(1-\omega)\pi$. Combination of these yields $x'=\omega^m x + n(1-\omega)\pi$. Repetition of this gives

$$x' = \omega^{m+p'}x + [q'\omega^m - q'\omega^{m+1} + n - n\omega]\pi,$$

or simply

$$x' = \omega^n x + (p + q\omega)\pi.$$

(Of course, m, n, p', q', p and q are integers.)

Consequently, x being an arbitrary point, the function must have the same value at the points

$$x + (p + q\omega)\pi, \quad \omega x + (p + q\omega)\pi, \quad \omega^2 x + (p + q\omega)\pi$$

that it has at x. A function having the same value at the points $x+(p+q\omega)\pi$ as at x is a doubly periodic function of primitive periods π and $\omega\pi$; call $g(x)$ such a function. Then $\psi(x) = g(x) + g(\omega x) + g(\omega^2 x)$ is a doubly periodic function of periods π and $\omega\pi$, and it is invariant under all of the above transformations. Hence $\psi(x)$ is a single valued function of x^3 and also of $(\pi - x)^3$, and $\psi'(x)$ can be expanded in both series.

The question of whether or not these series have a common region of convergence depends for its answer on the location of the poles of $\psi(x)$. It is clear that the poles may be situated so that the two triangles of convergence* have no common region in their interiors. On the other hand, it is not hard to put down the poles of $g(x)$ so that the triangles will have a region interior to both, as the following example shows.

Let $g(x)$ have a double pole (of course with zero residue) at the point $x = \pi/2 + \pi i/2(3)^{1/2}$, and no other singularity in the parallelogram three of whose vertices are at 0, π, $\omega\pi$. Then $g(\omega x)$ and $g(\omega^2 x)$ will also have double poles at this point and nowhere else in this parallelogram. Hence $\psi(x)$ will have double poles at the points $x = \pi/2 + \pi i/2(3)^{1/2} + p\pi + q\omega\pi$ and nowhere else, and the same is true of $\psi'(x)$. No singularity of $\psi'(x)$ is nearer the origin than the pole at $x = \pi/2 + \pi i/2(3)^{1/2}$, and hence the circle of Theorem I may be given a radius slightly smaller than $\pi/3^{1/2}$ but greater than $\pi/2$. The circle of Theorem II may be given the same radius. Then these circles will have a region interior to both of them, which will contain a portion of the axis of reals, along which the series mentioned in both theorems will converge uniformly to $\psi'(x)$.

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* Hopkins showed in his special case that the region of uniform convergence is the interior of an equilateral triangle centered at $x = 0$ and having one vertex on the positive axis of reals.

INVARIANTS OF A PORISTIC SYSTEM OF TRIANGLES*

BY J. H. WEAVER

1. *Introduction.* In the last quarter of a century considerable work has been done on various systems of triangles.† Several fixed circles have been discovered which are associated with poristic systems of triangles. However, so far as the writer knows, no attempt has been made to determine all the invariant curves of any degree for any system of triangles. The present paper determines the invariant curves of the first and second degrees for a poristic system of triangles which has a fixed incircle and a fixed circumcircle.

2. *General Considerations.* Let there be a triangle ABC with a fixed incircle, center I, and a fixed circumcircle, center O. If we consider OI as the X-axis and I as the origin, the equations of the sides of the triangle may be written in the normal form as follows:

$$(1)\qquad \begin{cases} BC: & x\cos A_1 + y\sin A_1 - r = 0, \\ CA: & x\cos A_2 + y\sin A_2 - r = 0, \\ AB: & x\cos A_3 + y\sin A_3 - r = 0, \end{cases}$$

where r denotes the radius of the incircle.

If $A'B'C'$ be any other triangle of the system the equations of its sides are

$$(2)\qquad \begin{cases} B'C': & x\cos A_1' + y\sin A_1' - r = 0, \\ C'A': & x\cos A_2' + y\sin A_2' - r = 0, \\ A'B': & x\cos A_3' + y\sin A_3' - r = 0. \end{cases}$$

Let $P(x', y')$ be any point in the plane of the triangles. Then the distances of P from the sides (1) and (2) are

$$(3)\qquad \alpha_i = x'\cos A_i + y'\sin A_i - r, \qquad (i = 1, 2, 3),$$

$$(4)\qquad \alpha_i' = x'\cos A_i' + y'\sin A_i' - r, \qquad (i = 1, 2, 3),$$

respectively.

* Presented to the Society, September 8, 1926.

† See Gallatley, *Modern Geometry of the Triangle*, second edition, London, 1910; and Coolidge, *Treatise on the Circle and Sphere*, Oxford, 1916.

Eliminating x' and y' from (3) and (4), we obtain

$$(5)\qquad \alpha_i' = \frac{|\alpha_1, \sin A_2, 1|}{|\cos A_1,\ \sin A_2, 1|}\cos A_i' + \frac{|\cos A_1, \alpha_2, 1|}{|\cos A_1,\ \sin A_2, 1|}\sin A_i' - r.$$

The angle A_2' determines A_1' and A_3'. Hence α_i' depends upon α_i and A_2', if A_1, A_2 and A_3 remain fixed.

Equations (5) then determine a one-parameter group of transformations, and by the Lie Theory of one-parameter groups we have for an invariant

$$(6)\qquad \sum_{i=1}^{3}\left[\frac{\partial \alpha_i'}{\partial A_2'}\right]_{A_2'=A_2}\frac{\partial}{\partial \alpha_i} \equiv 0.$$

Equation (6) for the particular case represented by equation (5) becomes

$$(6')\qquad \sum_{i=1}^{3}\left[-\frac{|\alpha_1,\ \sin A_2, 1|}{|\cos A_1,\ \sin A_2, 1|}\sin A_i\frac{dA_i}{dA_2} + \frac{|\cos A_1, \alpha_2, 1|}{|\cos A_1,\ \sin A_2, 1|}\cos A_i\frac{dA_i}{dA_2}\right]\frac{\partial}{\partial \alpha_i} \equiv 0.$$

If we choose for the triangle ABC the special position where B lies on the line OI, and to the right of I, equation (6′) will be much simplified without loss of generality. In fact it becomes

$$(7)\qquad \begin{cases} \left[(2\alpha_2 - \alpha_3 - \alpha_1)\cos^2\frac{B}{2}\cdot\frac{dA_1}{dA_2}\right. \\ \quad \left. + (\alpha_1 - \alpha_3)\left(\sin\frac{B}{2} + 1\right)\left(\sin\frac{B}{2}\right)\frac{dA_1}{dA_2}\right]\frac{\partial}{\partial \alpha_1} \\ \quad + (\alpha_3 - \alpha_1)\left(\sin\frac{B}{2} + 1\right)\frac{\partial}{\partial \alpha_2} \\ + \left[-(2\alpha_2 - \alpha_3 - \alpha_1)\cos^2\frac{B}{2}\cdot\frac{dA_3}{dA_2}\right. \\ \quad \left. + (\alpha_1 - \alpha_3)\left(\sin\frac{B}{2} + 1\right)\sin\frac{B}{2}\cdot\frac{dA_3}{dA_2}\right]\frac{\partial}{\partial \alpha_3} \equiv 0, \end{cases}$$

where $dA_3/dA_2 = dA_1/dA_2 = R/(R-d)$, in which R denotes the radius of the circumcircle, and $d = OI$.

3. *A First Degree Invariant.* Let

$$l\alpha_1 + m\alpha_2 + n\alpha_3 = 0 \tag{8}$$

represent any line in the plane of the triangle ABC. If we apply the differential operator (7) to the line (8), we obtain

$$\begin{aligned}[(k' - k)l - k''m + (k' + k)n]\alpha_1 + (2kl - 2kn)\alpha_2 \\ + [-(k' + k)l + k''m - (k' - k)n]\alpha_3 \equiv 0,\end{aligned} \tag{9}$$

where

$$k = \cos^2 \frac{B}{2} \cdot \frac{dA_1}{dA_2};$$

$$k' = \left(\sin \frac{B}{2} + 1\right) \sin \frac{B}{2} \cdot \frac{dA_1}{dA_2};$$

$$k'' = \sin \frac{B}{2} + 1.$$

In order that (9) shall be satisfied we must have

$$\begin{cases} (k' - k)l - k''m + (k' + k)n = 0, \\ l - n = 0, \\ -(k' + k)l + k''m - (k' - k)n = 0. \end{cases} \tag{10}$$

The matrix of the set of equations (10) is of rank 2, hence there is one solution for l, m, n, other than zeros. A solution may then be written $l = l$, $m = (2k'/k'')l$, $n = l$. But $2k'/k'' = 1$ and the equation of the single invariant line of the system is

$$\alpha_1 + \alpha_2 + \alpha_3 = 0.$$

This is the equation of the line in which the bisectors of the exterior angles of the triangle meet the opposite sides; hence we may state the following theorem.

THEOREM I. *In a poristic system of triangles, which has a fixed incircle and a fixed circumcircle, the line determined by the points of intersection of the bisectors of the exterior angles of the triangle with the opposite sides is a fixed line.*

4. *Second Degree Invariants.* The equation of any curve of the second degree in the plane of the triangle ABC may be written

$$l\alpha_1^2 + m\alpha_2^2 + n\alpha_3^2 + 2p\alpha_1\alpha_2 + 2s\alpha_1\alpha_3 + 2t\alpha_2\alpha_3 = 0. \tag{11}$$

If we apply the differential operator (7) to the equation (11) we obtain

$$\begin{aligned}&[l\alpha_1 + p\alpha_2 + s\alpha_3][(2\alpha_2 - \alpha_3 - \alpha_1)k + (\alpha_1 - \alpha_3)k'] \\ &+ (m\alpha_2 + p\alpha_1 + t\alpha_3)(\alpha_3 - \alpha_1)k'' \\ &+ [-(2\alpha_2 - \alpha_3 - \alpha_1)k + (\alpha_1 - \alpha_3)k'][n\alpha_3 + s\alpha_1 + t\alpha_2] \\ &\equiv 0.\end{aligned} \tag{12}$$

In order that equation (12) shall be valid we must have the following set of equations.

$$\begin{aligned} l(k' - k) - pk'' + s(k' + k) &= 0, \\ p &= t, \\ n(k' - k) - tk'' + s(k' + k) &= 0, \\ mk'' - 2nk - p(k' + k) + 2sk - t(k' - k) &= 0, \\ l(k' + k) - n(k' + k) - pk'' + tk'' &= 0, \\ 2lk - mk'' + p(k' - k) - 2ks + (k' + k)t &= 0. \end{aligned} \tag{13}$$

The matrix of equations (13) is of rank 4. Hence there is a single infinity of solutions for the set of equations. This set of solutions may be written in terms of l and p, as follows:

$$n = l,\ t = p,\quad s = \frac{R - d}{R}p + \frac{d}{R}l,\quad m = \frac{R + d}{R}l - \frac{d}{R}p.$$

Equation (12) may therefore be written in the form

$$\begin{aligned}&l\alpha_1^2 + \left(\frac{R+d}{R}l - \frac{d}{R}p\right)\alpha_2^2 + l\alpha_3^2 + 2p\alpha_1\alpha_2 \\ &+ 2\left(\frac{R-d}{R}p + \frac{d}{R}l\right)\alpha_1\alpha_3 + 2p\alpha_2\alpha_3 = 0.\end{aligned} \tag{14}$$

By finding the polar of the incenter with respect to the family (14) we obtain a result which may be stated in the following theorem.

THEOREM II. *The first polar of the incenter with respect to the family* (14) *is the line* $\alpha_1+\alpha_2+\alpha_3=0$.

Also if we write down the condition that the system (14) shall represent two lines we find that it is necessary that $l=p$.*

A substitution of this condition in (14) gives $(\alpha_1+\alpha_2+\alpha_3)=0$. Hence the straight lines of the system (14) consist of the first degree invariant counted twice.

If we set down the condition that equation (14) represent a circle we find

$$p = l\left(\frac{R+d}{R}\right).$$

A substitution of this condition in (14) gives for the equation of the circle the expression

$$\alpha_1^2 + \frac{2r}{R}\alpha_2^2 + \alpha_3^2 + 2\left(\frac{R+d}{R}\right)\alpha_1\alpha_2 + 2\left(\frac{2r+d}{R}\right)\alpha_1\alpha_3 + 2\left(\frac{R+d}{R}\right)\alpha_2\alpha_3 = 0. \tag{15}$$

The center of this circle lies on OI. And since we have chosen as our triangle of reference the triangle whose vertex B lies on the line OI, the equation of OI is

$$\alpha_1 - \alpha_3 = 0. \tag{16}$$

Also from the triangle itself we have

$$\alpha_1 + \left(\frac{R-d}{R}\right)\alpha_2 + \alpha_3 = \left(\frac{R-d}{R}\right)(R+d+r). \tag{17}$$

* See Whitworth, *Modern Analytical Geometry*, p. 244.

If we solve equations (15), (16), (17) for α_2 we find

$$\alpha_2 = \frac{d(R+d+r) \pm [Rd(R+d)(R+d+r)]^{1/2}}{d}.$$

The half sum of the two values of α_2 gives the distance of the center of the circle (15) from the side AC of the triangle. This distance is $R+d+r$. Hence we have the following theorem.

THEOREM III. *The center of the invariant circle* (15) *lies on the circumcircle of the variable system of triangles.*

The half difference of the values of α_2 gives the radius ρ of the circle (15). Consequently

$$\rho = \frac{[Rd(R+d)(R+d+r)]^{1/2}}{d}.$$

Using this result we may easily prove the following theorem.

THEOREM IV. *The line* $\alpha_1+\alpha_2+\alpha_3=0$ *is the radical axis of the incircle and the circle* (15).

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OSGOOD'S ADVANCED CALCULUS

Advanced Calculus. By William F. Osgood. New York, The Macmillan Company, 1925. xvi+530 pages.

Almost twenty years ago, in a presidential address delivered before the American Mathematical Society*, Professor Osgood outlined his conception of the aims and methods that should underly the teaching of the calculus. As between the formalists and the reformists of the Perry school, he pointed out that though the former were right in insisting on the necessity of rigorous training in formal work in order to acquire the ideas of the calculus, still this drill must appear to the student "as having for its direct object the power to solve some of the real problems of pure and applied mathematics, and these problems must always be kept before his eye." This idea was emphasized again with the words "That which is most central in the calculus is its *quantitative* character, through which it measures and estimates the things of the world of our senses. And instruction in the calculus that does not point out—not merely at the beginning or at the end, but all through the course—this close contact with nature, has not done its duty by the student." He felt, however, that too often those who had attempted to interpret physical phenomena mathematically had started from incomplete and vaguely stated hypotheses and had used methods so slipshod as to be beneath the respect of the undergraduate student of the calculus.

Later in the same year his *First Course in the Differential and Integral Calculus* was published. Here the program of his presidential address was carried out. There was a large amount of material for drill in technique; statements of hypotheses and conclusions were clear and accurate; proofs were carried through with all the rigor suitable for a first course in the calculus, and a large part of the book was devoted to applications, with the aim not merely to impart knowledge, but to foster appreciation of the spirit of the calculus and to give power to use it as a tool in the interpretation of nature. The book had a wide sale, and many of the younger generation of American mathematicians there acquired their first appreciation of what mathematics means. Still, however great the service thus rendered, it was surpassed by the influence for the better that this text exerted upon all succeeding works on the calculus published in this country, and on many published abroad. Few could successfully imitate the style of Osgood, many saw only imperfectly what he was aiming at, yet if they could not draw the bow of Ulysses they at least modeled their armory after his and shot as nearly as they could in the same direction.

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* This Bulletin, vol. 13, No. 9 (June, 1907), pp. 449–467.

Some instructors found the *First Course* too "hard"; others, with more reason, felt that the material presented was far too copious for a three-hour year course, but considerably short of enough for a second course. The author himself felt that rearrangement, revision, and extension were advisable. This plan was carried out by the publication in 1922 of his *Introduction to the Calculus* (in the previous year the first half had been brought out under the title *Elementary Calculus*), followed three years later by the volume which is the subject of this review.

The author does not here attempt to duplicate any part of his *Introduction*, hence no general restatement or summary of a first year course in the calculus is included. Instead there are numerous references to the *Introduction*. The instructor who uses this text must therefore see to it that the *Introduction* is accessible to his class, or he must be prepared to substitute other explanations and references. Even the syllabus of elementary solid analytic geometry that found a place in the *First Course* has been dropped, and the reader is referred to Osgood and Graustein's *Plane and Solid Analytic Geometry*. It may be surprising to many that an Advanced Calculus should contain almost nothing on the convergence or divergence of infinite series, and little on Maclaurin's and Taylor's developments. The author, has, however, preferred to devote considerable space in the *Introduction* to these topics and thus can dispense with their treatment here.

On the other hand he has included two chapters on general methods of integration and on reduction formulas, which ordinarily are considered a part of elementary rather than of advanced calculus. However, it is a satisfaction to find here a good proof of the theorems concerning partial fractions, a systematic treatment of rationalizing substitutions, and a proper appreciation of the waste of ingenuity involved in proofs of reduction formulas except as the inverse of differential relations. Otherwise the book is distinguished by its wealth of applications to physical problems and only here presents material beyond that covered by the normal class in advanced calculus.

The author explains that the order in which he has written is not a necessary order for the reader. The latter may begin with the chapter on Partial Differentiation, or with Double Integrals, or Differential Equations. Again, the omission of certain sections on first reading of a chapter is suggested. This very flexibility indicates clearly the author's perception of the important part the teacher must play in determining the success or failure of a course in the calculus. The instructor who, without looking ahead, assigns "the next four pages with all the odd-numbered problems", will have poor sailing and a mutinous crew before long. If, for instance, the lesson includes, without previous explanation, the proof of pages 73–74 that a density function in two dimensions must be continuous although this does not hold in one dimension, there may be trouble. Along with the easier and more formal problems will be found others that will tax the best in the class, others that anticipate later chapters. The instructor must be prepared to pick and choose according to the strength and needs of his class.

Although it is true that the author does not hesitate to include both proofs and problems of some difficulty, he does not follow the too prevalent plan of plunging through a stiff demonstration as quickly as possible and emerging to simpler matters calculated to reassure the bewildered neophyte. He is quite willing, in fact, to take a hard journey in several stages. Frequently there is a "heuristic proof", which is shown in its true light as a discussion pointing the way to the truth, followed by a demonstration of more satisfying rigor. For example, there are three proofs of the formula which reduces a double integral to iterated integrals in rectangular coordinates. The first is based on geometric intuition, the second is arithmetic but incomplete, the third, some distance on, indicates how the second is to be made satisfactory. Everywhere there is the greatest care to say precisely what is meant. The author does not suffer from the delusion that accuracy of statement makes a subject more difficult or less attractive.

There are enough exercises. Many are formal and relatively easy, but others will require more than ordinary ability for their solution. Some test the student's capacity to anticipate matters taken up later in the text. In fact there is no gentleman's agreement to make only backwards references. Both in text and exercises there is not infrequently a look forward, and sometimes results are used whose truth is for the time being assumed, but is proved later. There is a good supply of examples worked out. These often emphasize critical points as well as matters of routine. In fact exercises, solved or unsolved, in a few cases supply the proofs of formulas that follow in later pages of the text, thus affording the class an excellent opportunity to find whether the instructor knows what sort of a lesson he has assigned.

Up to the end of Chapter X much material has been taken from the *First Course*, but besides Chapters I and II, Chapter IX is new, as is practically all that follows the tenth chapter. There has been, however, much revision of the treatment of topics from the *First Course*, and nearly all the changes are, to the reviewer's mind, improvements. As one leaves behind this first part of the book, one cannot but sense that the author feels a certain relief in being through with this too familiar, often-worked-over field. There is an atmosphere of added enthusiasm, an interest more tense, as we pass on to chapters where applications are of major importance and technique is less stressed. There are passages also where the author unbends to make an apt remark or to indulge in a picturesque or even a homely phrase. To the reader we leave the pleasure of digging these out and chuckling over them. They are not too frequent, and are well balanced by the more serious exhortations against such transgressions as the uncritical use of infinitesimals or juggling proofs.

We have already indicated the subject matter of Chapters I and II, on general methods of integration and reduction formulas. Chapters III and IV consider double and triple integrals. Here much has been taken from the *First Course*, but there are additions and changes. The proofs depend largely on geometric intuition, as the author is careful to point out; more rigorous demonstrations are deferred to Chapter XII. Thus the student obtains some insight into the methods and uses of integration

before he encounters existence proofs of an arithmetic character. Besides the applications generally to be found in other texts, there are to be noted the sections on attractions, on density, pressure at a point and specific force, and on potential in three dimensions.

The next four chapters, from V to VIII, are concerned with partial differentiation and its applications, these latter being chiefly geometric. Part of the corresponding material of the *First Course* was used in the last nine pages of the *Introduction*; most of the rest is to be found here in revised form, with some important additions. Thus the former admirable treatment of the total differential is here somewhat expanded, and the approach to it has been made easier. In section 12 of Chapter V, existence theorems are carefully stated for implicit functions defined by one equation or by several simultaneous equations, but the reader is referred to other books for proofs. On pages 180–185, Chapter VII, there is a discussion of Lagrange's method of multipliers in the theory of maxima and minima. Chapter VIII, on envelopes, develops a sufficient condition that a family of curves have an envelope, whereas most texts stop with necessary conditions.

Elliptic integrals are treated in Chapter IX. Most of the thirteen pages are devoted to the standard forms for integrals of the first and second kind and the reduction of integrals to these forms. This is followed by a brief but skillful explanation of Landen's transformation, and by a half page defining the elliptic functions and giving references.

Chapter X, on indeterminate forms, has, in the main, been transferred from the *First Course*. It is probably more the fashion to take this subject up in elementary calculus, but here, at least, we have satisfactory proofs and a good sense of relative values.

In Chapter XI the notion of work done on a particle by a force leads up to definitions of line integrals. Green's Theorem in two dimensions is then discussed, and is used in treating integrals independent of path. In this connection the author pauses, for the space of three pages, to consider definitions and elementary properties of simply and multiply connected regions. He then passes on to line integrals and Green's Theorem in three dimensions (called Green's Lemma in a later chapter, page 287). Stokes' Theorem is next proved more carefully than is usual in other texts. Sections 11–17 develop the equations of the flow of heat and of electricity in conductors. A valuable feature is the clear distinction between mathematical proof and the inductive processes by which a physical law is inferred and expressed in a mathematical formula. It would be hard to improve on these sections, which are adaptations of material from the author's *Allgemeine Funktionentheorie*. In sections 18 and 19 there is a glance forward to related topics treated in later chapters.

Most readers will find Chapter XII, on the transformation of multiple integrals and on the equation of continuity, more difficult than any other in the book. There is no help for this; it could not be otherwise in a treatment at all adequate. The first section begins with a satisfactory arithmetized existence proof for the ordinary definite integral, with some novelty in its form, and so stated as to help in proving the existence theorem for

double integrals and the formulas for their reduction to iterated integrals considered in the following sections. The property of uniform continuity is explicitly assumed, and the property of zero area for the boundary is tacitly supposed to hold for all double integrals considered. This latter matter may trouble the reader of inquiring mind when he tries to verify the assertion "the further development of the proof" (for double integrals) "follows precisely the lines of the earlier case" (for the ordinary definite integral). The reader referred to may also be puzzled to find why the author assumes, on page 254, that $f(x)$ is positive. The transformation of double and triple integrals is next considered, a "heuristic proof" employing infinitesimals and Duhamel's Theorem being in each case followed by a logically more satisfactory one using, respectively, line and surface integrals. The last four sections of the chapter are devoted to obtaining in various forms the equation of continuity in hydrodynamics. Section 11 will be difficult reading for the average student, but the effort to master it will be well worth while. We may note again the author's insistence on a distinction between what is proof and what is not (see especially pages 278–279).

An elementary treatment of vector analysis occupies Chapter XIII. Green's and Stokes' Theorems are put in vector form, and there is a section on curvature and torsion of twisted curves. In section 9 there are very sensible remarks on limitations to the usefulness of vector notations.

The next chapter is a long one of 67 pages on differential equations, with emphasis on applications. Thus of the first third, on equations of the first order, seven pages are devoted to solutions of standard cases, while sixteen give applications. The other subdivisions are on linear equations, geometrical interpretation and singular solutions, solutions by series and by integrating factors, and partial differential equations. Section 14 will hardly be appreciated unless the reader is familiar with the pages of Appell's *Mécanique Rationnelle* given as a reference, and the top of page 348 may also prove troublesome owing, apparently, to an unexplained change of notation. The brief discussion of singular solutions is excellent. In sections 24 and 25 there is a treatment of partial differential equations of the first order by the method of characteristics. Naturally, this is not one of the simplest parts of the book.

Chapters XV and XVI are especially valuable, though many classes will omit one or both in a one-year course. The matters here presented can hardly be found elsewhere in such concise and elegant form. The former chapter begins with a treatment of forced vibrations and then derives in exact form the partial differential equations for both transverse and longitudinal motion of a vibrating string. The familiar simpler forms are obtained as approximations. The vibrating membrane is also briefly considered. Chapter XVI, on Fourier's series and orthogonal functions, considers formal developments in Fourier's series, power series as proceeding from problems of approximate representation, and series of orthogonal functions in general, especially as suggested by the principle of least squares. Five pages are given to zonal harmonics and Bessel's functions.

Chapter XVII, on the calculus of variations and Hamilton's principle, carries the first subject only as far as the derivation and solution of Euler's equation, i.e., the problem of finding extremals, with applications. Most of these applications are familiar enough, though the derivation of Laplace's equation in curvilinear coordinates is not so well known. Isoperimetric problems and variable end points are considered, also integrals in parametric form. The author's insistence on accurately stated definitions of variations is to be commended. Hamilton's Principle and the Principle of Least Action occupy the last fourteen pages. Their application to a number of relatively simple problems is a most valuable feature.

Chapter XVIII, on thermodynamics and entropy, will help clear up some fundamental notions in this field.

With Chapters XIX and XX we return to matters of a more formal sort. The first takes up definite integrals with parameters, improper integrals and tests for their convergence or divergence, the Gamma and Beta Functions, improper double integrals, and special methods for the evaluation of improper integrals. The last chapter gives a brief introduction to the elementary theory of functions of a complex variable.

There are some misprints, though not an undue number. As part or all may have been corrected in the plates since the first printing, a list of them would be useless here.

D. R. Curtiss

HEATH'S EUCLID

The Thirteen Books of Euclid's Elements translated from the text of Heiberg with introduction and commentary. By Sir Thomas L. Heath, K.C.B., K.C.V.O., F.R.S. Cambridge, University Press, 1926. 8 vo. 3 volumes, pp. xii+432; 436; iv+546. £3, 10 s.

It is about eighteen years ago that this writer published in this periodical (this Bulletin, (2), vol. 15, pp. 386-391) a review of the first edition of this noteworthy example of English scholarship and power of exposition. The titlepage then gave the author as T. L. Heath, C. B., Sc. D.; it now records honors which this and other works upon the history of mathematics have brought to the author, and with the approval of the whole scientific world. As regards the general treatment of the subject, the significance of such a publication, and its influence upon modern education, little need be added to the review above mentioned. The only matter demanding special consideration at this time relates to the changes which characterize the new edition.

In general, the work is a reprint of the first impression, with such corrections and minor changes as are naturally desirable after such a lapse of years. It is a gratifying tribute to the scholarship of the author that the changes in the original text are so few, as also that the demand for the work has made this edition necessary. It should not be thought, however, that the text has not been thoroughly revised or that it fails to include the latest information relating to discoveries in the field considered. The care shown in the revision is seen in numerous changes in the foot-

notes as well as in the pages themselves. A considerable number of these changes were made for the purpose of embodying in the text certain addenda and corrigenda given in volume III of the first edition, but there are numerous other additions which were rendered necessary by recent studies of Greek literature.

Some idea of the nature of the changes may be had from the following brief summary of the more important ones:

Vol. I, pp. 8, 9. The statement regarding Euclid's *Book on the division of figures*, necessitated by Professor Archibald's edition (Cambridge, 1915) of this work.

Pp. 20–22. A reconsideration of the probable date of Heron. The author rightly speaks of it as "still a vexed question," but concludes with the statement that "the net result, then, of the most recent research is to place Heron in the third century A. D., and perhaps a little earlier than Pappus," a conclusion which he had already expressed in his *History of Greek Mathematics*. This result of his summation of the arguments will probably be accepted as conclusive unless and until new evidence is found to invalidate it.

P. 32. The question of whether or not Proclus wrote any commentaries upon Euclid's *Elements* beyond Book I,—already considered in the addenda of volume III of the first edition.

P. 39. On the origin of the name Geminus.

P. 74. On the origin of the scholia of Euclid.

P. 113. Certain recent editions of Euclid.

P. 351. Further notes on Euclid I, 47, largely from the addenda of the first edition.

P. 352. The seeming approval of Professor Peet's assertion that nothing in Egyptian mathematics "suggests" that the Egyptians were acquainted with the fact that the 3–4–5 triangle was right-angled,—an assertion which will certainly be considered further by future historians, and which (as to the word "suggested") is even now open to doubt.

P. 362. The possibility that the Pythagorean relation $a^2+b^2=c^2$ had its origin in India or in China.

The most extended additions to the text of volume I consist of two excursus. The first, which was for the most part given among the addenda of volume III of the first edition, relates to Pythagoras and the Pythagoreans and considers chiefly the arguments advanced by Junge (1907) and Vogt (1908) to the effect that the Greek work on irrationals was due rather to the later Pythagoreans than to the early founders of the brotherhood, a theory which the author believes to be untenable. He agrees, however, with Vogt that it is not likely that the early Pythagoreans actually constructed the five regular solids in the manner seen in Euclid XIII.

The second excursus will have more popular interest, since it is concerned with the history of certain fanciful names that have attached themselves to some of Euclid's theorems. These are the *pons asinorum* (properly, Euclid I, 5); "the theorem of the bride," "the bride's chair," the *dulcarnon*, or the *Francisci tunica* ("the Franciscan's cowl;" Euclid I, 47), and the

pes anseris ("goose's foot") or *cauda pavonis* ("peacock's tail;" Euclid III, 7 and 8).

The important changes in volume II are largely from the addenda of the first edition and are found on pages 190 (on Euclid VI, def. 5) and 425 (on perfect numbers).

Volume III is practically unchanged.

The preface to this edition refers to two distinct movements "in the domain of geometrical teaching." The first relates to a "widespread desire among teachers for the establishment of an agreed sequence to be generally adopted in teaching the subject," and the second to "the movement in favor of reviving, in modified form, the proposal made by Wallis in 1663 to replace Euclid's Parallel-Postulate by a Postulate of Similarity." Such movements may be commanding the attention of teachers in England; but in the United States the movement in the first case is decidedly in the opposite direction, and the second case has attracted no attention on the part of our schools. Furthermore, it is very doubtful if a study of current textbooks would show that either of these tendencies is very pronounced in the continental countries.

In any case, however, no teacher of geometry can afford to be ignorant of the spirit of Euclid, since it is this spirit which constitutes the essence of all demonstrative geometry. It is probable that no equal expenditure of money would produce as good results in the teaching of geometry as the purchase of sets of these volumes and the placing of them in the libraries of our high schools. In comparison with the rather ephemeral educational literature of the time, a work of this kind has a permanent value and sets a scholarly standard that will encourage better teaching in the one branch of study in our secondary school that gives real insight into the significance of mathematics.

David Eugene Smith

SHORTER NOTICES

Dynamik. By Dr. Wilhelm Müller. Berlin and Leipzig, Walter de Gruyter, 1925. Part I, *Dynamik des Einzelkörpers*, 160 pp. Part II, *Dynamik von Körpersystemen*, 137 pp.

Précis de Mécanique Rationnelle. By Georges Bouligand. Vol. I. Paris, Librairie Vuibert, 1925. viii+282 pp.

Introduction Géométrique à la Mécanique Rationnelle. By Charles Cailler. Ouvrage publié par H. Fehr et R. Wavre. Genève, George et Cie., Paris Gauthier-Villars, 1924. ix+627 pp.

The two small volumes by Müller follow the aim of the Göschen series to give with brevity the essentials of a subject. The evident purpose has been to fit these two books to the technical student as well as the mathematical student who is desirous of becoming acquainted with dynamics as an application of his science. Thus each chapter opens with a theoretical discussion and closes with a problem as practical as possible. Vectors are introduced immediately and used everywhere to add to the compactness of the treatment.

The first chapter treats of the motion of a point in terms of velocity and acceleration ending with curvilinear motion in space. In the second chapter, mass and force are added and Newton's laws of motion developed. Motion, involving only translation, in a gravitational field, on a surface, periodic, including both damped and forced vibrations, are typical examples of this chapter. The final chapter studies the general motion of a rigid body thus including rotation. The examples include rolling motion, the engine regulator, the pendulum, closing with the theory of the top.

The second volume of Müller treats of dynamical systems of rigid bodies. In the first chapter the concept of a system is defined, and the various kinds of forces involved are considered, friction in detail on account of its technical importance. The equations of motion are considered for the various portions of a system with such examples as a cylinder rolling on a moving inclined plane, complicated pulley systems, etc.

In the second chapter the general analytic methods are developed drawing on D'Alembert's principle, and the Lagrangian equations are derived. Examples considered are the steam engine with its regulator, the double pendulum, the precession and nutation of the top, and the gyroscopic stabilizer for ships. The third chapter includes the applications of the calculus of variations to dynamics. Much ground is covered well for so small a book.

The *Mécanique Rationnelle* by Bouligand centers its development upon meeting the demands of the French examinations. A perusal of this book will give an idea of the skill in manipulation and the range of topics re-

quired of the candidates for the exacting examinations of the "agrégation" and the "licence". The sound advice is given to the student who must be his own instructor to push on to the problems during a first reading, and then with the aid of the knowledge acquired to return to the study of the theory.

The first seven chapters give a rapid introduction into the dynamics of systems. Vectors, motion of a point and of a solid body, center of mass, principles and general theory of dynamics with applications to a solid body, follow in rapid order. The calculus of variations is introduced and illustrated by the geodesics of a surface, and then it is shown how the equations of motion can be considered as extremals of an integral. Constant appeal is made to the theory of Lagrange.

Chapter VIII, the most important in the book, gives a résumé of the theoretical results, and then proceeds to show in detail the methods of dealing with examples which are classified according to the number of the degrees of freedom. Each of the degrees from one to five is illustrated by examples of considerable difficulty, many of which are taken from past examinations. Among the topics of special interest is the use of analysis situs to exhibit the differences of the so-called equivalent dynamical systems. A second volume of this treatise is promised.

After a long period of teaching mathematics and mechanics at the University of Geneva, Charles Cailler retired in 1921. While busy in the preparation of the *Mécanique Rationnelle* he was overtaken by death in 1922. The manuscript in a partly incomplete form was turned over to H. Fehr and R. Wavre for editing. Cailler's object was to give a new presentation of the subjects of kinematics, statics, and dynamics. Fortunately the co-editors were familiar with his point of view, since they were not only his colleagues at the University but also his former pupils. The high esteem in which Cailler was held by all and in particular by the co-editors is expressed with grace in their introduction.

The author makes his treatise depend upon determinants and linear transformations not limiting himself to a euclidean space of three dimensions but assuming a space of n dimensions in which motion without deformation is possible. The book is divided into four parts. The first part is devoted to the laying of this algebraic foundation for which the author claims a double advantage: first that it unifies the whole subject of rational mechanics and second that it makes easier the approach to the relativity theory, a subject which goes beyond the scope of the present volume. The three chapters of this section are, in order: linear forms, quadratic forms, and the theory of linear transformations. These are all developed in detail with constant geometrical interpretation.

The second part treats of the geometry of vectors, forces, and line geometry.

The third part develops kinematics and finite motions with an extended section on quaternions.

The final part is again kinematics and infinitesmal motions. The discussion ranges from simple topics as parabolic motion, harmonic

motion, Lissajous' curves to those more advanced, as the transformations of Lagrange, the instantaneous motion of a solid body, the general theory of rolling motion.

The examples and applications are indeed numerous and well chosen throughout the 627 pages and represent the fruits of a life spent in the cultivation of this field. The co-authors consider the correspondence set up between line geometry and the point geometry on a sphere by means of the complex coordinates of a line to be the most interesting and original part of the work.

G. M. Conwell

Höhere Algebra. By Helmut Hasse. Vol. I, *Lineare Gleichungen.* Sammlung Göschen. Berlin and Leipzig, Walter de Gruyter, 1926. 160 pp.

The author states in the introduction that the fundamental problem of algebra is the development of general methods by which equations, formed by the four elementary calculation operations from the known and unknown elements of a *Körper*, may be solved. This volume treats the solution of linear equations and volume II is to treat equations of higher degree.

In Chapter I the theory of *Körper* and *Integritätsbereiche* is developed, and rational and integral rational functions defined in relation to the elements of *Körper* and *Integritätsbereiche*. Chapter II contains the elements of the abstract theory of groups, developed in a manner analogous to the theory of *Körper*.

Chapters III and IV are devoted to the actual solution of the problem. Chapter III contains a complete solution from the theoretical standpoint, namely, the Toeplitz treatment of linear equations. This method consists essentially in reducing the problem of the solution of a given system of linear equations, satisfying the necessary condition for solution, to that of the solution of an equivalent system of linearly independent equations for which the necessary condition is proved to be sufficient. A vector is defined in this chapter as the coefficients of a linear form, and a matrix as m vectors, each of which is n-membered. Chapter IV differs but little in the material included from the usual chapter on determinants and linear equations contained in elementary texts on the theory of equations. The treatment, owing to the excellent foundations laid in the first two chapters, is most rigorous. It is introduced by a brief treatment of permutation groups.

Notwithstanding the small size of this volume, the author sacrifices nothing of rigor and clarity to compactness. There are several sets of examples in the first two chapters; but, if the reader wishes to apply the theory, he is forced to look elsewhere for problems.

L. T. Moore

NOTES

The opening number of volume 28, series 2, of the Annals of Mathematics (December, 1926) contains: *On the problem of coloring maps in four colors*, I, by C. N. Reynolds; *Some theorems on deducibility*, by C. H. Langford; *Note on certain associated systems of linear equalities and inequalities*, by L. L. Dines; *Some further theorems concerning the formation of chains*, by A. Arwin; *Note on the extensions of groups to obtain nth roots*, by M. H. Ingraham; *Periodic solutions of linear differential equations*, by W. B. Fite; *On the inversion of the order of integration of a two fold iterated integral*, by H. J. Ettlinger; *The summability of single and multiple Fourier series*, by H. W. Bailey; *The arithmetic of a general algebra*, by O. C. Hazlett.

The Physico-Mathematical Society of Leningrad has begun, in 1926, the publication of a Journal. Most of the articles are in Russian, but abstracts in French, German, or English are provided.

Professor J. C. Fields, president of the international mathematical congress held at Toronto in 1924, reports that the printing of the Proceedings of this congress is now well under way. The completed work will be in two volumes, aggregating about 1800 quarto pages; it is hoped that it will appear in the summer of 1927.

At the annual meeting of the London Mathematical Society, held November 11, 1926, the following officers were elected: Professor G. H. Hardy, president; Professor S. Chapman, Professor A. L. Dixon, and Mr. J. E. Littlewood, vice-presidents; Dr. A. E. Western, treasurer; Professor H. Hilton, librarian; Professor G. N. Watson and Mr. F. P. White, secretaries.

The Paris Academy of Sciences announces the award of the following prizes for 1926: the Poncelet prize to Paul Montel, for his mathematical work; the Francœur prize to Gaston Julia, for his work in the theory of functions; the Montyon prize to Kyrille Popoff, of the University of Sofia, for his *Les Méthodes d'Intégration de Poincaré et le Problème de la Balistique Extérieure*; the Bordin prize to Serge Bernstein, of the University of Kharkoff, for his *Leçons sur les Propriétés Extrémales et la Meilleure Approximation des Fonctions Analytiques d'une Variable Réelle*; the de Parville prize to Antoine Alayrac, for his *Mécanique de l'Aviation*; the Valz prize to Frank Schlesinger, of Yale University, for his astronomical work, especially in the measurement of stellar parallaxes; the Grand Prize in mathematics to E. B. de Fontviolant, for his work on the resistance of materials; the de Joest prize to A. Quemper de Lanascol, for his *Géométrie du Compas*; the Jules Mahyer prize to Louis de Broglie, for his work in

quantum theory; the Jérome Ponti prize to Maurice Fréchet, for his work in the theory of assemblages and in the calculus of functionals; a prize from the Gegner foundation to René Baire, for his mathematical work; a prize from the Becquerel foundation to Georges Bruhat, for his work in theoretical physics.

The New York Academy of Sciences has awarded its A. Cressy Morrison prize for a research paper on the source of solar energy (see this Bulletin, vol. 32, p. 722) to Donald H. Menzel, of the Lick Observatory.

Associate Professor Emil Artin, of the University of Hamburg, has been promoted to a full professorship of mathematics.

Dr. Erich Kamke, of the University of Münster, has been appointed associate professor of mathematics at the University of Tübingen.

The Daniel Guggenheim Fund for Research in Aeronautics has made a grant of $78,000 to the University of Michigan; this will be used to complete the construction of wind tunnels and to establish a new professorship in aeronautics.

At the University of Wisconsin, Professor E. Schrödinger, of the University of Zurich, delivered a course of lectures in January, 1927, on recent developments in quantum mechanics; Professor Peter Debye, of the Zurich Technical School, has been appointed acting professor of mathematical physics for the second semester of 1926-1927.

Mr. L. G. Butler has been appointed professor of mathematics at Albany College, Albany, Oregon.

Professor Alan D. Campbell, of the University of Arkansas, has accepted an associate professorship at Syracuse University, and will assume his duties there next September.

Dr. D. R. Davis has been appointed assistant professor of mathematics at the University of Oregon.

Mr. P. S. Dwyer has been appointed assistant professor of mathematics at Antioch College.

Dr. W. W. Elliott has been appointed assistant professor of mathematics at Duke University.

The following appointments to instructorships in mathematics are announced: Columbia University, Dr. Edgar Dehn; Lehigh University, Dr. George Riddle; Yale University, Mr. T. W. Moore.

Dr. Joaquin de Mendizàbal-Tamborrel, geographical and military engineer in the Mexican army, and author of several books of mathematical tables, died September 8, 1926. Dr. de Mendizàbal-Tamborrel had been a member of this Society since 1892.

NEW PUBLICATIONS

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BROSE (H. L.). See SOMMERFELD (A.).

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THE FEBRUARY MEETING IN NEW YORK

The two hundred fifty-third regular meeting of the Society was held at Columbia University, on Saturday, February 26, 1927, extending through the usual morning and afternoon sessions. The attendance included the following seventy-two members:

W. L. Ayres, W. M. Bond, Carrie, L. D. Cummings, Dadourian, Dehn, Doermann, Douglas, Eisenhart, Ettlinger, Fiske, Fite, D. A. Flanders, Fort, Gehman, Gill, Glenn, Gronwall, C. C. Grove, Guggenbühl, Haskins, Hedlund, Himwich, Joffe, R. A. Johnson, Kasner, Kellogg, Kholodovsky, Kline, Koopman, Kuhn, Langman, Lefschetz, Littauer, Luby, MacColl, MacNeish, Michael, T. W. Moore, Mullins, F. H. Murray, K. E. O'Brien, Pederson, Pfeiffer, Pierpont, Pooler, Post, R. G. Putnam, Raudenbush, Reddick, Reid, R. G. D. Richardson, D. E. Richmond, Ritt, Seely, Skolnik, Smail, M. H. Stone, Takasu, E. M. Thomas, T. Y. Thomas, Tracey, Vallarta, Veblen, Evelyn Walker, Wedderburn, Weida, Weisner, Whittemore, Wiener, W. A. Wilson, J. W. Young.

There was no meeting of the Council or of the Trustees of the Society.

Professor Wiener presided at the morning session, and Vice-President Kellogg in the afternoon.

At the request of the Program Committee, Professor D. J. Struik, of the Deft Technical School, now visiting professor at the Massachusetts Institute of Technology, delivered an address, at the beginning of the afternoon session, entitled *The geometry of linear displacements*; this paper will be published in full in an early issue of this Bulletin.

Titles and abstracts of the other papers read at this meeting follow below. The papers of Ayers (first paper), Cramlet, Franklin, Garver, Hollcroft, Rosebrugh, Silverman and Tamarkin, Stone, Thomas, G. T. Whyburn, W. M. Whyburn, Widder, Wiener (first three papers), and Wilder were read by title. Mr. Gourin was introduced by Professor Ritt, and Professor Rosebrugh by the Secretary.

1. Professor H. J. Ettlinger: *On sequences of horizontal (step) functions.*

If E is a subset of points of a cube S in m dimensions, and S is subdivided into subcells by means of m sets of n hyperplanes parallel to the fundamental coordinate planes, then a horizontal function of index $k=n^m$ is a function defined on E which is constant at all points of the subset E contained in a given subcell of S. An infinite sequence of horizontal functions is a set of horizontal functions defined for $n=1, 2, 3, \cdots$. If a function defined on E is the limit almost everywhere of an infinite sequence of horizontal functions, it is called a function of class M. It is the object of the present paper to make a systematic study of the properties of functions of class M.

2. Dr. T. H. Gronwall: *Longitudinal vibrations of a liquid contained in a tube with elastic walls.*

An experimental method for determining the velocity of sound in a liquid consists in closing the lower end of a vertical metal tube by a membrane which is excited by an alternating current of known frequency, and filling the tube with liquid up to the point where resonance is obtained. The characteristic equation determining the frequencies of the vibrations of the liquid is set up, in the form of an infinite determinant, the elements of which contain products of Bessel functions, and, from this, approximate formulas of simple form are derived, which are directly applicable to the experimental data.

3. Dr. T. H. Gronwall: *On the existence and properties of the solutions of a certain differential equation of the second order.*

Under certain very general assumptions on $f(x, y)$ and $g(x, y)$, it is shown that the differential equation $(d/dx)[g(x, y)dy/dx]=f(x, y)$ with any one of the four boundary conditions (A) $y(x_0)=y_1$, $y(x_1)=y_1$; (B) $g(x_0, y(x_0))y'(x_0)=\text{const.}$, $y(x_1)=y_1$; (C) $y(x_1)=y_1$, $y(x)\to 0$ as $x\to\infty$; (D) $g(x_0, y(x_0))y'(x_0)=\text{const.}$, $y(x)\to 0$ as $x\to\infty$, has a solution which is unique. It is shown that any solution has at most one extreme, and a very general comparison theorem for the solutions of two equations of this type is established. A special case of the above equation with boundary conditions (D) occurs in the Debye-Hückel theory of solutions of strong electrolytes.

4. Dr. T. H. Gronwall: *On the converse of Euler's theorem for homogeneous functions.*

A homogeneous function $\phi(x_1, x_2, \cdots, x_n)$ of degree k satisfies Euler's partial differential equation (where m is any integer)

$$\sum x_{\nu_1} x_{\nu_2} \cdots x_{\nu_m} \phi_{\nu_1, \nu_2, \ldots, \nu_m} = k(k-1)\cdots(k-m+1)\phi,$$

where the subscripts on ϕ denote partial derivatives, and the summation

extends over all subscripts of sum m. It is shown that the general solution of this equation is

$$\phi = \sum \phi^{(\rho)}(x_1, \cdots, x_m) + \log x_1 \cdot \sum{}' \psi^{(\rho)}(x_1, \cdots, x_m),$$

where $\phi^{(\rho)}$ and $\psi^{(\rho)}$ are arbitrary homogeneous functions of degree ρ, and the first sum extends over all the roots ρ of $\rho(\rho-1)\cdots(\rho-m+1) = k(k-1)\cdots(k-m+1)$, while the second sum extends the over all double roots (if any; roots of higher multiplicity do not occur).

5. Dr. Jesse Douglas (National Research Fellow): *Reduction to integral equations of the problem of Plateau for the case of two contours.*

Let $x_i = {}^1f_i(t)$, $x_i = {}^2f_i(t)$, $i = 1, 2, \cdots, n$, represent the given two contours Γ_1, Γ_2, supposed contained in a euclidean space of n dimensions. The $2n$ functions f are presumed to be periodic of common least positive period Ω. The problem of Plateau is formulated so as to require a doubly-connected portion M of a minimal surface bounded by Γ_1, Γ_2, and at the same time a conformal map of M on a circular ring, outer radius 1, inner radius, originally unknown, q. When Γ_1, Γ_2 are the boundaries of a doubly-connected portion of a plane, this reduces to the Riemann mapping problem. This paper reduces the Plateau problem to a system of two integral equations, for two functions $\phi_1(t)$, $\phi_2(t)$ of the points of Γ_1, Γ_2 respectively, together with a third equation for the determination of $p = \log(1/q)$.

6. Dr. Jesse Douglas: *Extremals and transversality of the general calculus of variations problem of the first order in space.*

Every calculus of variations problem,

$$(1) \quad \int F(x, y, z, y', z')dx = \text{minimum},$$

defines two geometric objects: first, its extremals, a family F of ∞^4 curves in space; second, its transversality T, which is essentially a certain correspondence between lineal elements λ and surface elements σ such that corresponding elements have the same base point. The present paper establishes a simple criterion for determining of a given family F of ∞^4 curves, together with a given transversality T, whether or not they can be identified respectively with the extremals and transversality of a problem of the form (1). The criterion is that the ∞^2 curves of F which meet any one surface (union) Σ transversality must always admit ∞^1 transversal surfaces. The proof is rather long, and will be published in the April, 1927, number of the Transactions of this Society.

7. Mr. Eli Gourin: *On the essentiality of arbitrary functions.*

Consider an analytic function $F(p, q, p', q')$, where p, q are arbitrary functions of x and p', q' their respective derivatives. The paper gives a set of conditions necessary and sufficient for the existence of a relation $F(p, q, p', q') = S(\alpha, \alpha')$, where α is a function of p and q and α' is $d\alpha/dx$.

8. Mr. H. W. Raudenbush: *Note on Hilbert's thirteenth problem.*

It is proved that f defined by the equation

$$f^7+xf^3+yf^2+zf+1=0$$

cannot have any one of the following forms: $F(\alpha(x, y), \beta(y, z))$, $F(\alpha(x, y), \beta(x, z))$, or $F(\alpha(x, z), \beta(y, z))$, where α, β, and $F(\alpha, \beta)$ are analytic functions.

9. Professor O. E. Glenn: *A theory of integers in relation to the iteration of algebraic functions.*

The content and methods of this paper are presented in the form of original algorithms on the number system. The object is to develop a theory of indices in which Δ of the fundamental relation $\Delta^r \equiv \Delta \pmod{n}$ instead of being an integer, is a functional operation upon an integer, and Δ^r is its rth iteration. The index r is determined with considerable generality as the serial number of a term in a sequence of resultants. The resultants are modular and appertain to the theory due to Stieltjes. Applications are noted, to problems relating to two-dimensional residue systems.

10. Mr. T. W. Moore: *On the invariant combinants of two binary quintics.* Preliminary report.

In a study of the rational space quintic curve the author has been led to the set of combinant forms of two binary quintics. From work of Gordan and Stroh it is known that the complete system of combinants of the two quintics is the same as the simultaneous system of covariant forms of the binary octavic and quartic, which are the first and third transvectants respectively of the two given forms. In turn, this system is identical, except for invariant factors, with that of another binary octavic apolar to the two quintics, and so the number of combinant forms is known. Starting with the three syzygies given by L. Berzolari in vol. 7 of the Rendiconti di Palermo, this paper sets up the complete system of invariant combinants of the two forms of the fifth order, together with the necessary covariant forms required for this purpose.

11. Dr. Louis Weisner: *Invariant theory of plane n-points and n-lines.*

The author investigates the projective invariant theory of complete n-points and n-lines. The work naturally falls into two parts: the consideration of concomitants expressed in terms of the coordinates of the vertices of the n-points and of the sides of the n-lines, and the consideration of concomitants expressed in terms of the coefficients of the degenerate ternary line-curves and point-curves defined by the n-points and n-lines respectively. Finiteness theorems are proved, a symbolic theory is developed, and a theorem somewhat analogous to Hermite's reciprocity law is proved.

12. Professor James Pierpont: *Optics in spaces of constant curvature.* Second paper.

In a paper read at the last Annual Meeting of the Society, the author showed that in space of constant non-vanishing curvature, central optical collineation gives only an image congruent with the object. Thus this elegant method is not available in elliptic or hyperbolic spaces. However, instead of rays of light one may use waves of light, adopting the methods laid down by C. S. Hastings (*On certain new methods and results in optics*, Memoirs of the National Academy vol. 6 (1893), p. 37). In general it is found that the results are similar to those of euclidean optics. As an example, the familiar formula for a convex lens, $(1/r)+(1/r_1)=((1-n)/n)((1/l)+(1/l_1))$, $n<1$, becomes for hyperbolic spaces, $\operatorname{ctnh}(\rho_1/R)+\operatorname{ctnh}(\rho/R)=((1-n)/n)(\operatorname{ctnh}(\lambda/R)+\operatorname{ctnh}(\lambda_1/R))$, and a similar formula for elliptic space.

13. Dr. M. H. Stone: *A characteristic property of certain sets of trigonometric functions.*

In this paper the nature of a closed normal orthogonal set of functions whose derivatives constitute an orthogonal set is discussed. It is shown that when certain auxiliary conditions are satisfied by the functions of the set they are necessarily trigonometric and satisfy a differential system of the second order.

14. Dr. M. H. Stone: *The normal frequency function and general frequency functions.*

The problem considered is that of approximating to an arbitrary function by a linear combination of functions of the form $e^{-\lambda(x-\mu)^2}$, where λ and μ are constants, $\lambda>0$. Three theorems concerning the possibility of such representation are obtained with the aid of the theorem of Riesz-Fischer.

15. Professor Norbert Wiener: *On the closure of sets of trigonometric functions.*

The author discusses the closure of the set of functions $\cos \lambda_n x$.

16. Professor Norbert Wiener: *A new definition of almost periodic functions.*

The author gives a new necessary and sufficient condition for a function to be almost periodic.

17. Professor Norbert Wiener: *Laplacians and linear functionals.*

The author develops a new theory of functions of limited total variation in n dimensions, connected with the Laplacian operator. He applies this to the theory of linear functionals and to that of multi-dimensional trigonometric developments.

18. Mr. G. T. Whyburn: *Concerning the complementary domains of continua.*

The author shows that (1) if the complementary domains of a boundel continuum M have the property that for any number $\epsilon>0$ at most a finite number of them are of diameter $>\epsilon$, and P is any point of M not belonging to the boundary of any complementary domain of M, then (a) for every number $d>0$, M contains a subcontinuum of diameter $<d$ which separates P from all those points of the plane whose distance from P is $>d$, and (b) M is regular at P; (2) the complementary domains of a bounded continuum M have the above mentioned property if and only if for every point P of M and every number $d>0$ there exists a continuum N of diameter $<d$ which separates P from all those points of the plane whose distances from P are $>d$ and which is a subset of $M+$ a finite number of the complementary domains of M; and (3) if the complementary domains of an indecomposable continuum M have the above mentioned property, then M is the boundary of at least one of its complementary domains.

19. Mr. G. T. Whyburn: *Concerning the sum of the boundaries of the complementary domains of a plane continuous curve.*

In this paper it is shown that if H denotes the point set obtained by adding together the boundaries of all the complementary domains of a continuous curve, M every subcontinuum of which is a continuous curve, then (1), the set $M-H$ contains no continuum, and (2), the set H is connected. It is further shown that if M is any continuous curve which contains no domain, then in order that the point set H obtained by adding together the boundaries of all the complementary domains of M should be connected, it is necessary and sufficient that the set $M-H$ should contain no closed subset which separates the plane.

20. Mr. G. T. Whyburn: *Concerning the open subsets of a continuous curve.*

A boundary point P of a domain D in a space E will be said to be *regularly accessible* from D provided it is true that for every positive number ϵ, a $\delta_\epsilon>0$ exists, such that every point X of D, whose distance from P is $<\delta_\epsilon$, may be joined to P by an arc XP of diameter $<\epsilon$, and such that $XP-P$ lies in D. In this paper it is shown that (1) if E is the euclidean plane, the above definition is equivalent to accessibility from all sides as introduced by Schoenflies; (2) if E is any plane continuous curve, every boundary point of a domain D is regularly accessible from D if and only if D has property S (R. L. Moore, Fundamenta Mathematicae, vol. 3, p. 232); (3) if E is a plane continuous curve, every subcontinuum of which is a continuous curve, every boundary point of any domain D is regularly accessible from D; (4) a connected open subset of a plane continuous curve is cyclically connected if and only if it has no cut point; (5) if M and N are continuous curves, $N\subset M$, then every maximal connected subset of $M-N$ has property S.

21. Professor W. M. Whyburn: *Existence and oscillation theorems for second-order differential systems whose boundary conditions contain definite integrals.*

In this paper, the pair of differential equations $dy/dx = K(x, \lambda)z$, $dz/dx = G(x, \lambda)y$, together with boundary conditions, one of which contains a definite integral (Lebesgue), is treated. The functions K and G are assumed continuous in λ for each x on the interval of definition, and summable in x for each λ on the λ interval. For every x and every λ on the intervals of definition, K and G are less than (or equal to) a summable function of x. In the presence of the restrictions that K/G and A/G be non-decreasing functions of x for each fixed λ, where $A(x, \lambda)$ is the coefficient of $y(x, \lambda)$ in the integral condition, the existence of an infinite set of characteristic values of λ is established. Oscillation theorems are given for the solutions of the system, for linear combinations of these solutions, and for indefinite integrals containing these solutions in their integrands. Similar theorems are proved for $z(x, \lambda)$.

22. Professor W. M. Whyburn and Mr. G. T. Whyburn: *On the Porter-Vitali theorem.*

The authors give a new and shorter proof of a theorem due to Arzela that every bounded sequence of functions which are analytic in a closed region R contains a subsequence which converges to an analytic limit function everywhere in R. On the basis of this theorem they obtain a very short proof of the Porter-Vitali theorem that if a uniformly bounded sequence of functions analytic in a region R converges over a set of points having a limit point in R, then the whole sequence converges throughout R to an analytic limit function. A new application of these theorems is given.

23. Professor R. L. Wilder: *A point set which has no true quasi-components, and which becomes connected upon the addition of a single point.*

Sierpinski has given (Fundamenta Mathematicae, vol. 2 (1921), pp. 81–85) an example of a point set which has no true quasi-components, but which, on the addition of a single point, contains a quasi-component. In the present paper the author shows, by an example, that there exists a point set M which has no true quasi-component, and which, on the addition of a single point, becomes connected. Incidentally, the set M furnishes another example of a certain type of point set recently constructed by Mazurkiewicz (Fundamenta Mathematicae, vol. 2, pp. 201–205); i. e., a quasi-connected set M of which every two points are separated in M.

24. Mr. W. L. Ayres: *Note on a theorem concerning continuous curves.*

In a number of recent applications the following theorem has been found useful: If M is a plane continuous curve (bounded or unbounded)

and N is a bounded continuous curve which is a subset of M, then, for any positive number ϵ, $M-N$ contains at most a finite number of maximal connected subsets of diameter greater than ϵ. This is an extension of a result previously presented (*Concerning the arcs and domains of a continuous curve*, this Bulletin, vol. 32 (1926), p. 37).

25. Professor T. R. Hollcroft: *On higher point correspondences.*

When both planes are multiple, only (2, 2) and (2, 3) point correspondences have been classified and studied in detail. In this paper, (3, 3) point correspondences between two planes are completely classified. Fifteen types of general (3, 3) point correspondences and fifteen types of (3, 3) compound involutions are found, and the characteristic features of each type are given. The number of types each of (2, 4), (3, 4), and (4, 4) point correspondences and the general characteristics of these correspondences are also found. As in the case of (1, 4) involutions when multiplicity of one or both planes is greater than four, no new features are introduced.

26. Dr. Raymond Garver: *A theorem concerning Tschirnhaus transformations.*

The following theorem is proved: Every Tschirnhaus transformation of degree $n-1$ which transforms a sufficiently general principal equation (in x) of degree $n>4$ into another principal equation can be expressed as a linear function of ϕ and ψ, where ϕ and ψ are polynomials (in x) with special properties. (By a principal equation we mean one in which the second and third coefficients are zero.) Transformations of this kind are especially convenient to work with, and we thus show that in certain problems we suffer no loss of generality by restricting our attention to them.

27. Dr. Raymond Garver: *Division algebras of order sixteen.*

In this paper a type of division algebra of order sixteen, based on a root of a quartic equation irreducible in the field of rational numbers and having the Galios group (1), (12) (34), (13) (24), (14) (23), is set up, in accordance with the conditions given in Dickson's *New division algebras* (Transactions of this Society, April, 1926). For this case there are six conditions, which restrict the coefficients of the quartic and the constants of multiplication of the algebra. Cases are exhibited where all the conditions are satisfied.

28. Mr. C. M. Cramlet: *The derivation of projective algebraic invariants by tensor algebra.*

In this paper it is proved that any algebraic invariant can be readily written as an inner product of the fundamental tensors, which are the coefficients of the forms, and the determinant tensor

$$\delta^{r_1 \cdots r_n}_{s_1 \cdots s_n}.$$

The methods apply also to non-symmetric tensors. Although invariants written in this manner are explicitly expressed, the transition to the classical symbolic form involves a mere change of notation.

29. Dr. D. V. Widder (National Research Fellow): *The transformations of the special relativity theory.*

In the special relativity theory it is desired to determine those real one-to-one transformations $x_i = x_i(y_1, y_2, y_3, y_4)$ which make

$$dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 = \mu(dy_1^2 + dy_2^2 + dy_3^2 - dy_4^2),$$

where μ is a function of y_1, y_2, y_3, y_4, in general different from zero. In this paper, all possible values of μ are determined, and the corresponding transformations are set up. It is found that the case $\mu =$ constant is the only one that leads to a transformation continuous throughout space. This continuity is a requirement of the relativity theory, so that in this way all other values of μ may be excluded.

30. Dr. J. M. Thomas (National Research Fellow): *On systems of total differential equations.*

It is shown that, in applying the theorem given by O. Veblen and the writer (Annals of Mathematics, (2), vol. 27 (1926), p. 290) to systems of tensor differential equations, the process of differentiation can be chosen as partial or covariant at will. By adopting the latter alternative, the theorems for the existence of homogeneous first integrals in the geometry of paths (O. Veblen and T. Y. Thomas, Transactions of this Society, vol. 25 (1923), pp. 551–608) are readily deduced as special cases.

31. Professor J. W. Young: *On the definition of the Scorza-Dickson algebras.*

The definition of "an algebra A over a field F" given by Dickson in his *Algebras and their Arithmetics*, pp. 9, 10, consists of five groups of postulates I-V, of which groups II and III refer exclusively to multiplication between the elements of A and the marks of F. In the present paper it is shown that for a large part at least, if not for all, of the later theory the postulate groups II and III are unnecessary and all reference to a field F can be avoided. A slight modification of postulate group I (insuring merely that A forms an abelian group with respect to addition) and postulate groups IV and V constitute a sufficient basis from which to derive all the theorems in the first three chapters of Dickson's book. This is accomplished by using the idea of the partitions of the addition group (see abstract of a paper by the author presented to the Society September 9, 1926, this Bulletin, vol. 32, p. 588).

32. Professor Norbert Wiener and Dr. Robert Schmidt: *A general form of Tauberian theorem.*

By using the Fourier integral and the related theory of complex moments, the authors obtain a very general Tauberian theorem, in-

cluding most known Tauberian theorems as special cases, in particular, the Tauberian theorem for Lambert series, from which the prime number theorem may be deduced, as Hardy has shown.

33. Professor Norbert Wiener: *The approximation theorem for almost periodic functions.*

The author gives a new proof of the approximation theorem for almost periodic functions.

34. Mr. W. L. Ayres: *On the separation of points of a continuous curve by arcs and simple closed curves.*

If x and y are distinct points of a plane continuous curve M (bounded or unbounded) and M_{xy} denotes the set of all points P, such that P lies on some simple continuous arc of M whose end points are x and y, then M_{xy} is a continuous curve. With the use of this result, the following two theorems are proved: (1) a necessary and sufficient condition that M contain a point which separates x and y in M is that every two simple continuous arcs of M whose end points are x and y have a point distinct from x and y in common; (2) if M is bounded, a necessary and sufficient condition that M contain a *simple continuous arc* which separates x and y in M is that $M-(x+y)$ contain only one maximal connected subset which has both x and y as limit points; (3) if M is bounded, a necessary and sufficient condition that M contain a simple closed curve J, such that one of the two points x or y is interior to J and the other is exterior, is that both x and y do not lie on the boundary of the same complementary domain of M. In (2), a point is considered to be a special case of an arc.

35. Mr. W. L. Ayres: *Concerning the boundaries of domains of a continuous curve.*

If D is a domain of a plane continuous curve M, B is its M-boundary, and P is a point of $M-(D+B)$, then the M-boundary of the maximal connected subset of $M-(D+B)$ containing P is called the M-boundary of D with respect to P. These theorems are then proved: if N is a continuous curve which is a subset of M, every closed and connected subset of the M-boundary of a complementary M-domain of M is a continuous curve; (2) if β is the M-boundary of D with respect to P, then β is the entire M-boundary of some M-domain which contains D; (3) if every maximal connected subset of B is a continuous curve and either D or the maximal connected subset of $M-(D+B)$ which contains P is bounded, then every maximal connected subset of the M-boundary of D with respect to P is either a point, an arc, or a simple closed curve, and if one maximal connected subset is a simple closed curve J, then J is the entire M-boundary of D with respect to P; (4) if β is the M-boundary of D with respect to P, R is the maximal connected subset of $M-(D+B)$ containing the point P, and Q is any point of the maximal connected subset of $M-(R+\beta)$ which contains D, then β is the M-boudnary of R with respect to Q.

36. Professor Edward Kasner: *Equilong geometry.*

In previous papers, the author has studied various questions of equilong geometry mainly with the object of comparison with the analogous (but in general more difficult) questions of conformal geometry (see Proceedings of the Fifth International Congress, Cambridge, 1912, vol. 2, pp. 81–87, and this Bulletin, vol. 23 (1917), pp. 341–347). In the present paper, he discusses more fully the analogy of isothermal systems of curves in equilong geometry, equilong symmetry (see this Bulletin, vol. 24 (1918), p. 471), and the invariants of irregular analytic curves. Some types have invariants and some have not, in general as in the conformal theory (see Transactions of this Society, vol. 16 (1915), p. 339), but the detailed results are different.

37. Professor Edward Kasner: *Irregular differential invariants.* I: *General theory.*

The ordinary concept of differential element and differential invariant relates to curves as represented by polynomials or power series with integer exponents. The author has extended this theory by allowing irregular elements or curves where the series involve fractional exponents. This theory has been carried out for the conformal group of the plane in his paper in the Transactions of this Society, vol. 16 (1915), pp. 333–349 (see also this Bulletin, vol. 21 (1915), p. 280), and for the equilong group in his first paper read at this meeting of the Society. In the present paper he considers continuous groups in general, and in later papers will discuss the irregular invariants of the infinite groups of contact transformations and of point transformations, as well as the finite projective and affine groups.

38. Professor Philip Franklin: *Analytic functions with assigned values at an infinite number of points.*

The question of determining when the values of a function at an infinite set of points in a finite region determine a function analytic in this region, and if so the function in question, has recently been raised by T. H. Hildebrandt (this Bulletin, vol. 32 (1926), p. 552). This question is answered in the present note.

39. Professors L. L. Silverman and J. D. Tamarkin: *On the generalization of Abel's theorem for certain definitions of summability.*

Abel's classical theorem on power series was first generalized by Frobenius, for the case when the series is summable by arithmetic means. Subsequently, Abel's theorem has been extended by various authors to other definitions of summability: Cesàro's and Hölder's of all orders, Borel's and Euler's. The purpose of the present paper is to discuss the extension of Abel's theorem to the general class of analytically regular transformations (Hurwitz and Silverman, Transactions of this Society,

vol. 18 (1917), pp. 1–20); the result obtained is that among these transformations only those equivalent to Hölder's admit of the generalization of Abel's theorem. The general regular transformations permutable with M, as given by Hausdorff, are also discussed, and, finally, Abel's theorem is extended to the definition of summability of Nörlund.

40. Professor T. R. Rosebrugh: *Quantic determinants.*

In this discussion, any number of linear substitutions are given, independent of each other as to coefficients and variables. The substitution is formed which gives all corresponding new monomials linearly in terms of all the possible old monomials whose factors are taken from all substitutions, and from each to a total degree assigned to that substitution. The algebraic composition of this determinant is found without assumptions by examination of the effect of certain linear partial-derivative operators upon it. From this general result, by specialization in different directions, theorems previously given by Kronecker, Faà di Bruno, and Scholtz may be obtained.

R. G. D. Richardson,
Secretary

A CORRECTION

In the last line of page 140 of the March-April issue (this Bulletin, vol. 33 (1927)),

instead of $\{f(x)-[T_n(x)]\}^2$, read $\{f(x)-[T_n(x)]^2\}^2$.

THE FIFTY-FIRST REGULAR MEETING OF THE SAN FRANCISCO SECTION

The fifty-first regular meeting of the San Francisco Section of the Society was held at Stanford University on Saturday, April 2, 1927. The Chairman, Professor Allardice, presided. The total attendance was 45, including the following 29 members of the Society:

Allardice, Barter, Bernstein, Blichfeldt, Buck, Cairns, Cajori, Collier, Corbin, Eells, R. L. Green, M. W. Haskell, E. R. Hedrick, Hoskins, Hotelling, Glenn James, Vern James, D. H. Lehmer, D. N. Lehmer, S. H. Levy, W. A. Manning, Moreno, Neikirk, Noble, T. M. Putnam, Pauline Sperry, Stager, A. R. Williams, Wong.

It was decided to hold the next Spring meeting at Stanford University on Saturday, April 7, 1928.

Titles and abstracts of papers read at the meeting follow. Professor Bell's paper was read by title.

1. Dr. J. D. Barter: *On differential p-vectors.*

Those of the fundamental properties of differential p-vectors which appear pertinent to the theory of partial differential equations and of integral invariants are developed. Since a differential is characterized by the algebraic invariants of itself and of its vector derivative, considerable attention is given to these. Exception is taken to certain theorems and proofs of Gegenbauer, Schouten, and others, and corrections suggested. The nature of the automorphic transformations of differential p-vectors is specified and consequences indicated.

2. Dr. J. D. Barter: *Powers of determinants.*

It has seemed desirable to place on record certain determinant forms which may be given to the powers of determinants since the direct proof of the relations in question appears difficult and useful applications seem probable.

3. Dr. J. D. Barter: *On covariant differentiation.* Preliminary communication.

The two types of covariant differentiation, termed respectively albraic and vector, are derived and their mutual relation developed. Certain apparently novel results are obtained and applied to the theory of Jacobi's last multiplier and its generalization.

4. Professor E. T. Bell: *The arithmetic of logic.*

It is shown that the algebra L of logic is abstractly identical with certain parts of the arithmetic A of the rational integers. Formal equiva-

lents in L of the G.C.D., L.C.M., congruences, primes, arithmetical divisibility and the unique factorization law in A are developed. Lasker's concept of the residual in modular systems is necessary to complete the theory of congruences in L; it gives the equivalent in L of dividing (when possible) both members of a congruence in A by a common factor. There are two abstractly identical arithmetics of L, corresponding to the Peirce-Schröder dualism in L. The theory has applications to the classification of "doctrinal functions" (Keyser), or "system functions" (Sheffer). This paper will appear in the Transactions of this Society.

5. Professor W. D. Cairns: *Development in a system of approximately orthogonal functions.*

As is well known the terms of the expansion of $(1+1)^{2n}$, multiplied by a conveniently chosen factor, approach the ordinates of the curve $y=e^{-x^2/2}$ as n becomes infinite. This series of terms, and the principal parts of the successive differences formed therefrom, give polynomial functions which are dealt with in this paper. When combined by a summation analogous to the integration of orthogonal functions, these functions are shown to be orthogonal to within an error involving terms in $1/n$. A development of suitably restricted functions in terms of these functions is given, the approximation holding to within the order $1/n$.

6. Professor Florian Cajori: *The earliest arithmetic published in America.*

The author gives a description of the arithmetic of Pedro de Paz, a copy of which (partly in the original and partly in photostat reproduction) has been secured by the Library of the University of California. It was published in the city of Mexico in 1623. It is the earliest volume, on arithmetic exclusively, printed in America.

7. Professor Florian Cajori: *Unpublished letters of C. H. Schumacher and W. Struve to F. R. Hassler.*

The author gives an account of letters written by C. H. Schumacher and W. Struve to F. R. Hassler relating to astronomical publications, the design of improved geodetic instruments, and the procuring from Europe of expert copper engravers.

8. Professor Florian Cajori: *The logarithms of Napier.*

The author, with the aid of quotations from Napier, points out the invalidity of the argument against the accuracy of the formula Nap. $\log x = 10^7 \log (10^7/x)$, given on page 538 of the Proceedings of the National Academy of Sciences (September, 1926).

9. Professor E. R. Hedrick: *On certain expansive properties of neighborhoods in general spaces.*

Let us define as an expansive property of a region any property that holds true in any region provided it holds true in some region interior to the first one. Blumberg has recently presented a paper to the Society

(Philadelphia, Dec. 29, 1926) regarding extensions of the theorem that any derived set is closed. If a set is defined in any space that has the enclosable property, by means of an expansive property of the enclosing regions about each point of the set, it is shown in the present paper that the set is closed. Examples are cited, and applications to the theory, particularly to the validity of a Borel theorem, are mentioned.

10. Dr. Harold Hotelling: *Differential equations subject to error.*

A generalization of the theory of differential equations is obtained if we associate with every point, instead of a fixed direction, a frequency distribution of directions. In this way a frequency distribution can be be found for curves. It is argued in this paper that when an empirical law has been derived from a differential equation and statistical verification is sought, a more accurate test may often be applied to the differential equation itself than to its integral. The constants in the differential equation are then best determined first, the constants of integration afterward. These considerations are applied to the "logistic" hypothesis and the numerical work carried out for the population of the United States. Intercensal interpolation becomes a problem in the calculus of variations. For estimates by means of this theory the determination of probable errors is connected with the theories of random migration and heat conduction.

11. Professor Glenn James: *A theorem on monotonic decreasing functions.*

In evaluating certain monotonic decreasing series, need for further classification of such series arises. Toward meeting this need we formulate and prove the following theorem. *If* $\phi_0(x)$ *is positive and possesses derivatives to the nth order, those of even order being positive or zero, and those of odd order being negative or zero, and* $\phi_i(x)$ *denotes* $\phi_{i-1}(x)-\int_x^{x+b}\phi_{i-1}(z)dz$, $1\geqq b\geqq 0$, *then the same rule of signs applies to* $\phi_{i-1}(x)(i=1,2,3,\cdots,n)$ *and its first* $n-i$ *derivatives.* The theorem can be extended to functions of several variables.

12. Mr. D. H. Lehmer: *Note on the largest Mersenne number.*

The remarkable accuracy with which the famous statement of Mersenne in 1644 represents the facts concerning the primality of 2^n-1 is well known. The present author finds however that the largest number, $2^{257}-1$, given by Mersenne as a prime is really composite. This fact is revealed by the application of Lucas' test to the number. It was found that the 256th term of the recurring series 4, 14, 194, is not divisible by $2^{257}-1$. It follows then this 78-digit number is composite although it is impossible to say what its factors are. The test was undertaken in order to verify a similar calculation made by M. Kraitchik leading to the same general result. It has not been possible as yet to compare the two calculations.

13. Mr. D. H. Lehmer: *On the converse of Fermat's theorem.*

This paper appears in full in the present issue of this Bulletin.

14. Professor L. I. Neikirk: *A class of continuous curves defined by motion which has no tangent lines.*

The first example of a continuous curve without tangents was given by Weierstrass in 1872. Since then several others have been given. The most notable are those given by Peano, by Hilbert, by E. H. Moore, (*On certain crinkly curves*), and by W. F. Osgood, (*A Jordan curve of positive area*). The novel feature of the curve here discussed is its definition by linear and rotational motion which makes the lack of tangents geometrically evident. The paper includes an analytic proof of this fact.

15. Dr. A. R. Williams: *The quintic surface with two double straight lines.*

The quintic surface with two double straight lines is obviously universal. The plane system is composed of quintics having in common two double points and twelve simple points. There must be a relation between these fourteen points; for if taken arbitrarily they would impose eighteen conditions on the quintics. It is shown that they are the base points of a pencil of quartics that have a common tangent at the two which are the double points of the quintics. The class of the surface is in general thirty-four. There are twelve pinch points, six on each double line. There are, in general, thirteen lines which meet both double lines. Certain special cases are noted, wherein nodes or additional lines are produced by particular positions of the base points.

16. Dr. B. C. Wong: *Sextic surfaces with a double septic curve.*

By an involutorial quartic transformation by means of four hyperquadric surfaces in 4-space a plane σ is transformed into a two-dimensional variety $\sum_2^6$ of order 6. Projecting $\sum_2^6$ from any point z in S_4 on to a S_3 we obtain a surface of order 6 with a double septic curve. By taking z in positions related to $\sum_2^6$ or by having the base points, 10 in number, in σ take special relative positions, we obtain numerous special cases of the surface. This paper will appear in the Annals of Mathematics.

B. A. Bernstein,
Secretary of the Section.

MATHEMATICS AND THE BIOLOGICAL SCIENCES*

BY H. B. WILLIAMS

We meet this afternoon to do honor to the memory of one whose influence on the development of scientific thought and achievement has been in many ways remarkable. The auspices under which this assembly has gathered is in itself an earnest of the regard in which the name of Josiah Willard Gibbs is held by the mathematicians of America. Forty years ago he delivered the vice-presidential address before the Section of Mathematics and Astronomy of the American Association for the Advancement of Science. The topic was *Multiple Algebra*.† His own contributions to this subject, particularly his development of the theory of dyadics, would be sufficient to establish without question his standing as a pure mathematician, while his vector analysis with its practical and convenient notation has been of no small service to the cultivators of mixed mathematics.

The soundness of his judgment in the field of physics is attested by the fact that he was among the first to take up and extend Maxwell's electromagnetic theory of light. In his obituary of Professor Gibbs it is remarked by Professor Bumstead‡ that these optical papers are noteworthy for the entire absence of special hypotheses regarding the connection between matter and ether. It seems to have been a characteristic trait with him to strip his problems of every unnecessary element before commencing extensive treatment of them.

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* The Fourth Josiah Willard Gibbs Lecture, read at Philadelphia, December 28, 1926, before a joint session of the American Mathematical Society and the American Association for the Advancement of Science.

† J. Willard Gibbs, *On multiple algebra*, Proceedings A. A. A. S., vol. 35 (1886), pp. 37–66. Also Gibbs, Collected Scientific Papers, vol. 2, p. 91.

‡ Bumstead, Henry A., *Josiah Willard Gibbs*, American Journal of Science, (4), vol. 16 (Sept. 1903). Also Gibbs, Scientific Papers, vol. 1.

Professor Hastings relates that Gibbs once replied to a complimentary remark about his work that if he had met with any success it was largely because he had found ways to avoid the mathematical difficulties! I am inclined to think that this was not intended altogether as a pleasantry. For although Gibbs met and overcame many serious mathematical difficulties in his work, he probably was well aware that he had avoided much unnecessary labor by this habit of first eliminating all but the bare essentials of his problem. Professor Bumstead remarks (loc. cit.) of these optical papers that it seems likely the considerations advanced in them would have sufficed firmly to establish the electromagnetic theory even if the experimental discoveries of Hertz had not supplied a more direct proof of its validity.

It is, however, his work in thermodynamics, particularly the great work, *On the equilibrium of heterogeneous substances*,* which has made the name of Josiah Willard Gibbs familiar in every part of the world where science is cultivated. The present year marks the fiftieth anniversary of the publication of the first part of this paper, the importance of which was not at once recognized. We shall hardly be surprised at this when we remember that the subjects treated are chiefly important to the chemist. Indeed this work may justly be said to have laid the foundation for theoretical chemistry. But in 1876, and for a good many years thereafter the number of chemists who had more than a nodding acquaintance with even the more elementary branches of mathematics was comparatively small. The paper is not only highly mathematical, but the methods employed are such as to make its reading a privilege open only to those who have had the foresight and industry to acquire a considerable mathematical armamentarium.

It is usually stated that the paper remained in comparative oblivion until its value was discovered by Professor Ostwald,

* Transactions of the Connecticut Academy of Arts and Sciences, vol. 3, pp. 108–248 and pp. 343–524. Also Gibbs, Scientific Papers, vol. 1, No. 3, pp. 56–353.

who, in 1891, translated it into German. However, I find in the presidential address of the late Lord Rayleigh,* given before the British Association at Montreal in 1884, the following comment, made in connection with his discussion of the dissipation of energy: "The foundations laid by Thomson now bear an edifice of no mean proportions, thanks to the labors of several physicists, among whom must be especially mentioned Willard Gibbs and Helmholtz. The former has elaborated a theory of the equilibrium of heterogeneous substances wide in its principles, and we cannot doubt, far reaching in its consequences." It thus appears that Lord Rayleigh had read and was fully cognizant of the importance of this paper prior to the Montreal meeting of 1884. His comment is at once a tribute to the genius of Willard Gibbs and a proof of the catholicity of his own scientific interests. In 1899 this paper was translated into French by LeChatelier and in 1906 it became more widely available to English readers through the publication of Professor Gibbs' collected Scientific Papers. To the influence of this paper it is largely due that chemistry, then an almost un-mathematical science, has so developed that the mathematical equipment now required by the student of chemistry differs but little from that which is requisite for the student of physics. The two subjects tend ever more and more to overlap.

Nor is it alone in theoretical chemistry that the influence of Gibbs is felt. It is hardly too much to say that chemical engineering and the whole vast edifice of chemical industry rest upon his work as their secure foundation.

The esteem in which Gibbs was held is reflected in many comments which may be found in the scientific literature. It is somewhat unusual to apply adjectives of commendatory character when quoting the work of another investigator and when this *is* done by a recognized master in his field, it signifies more than ordinary appreciation of the validity and

* Lord Rayleigh, Presidential Address, British Association Report, Montreal, 1884, pp. 1–23. Also Rayleigh, Scientific Papers, vol. 2, p. 342.

importance of the work. Quoting again from the works of Lord Rayleigh, in his paper, *On reflexion of light at a twin plane of a crystal*,* we find mention of "Professor Gibbs' excellent comparison of the elastic and electric theories of light with respect to the law of double refraction and dispersion of colors." Again in the paper on *Foam*† Lord Rayleigh alludes to "the masterly discussion of liquid films by Professor Willard Gibbs." In the review of Gibbs' collected papers by Jeans is the following: "The publication of the collected works of a great scientist serves the double purpose of forming a memorial to the genius of the writer and of increasing the usefulness of his labors by making his writings available to other workers. It is in connection with the second of these purposes rather than the first that one is inclined to consider the present volumes. They form, it is true, a memorial, and a fitting memorial to the greatness of Professor Gibbs' scientific work, but we feel that a memorial was hardly needed—his greatness is too well established and his work too well known for either to be enhanced at this stage by the publication of volumes of paper and ink."

In view of the fact that Professor Gibbs has exerted his greatest influence through the application of mathematical methods to the problems of a branch of science which, prior to his work, had not been regarded as a fertile field for theoretical cultivation, it seems not inappropriate that in meeting here to renew our recognition of the fundamental importance of his work we should turn our attention to the possibility of extending profitably the application of mathematical modes of thought to yet another field of scientific endeavor. To many the idea of applying mathematical methods to biological investigation may seem quite as unusual as did the idea of their application to chemistry at the

* Philosophical Magazine, vol. 26, p. 241; and Scientific Papers, vol. 3, p. 190.

† Proceedings of the Royal Institution, vol. 13 (1890), pp. 85–87. Also Scientific Papers, vol. 3, p. 359.

beginning of the last quarter of the nineteenth century. A tacit implication that the biological sciences are essentially un-mathematical subjects is conveyed by the titles under which the two sections of the Proceedings of the Royal Society of London are published. Section A is supposed to contain the mathematical and physical papers, while section B is the biological section. Notwithstanding this implication and the rather widespread impression that the problems of biology are too complicated or too indefinite, or both, to permit of mathematical treatment, certain of its questions have been so investigated and such studies still continue. Among those who have been successful in the application of mathematics to physiology may be mentioned no less a personage than Helmholtz. The mathematicians in my hearing all know of Helmholtz as a mathematician and the physicists know that he was regarded as one of the ablest physicists of his day, but because of his eminence in these fields it is easy to forget that he began his career as a physician and that his first scientific post was the professorship of physiology at Koenigsberg. From Koenigsberg he went to Bonn as professor of anatomy and physiology and from Bonn he was called to the professorship of physiology at Heidelberg. It was during these years that the ophthalmoscope was discovered and the great works on *The Science of Tone Perception* and on *Physiological Optics* were written. The latter work has recently been translated into English fifty years after its original publication. Although the translation was undertaken partly in commemoration of the centenary of his birth, it was welcomed more because this book is still, in many aspects of the subjects treated, the most authoritative work available. That it should remain a living book during all this period of intense scientific activity is probably due in no small measure to the precise habits of thought of the great investigator who wrote it. In this and in the work on sound will be found a wealth of examples of the application of mathematics to physiological problems. One might name a goodly number of men among the older

school of physiologists who made use of mathematics in various aspects of their work.

Shortly before the middle of the nineteenth century there began an epoch of great activity and productivity in organic chemistry. It was not long before the trend of thought and investigation in physiology began to show the influence of this activity in chemistry and physiological investigations have ever since tended to proceed more and more along chemical lines. As chemistry had not yet evolved into a mathematical science, it was not unnatural that physiologists should have tended away from mathematics in their training and there was a considerable period when this was largely true. There have always been a few individuals who either by chance or from personal predilection have brought to the study of physiology a considerable mathematical training and an occasional one who did so from conviction that this training is important for the most successful pursuit of his profession. In more recent years, and particularly since the advent of theoretical chemistry, there has been a growing tendency to reinstate mathematics as an essential part of the physiologist's intellectual equipment.

Looking now to the work of contemporaneous writers, the name of Gullstrand is pre-eminent in the field of physiological optics. One can hardly be said to be fully conversant with the present state of thought in this subject unless he has read the more important of Gullstrand's papers, * yet there are few ophthalmologists and not many physiologists who are able to read them and for the same reason that the chemists of Gibbs' day were unable to read his work. Gullstrand's

* Allvar Gullstrand, *Die reelle optische Abbildung*, Kungl. Svenska Vetenskapsakademiens Handlingar, vol. 41 (1906) No. 3; *Die optische Abbildung in heterogenen Medien und die Dioptrik der Kristal-linse des Menschen*, ibidem, vol. 43 (1908), No. 2; *Über Astigmatismus, Koma und Aberration*, Annalen der Physik, (4), vol. 18 (1905), pp. 941–973; *Tatsachen und Fiktionen in der Lehre von der optischen Abbildung*, Archiv für Optik, vol.1 (1907), p. 2; *Preparation of non spherical surfaces for optical instruments*, Kgl. Svenska Vetenskapsakademiens Handlingar, vol. 60 (1919), p. 155, abstracted in Zeitschrift für Instrumentenkunde, vol. 41 (1921), pp. 123–25.

investigations have already led to important practical results. I need only mention his ophthalmoscope, his special lenses for patients who have undergone operation for cataract and the slit-lamp microscopy of the living eye. The influence of his work would undoubtedly be more widespread were there more among the group of men interested who could read his papers. In one of the appendices to the recent translation of Helmholtz's *Physiological Optics*, Gullstrand has presented some of his conclusions freed to a great extent from the more difficult mathematics, but it is unfortunately not possible to present such a subject in the most convincing manner without the logical processes whereby the theory has been built up.

In most of our colleges and universities we require of science students, particularly those who are entering the graduate and professional schools, a reading knowledge of German and French. This is an admirable requirement and the reason for it is too well understood to need explanation. It is less well understood that not alone the chemist and physicist, but the biologist as well, must be able to read mathematical papers if he is not to be cut off from the possibility of understanding important communications in his own field of science. And the situation here is worse than it is in the case of inability to read a foreign language. For a paper in a foreign language may be translated, but in many cases it is impossible to express in ordinary language symbols the content of a mathematical paper in such a way as to convey a knowledge of the logical process by which the conclusions have been reached. The result may be accepted on faith, but this is repugnant to the scientifically trained mind.

Among recent outstanding contributions to physiology is the work of Professor A. V. Hill on the phenomena of muscle contraction which was crowned with the Nobel prize.* One might have thought it unlikely that new information of

* A. V. Hill and W. Hartree, *The four phases of heat-production of muscle*, Journal of Physiology, vol. 54 (1920), pp. 84–128.

fundamental importance would be unearthed in a field on which so much labor had already been bestowed. Our interest lies not so much in the ingenious methods devised by Hill as in the fact that they could have been devised and employed for the successful accomplishment of the purpose only by one who brought to the task the viewpoint and power of the skilled mathematician, trained in the concepts and methods of physics and physical chemistry. The absence of any formidable array of mathematical symbols from Professor Hill's papers might lead the casual reader to surmise that they are of the usual descriptive biological type. A more careful perusal will show that he has endeavored to present his results so far as possible without the use of mathematical symbols and phraseology in order that the papers may be intelligible to the physiologists of the present day. To do this he has been obliged to omit certain details, the nature of which can be readily surmised by one with some mathematical experience.* Professor Hill's training in mathematics and physics followed the well known traditions of Cambridge. That with this preparation, so unusual at the present time for a biologist, he should, from the very beginning of his productive work, have secured results of far reaching importance, carries a lesson which deserves to be well pondered by those whose duty it is to advise young men preparing for a lifework in biological research.

In any enumeration of the applications of mathematics to biological investigation the statistical treatment of biological data should occupy a prominent position. This field is at once one of the most important and one of the most treacherous into which the biological mathematician can

* Since the above was written, it has been possible to verify the accuracy of this surmise. Furthermore, though no mention of it is made in Dr. Hill's paper, the solution of an integral equation was part of the underlying mathematical structure essential to the success of the investigation. Although the inclusion of these details might have deterred many contemporaneous biologists from attempting to read the paper, it seems proper that the facts should be known because of the influence which they may, and should, exert on the rising generation.

venture. Gibbs frequently quoted in this connection a warning of Maxwell and I can do no better than repeat a passage taken from his obituary of Professor H. A. Newton, the astronomer. This was read by Gibbs before the National Academy of Sciences in 1897.* He says "This kind of investigation Maxwell has called statistical, and has in more than one passage signalized its difficulties. The writer recollects a passage of Maxwell which was pointed out to him by Professor Newton, in which the author says, that serious errors have been made in such inquiries by men whose competency in other branches of mathematics was unquestioned." Gibbs was himself a master of this type of investigation and the biological sciences are fortunate in having secured the services of a former pupil and later distinguished colleague of Gibbs, Professor E. B. Wilson, who is now connected with the Harvard Medical School. Time will not permit a lengthy enumeration of the problems which have been approached by this method. Those who are interested will find ample references in the works of Professor Karl Pearson and in those of Professor Raymond Pearl. A recent interesting achievement in the field of statistical biology is the development by Mr. Arne Fisher† of a method of constructing from biological data a series of frequency curves for the death rates at various ages from certain groups of causes. These he has been able to combine into a compound frequency curve from which may be predicted the expected deaths at various ages in an entire population without knowledge of the total number of lives exposed to risk. The resulting curves have been shown to be in good agreement with curves established by the usual actuarial methods for the same populations. The method has been

* J. Willard Gibbs, *Hubert Anson Newton*, American Journal of Science, (4), vol. 3 (1897), pp. 359–376. Also Gibbs, Scientific Papers, vol. 2, pp. 268–284.

† Arne Fisher, *Frequency Curves*, New York, The Macmillan Co., 1922; *Note on a new method of construction of mortality tables when the number of lives exposed to risk is unknown*, Skandinavisk Aktuarietidskrift, 1925 pp. 163–215.

subjected to vigorous attack by actuaries unfamiliar with the biological principles involved, but there can be little doubt of the essential soundness of the procedure and its applicability is by no means limited to actuarial studies.

The use of statistical methods in biological studies has hitherto been confined largely to such matters as are included in the term biometry. Among the various items would be such as birth and death rates, incidence of morbidity, curves of population growth, correlations between various classes of biometric data and the like. These subjects are of great importance, an importance which I recognize fully at the same time that I feel impelled to offer the suggestion that statistical mathematics will perform its greatest service for the biological sciences in a very different field, a field as yet practically untouched. It may even be premature to suggest this application in the present state of our quantitative biological data. In a mathematical study of any system in which it is possible to measure the total integrated changes of energy or of material state, but impossible, on account of its complexity, the smallness of its parts or their inaccessibility to our methods of investigation, to know in detail all of the elements which contribute to the total change, we must have recourse to the statistical method. The brilliant success with which Maxwell has applied this method to the kinetic theory of gases and its power in the hands of Gibbs in investigating the energetics of chemical reactions, entices us toward the thought—the hope, perhaps—that one day it may come to be applied to such studies as the energetics of secretion and other biological processes. If the whole complicated mammalian organism operates in accordance with the principle of the conservation of energy, as experiments with the respiration calorimeters show it to do, it seems altogether likely that the minute details of the energy exchanges will be found to obey the same laws that hold in the world of the non-living. Only in the living organismone must reckon with the effect of the structure. This seems to be the chief peculiarity of living things and in any attempt

to apply the methods of chemistry and physics to the problems of biology one must keep constantly in sight the fact that structure and its consequences is an integral part of the problem. It is an item which seems to increase the difficulties enormously. Much data have already been accumulated. It is possible that some further information regarding *structure* is needed before the next step can be taken, possibly something about finer structure than we have been able to study with the microscope hitherto.

My colleague, Professor E. L. Scott, has recently carried out an investigation on the *number* of experiments which must be performed before the results acquire real significance. Such a study is partly mathematical, but there must enter into it experimental data, since the answer is intimately bound up with the uniformity with which the experimental conditions can be controlled and repeated and the certainty with which the experimenter can know that he has controlled the conditions within any specified limits of uniformity. In three series of observations, each in a different biological class, Dr. Scott finds that the number of experiments required for reasonable certainty is of the order of fifty. It is too early to predict whether this order of magnitude applies to any considerable number of types of biological experiment. However, it raises at once the question whether in any short series of biological observations the variations which occur can be regarded as due necessarily to experimental procedure. The variability which Professor Scott has found in the three types studied would seem to make it incumbent upon any investigator who desires to draw important conclusions from short series of observations to show that in the type of work he is doing, the variability is small enough to justify his procedure. In order to do this he must run at least one much longer series of experiments, since at times the larger variations may fail to occur in the first five or ten observations. A small dispersion in a series of five experiments may be sadly misleading.

If we are to begin to make biology an exact science, one of the first requisites will be reliable data. Familiarity with the theory of errors and a critical attitude in drawing conclusions from measurements are characteristics of the men who cultivate those sciences we now regard as "exact." We shall have taken an important step when biologists regard their results in the same critical light. Teachers of mathematics can probably further this desirable situation by explaining to students, who are known to contemplate a career in one of the biological sciences, the importance of a knowledge of the theory of measurements.

From time to time physiologists are constrained to enter upon mathematical investigations not so directly related to biological processes themselves. It sometimes happens that correct interpretation of experimental results requires a study of the theory of the instruments used in making the measurements. At times these inquiries have resulted in substantial improvements in methods of work and design of instruments. Notable examples are to be found in the work of Professor Einthoven in the field of electrophysiology and in that of Professor Otto Frank in hemodynamics.

Physiology is sometimes defined as the physics and chemistry of living things. While this definition resembles most definitions in that it is somewhat lacking in completeness, it serves to direct attention to the particular phases of physiology in which it appears most probable the greatest progress of the immediate future will be made. This is not my personal view alone. A distinguished zoologist recently voiced quite similar opinions in my hearing. That the plant physiologists have reached similar conclusions is evidenced by the attitude of Hampton and Gordon, * who in a recent issue of Science, plead for better preparation in mathematics, physics and chemistry as a foundation for research in plant physiology. They quote Lepeschkin as holding the view that plant physiology must develop hand in hand with

* H. C. Hampton and S. C. Gordon, *A suggested course in plant physiology*, Science, vol. 64 (1926), pp. 417–419.

physics and chemistry. I can see no escape from the conclusion that the physiologist of the immediate future must be to a considerable extent a chemist and physicist and these he can be only by building from a suitable mathematical foundation. Nor is this a hopeless prospect. Textbooks of physics of seventy-five years ago were much larger than at present. This in spite of the enormous additions since made to our knowledge of the subject. But these older books were voluminous because of minute descriptions of phenomena which we now recognize as what a mathematician would call particular cases, comprehended under broad general principles. Chemistry has also grown greatly in detail, but study of the phenomena of radioactivity and the discovery of the isotopes have already done much to simplify its concepts and we may look forward confidently to a time when our successors will be able to learn the broad principles of chemistry much more readily than can be done at present.

The ultimate object of all scientific study is to enable us to predict that in certain circumstances certain events will happen. The old saying, "History repeats itself," is tantamount to recognition that even so complicated a system as large groups of human beings reacts similarly in similar circumstances. We are not yet able to predict in biological matters to the same order of precision as in physics and chemistry, but that is no proof of lack of precision in biological reactions. Rather it points to our inability to control conditions and to recognize small, but significant, points of difference. Perhaps we may look forward to a day when the history of generalization in chemistry and physics may repeat itself in the biological sciences.

Cognate with the problem of securing a more general appreciation of the value of mathematical methods in biological investigation is that of overcoming a certain feeling of aloofness from these methods which in many people amounts almost to a pathological phobia. That the study of mathematics presents difficulties no one will deny, but that the difficulties are so great as many imagine them to be is

certainly *not* true of those branches most applicable to the problems of the physical world as we see them today.

There seems to be a rather widespread opinion that some individuals are endowed with an innate capacity for mathematical thinking which is quite as definitely lacking in others. There is probably *some* basis for this opinion. If we think of natural adaptations of more evident type we can find plenty of analogies. For example, no amount of training would enable a heavy Clydesdale horse to compete successfully against a thoroughbred in a running race. Nevertheless the Clydesdale *can* run and given a little more time would cover the same ground as the animal better adapted for speed. I fear that many students, quickly fatigued by the sustained mental effort requisite to follow the reasoning in a new branch of mathematics, conclude too readily that they are of the group who have not an hereditary capacity for this discipline. There are probably not a few professional mathematicians whose inherited capacity is actually less than that of men who have quit the subject in despair. The man of high courage and determination often succeeds in spite of heavy handicaps, while in every field of human endeavor we see failure, often after an auspicious beginning, where these important qualities are lacking. Where courage and industrious habit is combined with large natural capacity we may expect an unusual measure of success, but I am inclined to think that there are few men—or women—whose intellectual capacity is sufficient to permit their successful cultivation of any branch of science, who could not, if they felt an urgent need for it, obtain a sufficient mastery of those branches of mathematics most requisite for the study of the natural sciences.

Before any person of sound judgment will undertake a difficult task he must first be convinced that it is worth doing and that he has a reasonable chance of success if he undertakes it. I have known more than one man of mature years with very meager training in elementary mathematics who, finding himself checked by this lack, has acquired the

desired facility by study, aided only by books, and usually crowded into the routine of a busy life. The rapidity with which such a student often progresses depends, it seems to me, very greatly upon the maturity of his mental processes. He knows exactly why he wants the information; he sees beyond the task of the moment and has from the beginning a rather clear conception whither his study is to lead him. Furthermore he is likely to have attained to a certain measure of self-discipline and is able to keep his thoughts from wandering. This requires no effort at all once a real interest in the subject has been awakened. I have known of one instance where such auto-instruction has been attended with marked success in a man who as a youth regarded himself and was regarded by his teachers as one of those who have no capacity for advanced mathematical study. When it becomes necessary to spend the time of maturer years in thus acquiring what might have been secured in youth, the process always involves a certain waste. The individual has been deprived for a number of years of an intellectual aid which he might have had at his service, and the time of his maturity might be better employed in occupations more suited to his riper experience. It seems that the lesson to be derived is that the awakening of a very real interest on the part of the student in the subject of his study is quite as necessary for his success as is the possession of innate capacity.

One aspect of mathematical study which it seems does much to discourage the beginner is the smooth and finished character of every demonstration he meets. Realizing keenly that he could hardly hope to carry through any such perfectly finished logical reasoning on an original problem, he becomes discouraged. He thinks that these textbook demonstrations are models of the way in which mathematicians do their work. Even the artifices employed awaken mingled feelings of admiration for the mind that can conceive and apply them with such wonderful facility, and despair that he will ever be able to match this performance when confronted with an original problem with no one at hand to suggest the

manner of approach. The wise teacher can do a great deal at this stage for his encouragement. He may be told that even a finished mathematician usually writes down on his work-paper much more than he subsequently permits to be printed. Also that not infrequently when a result has been reached by devious ways and after expenditure of much time and labor, it is afterward seen to be capable of direct approach. He will thus become aware that the smooth and concise demonstration is not necessarily conceived in that finished form. I wonder if, for its educational value, it might not be better at times to let students see how some of these results which can be expressed so compactly and neatly when we know just how to do it, were actually reached originally. Beginners can hardly be expected to go to the primary sources, but the teacher who is familiar with the original literature can often perform a real service to his students by bringing to their attention passages from the older authors who, writing at a time when knowledge of their subject had not yet been cast into its present condensed form, sometimes shed light on points which give difficulty in the modern standard treatises.

I am reminded of a passage in Maxwell's *Electricity and Magnetism*.* Referring to the brilliant demonstration by Ampère of the laws of action between electric currents, he says;

"The experimental investigation by which Ampère established the laws of the mechanical action between electric currents is one of the most brilliant achievements in science. The whole, theory and experiment, seems as if it had leaped full grown and full armed from the brain of the 'Newton of electricity'. It is perfect in form, unassailable in accuracy and it is summed up in a formula from which all the phenomena may be deduced, and which will always remain the cardinal formula of electrodynamics. The method of Ampère, however, though cast into an inductive form, does not allow us to

* J. C. Maxwell, *A Treatise on Electricity and Magnetism*, Oxford, Clarendon Press, 1873, vol. 2, p. 162, § 528.

trace the formation of the ideas which guided it. We can scarcely believe that Ampère discovered the law of action by means of the experiment which he describes. We are led to suspect, what indeed, he tells us himself, that he discovered the law by some process which he has not shewn us, and that when he had afterwards built up a perfect demonstration, he removed all traces of the scaffolding by which he raised it. Faraday on the other hand, shews us his unsuccessful as well as his successful experiments, and his crude ideas as well as his developed ones, and the reader, however inferior to him in inductive power, feels sympathy even more than admiration, and is tempted to believe that, if he had the opportunity, he too would be a discoverer. Every student therefore should read Ampère's research as a splendid example of scientific style in the statement of a discovery, but he should also study Faraday for the cultivation of a scientific spirit, by means of the action and reaction which will take place between newly discovered facts and nascent ideas in his own mind." Although written by a physicist about a question in physics, it seems to me similar considerations apply to mathematics. If the student can be made to realize at the outset of his work that the beautiful edifice he admires has not arisen by some species of black magic out of nothing, but was originally cumbered by the scaffolding and debris of construction, he will learn that he must expect to accumulate a certain amount of debris when he undertakes to produce something of an original character.

A difficulty which not infrequently stops the novice in reading mathematical papers lies in his attempting to read from formula to formula, thinking that because the writer has written them in immediate succession, he should be able to see the derivation at once without the necessity of performing any work with paper and pencil. A reader of long experience with the subject under discussion may be able at times to do this, but the student should be cautioned in every case where it is not "evident" to start manipulating with paper and pencil the last formula he understands until the

next succeeding one has been derived. In writing papers which will probably be read only by professional mathematicians, authors not infrequently omit so many intermediate steps in order to condense their papers that the filling in of the gaps even by industrious use of paper and pencil may become no inconsiderable labor, especially to one approaching the subject for the first time. In reviewing the Electrical Papers of Oliver Heaviside, Professor FitzGerald remarks:* "In his most deliberate attempts to be elementary, he jumps deep double fences and introduces short cut expressions that are woeful stumbling-blocks to the slow-paced mind of the average man. . . ." Those who write for biological readers of the present time should bear in mind that the saving of effort which the insertion of a few, possibly not altogether necessary, formulas will effect may make all the difference between success in comprehending the paper and utter discouragement. Since papers are printed to be read and to get results, the saving of a little paper and ink at the expense of great loss of clarity is surely no gain.

It may not be wholly uninteresting to the mathematicians and possibly may shed light on the value of mathematics for those less familiar with its possibilities to advert briefly to some considerations of the subject colored by a physiological background. It is well known that man is set apart from other animals by reason of his greater intellectual powers. Wherein is the source of these powers? There are structural differences between the brains of the most highly developed of the lower animals and that of man, but when one considers the great disparity of function he is struck by the relative smallness of the structural difference. There is evidence, which I cannot stop to present, that the memory of the lower animals is to a great extent, if not entirely, a memory of particular concrete instances. Through the development of language man has substituted for these, abstract and general symbols which represent any object or event of a

* G. F. FitzGerald, *Heaviside's electrical papers*, Electrician, August 11, 1893. Also G. F. FitzGerald, Collected Scientific Writings, No. 61, p. 293.

given class. The phenomena of aphasia give evidence that certain limited parts of the brain are concerned with the language function and that thinking is ordinarily done in language symbols, usually in sound symbols. It is evident how much more easily the processes of thought can be extended in general symbols than would be the case if all thought had to deal with memories of particular and concrete objects. Now mathematics is both a body of truth and a special language, a language more carefully defined and more highly abstracted than our ordinary medium of thought and expression. Also it differs from ordinary languages in this important particular: it is subject to rules of manipulation. Once a statement is cast into mathematical form it may be manipulated in accordance with these rules and every configuration of the symbols will represent facts in harmony with and dependent on those contained in the original statement. Now this comes very close to what we conceive the action of the brain structures to be in performing intellectual acts with the symbols of ordinary language. In a sense, therefore, the mathematician has been able to perfect a device through which a part of the labor of logical thought is carried on outside the central nervous system with only that supervision which is requisite to manipulate the symbols in accordance with the rules. It is not even necessary during the intermediate mathematical steps to inquire what meaning is to be attached to the groupings of symbols. The thought which I have here in mind has been well expressed by Maxwell,* in referring to the work of Lagrange. Maxwell says: "The aim of Lagrange was to bring dynamics under the power of the calculus. He began by expressing the elementary dynamical relations in terms of the corresponding relations of pure algebraical quantities, and from the equations thus obtained he deduced his final equations by a purely algebraical process. Certain quantities (expressing the reactions between the parts of the system called into play by

* J. C. Maxwell, *Electricity and Magnetism*, Ed. of 1873, vol. 2, p. 184, § 554.

its physical connexions) appear in the equations of motion of the component parts of the system, and Lagrange's investigation, as seen from a mathematical point of view, is a method of eliminating these quantities from the final equations. In following the steps of this elimination the mind is exercised in calculation, and should therefore be kept free from the intrusion of dynamical ideas. Our aim, on the other hand, is to cultivate our dynamical ideas. We therefore avail ourselves of the labours of the mathematicians, and retranslate their results from the language of the calculus into the language of dynamics, so that our words may call up the mental image, not of some algebraical process, but of some property of moving bodies." These remarks of Maxwell would apply just as truly had the particular dynamical phenomena he had in mind been those of protoplasm.

One often hears it remarked that no new knowledge can be obtained by mathematical processes. The same may be as truly said of other modes of thought in the only sense in which it is true of mathematics. Thought cannot begin until there is material to think about, but the same material is presented alike to the animal, the savage, the child, and the scholar. No one who has given the matter serious thought will deny that the intellectual processes, whether mathematical or otherwise, do discover important truth in handling the material presented by the external world. It is the important *relationships* between the facts of observation which are discovered by thought and form the basis of our intellectual lives. Of all the aids to thought that man has devised the most powerful is mathematics. So long as thinking must be carried on solely in memories of particular things or of their corresponding sound symbols it is difficult to be certain that one has made a reasonably complete survey of the relationships between the facts, that he has realized all of the implications contained in them. Formulation of the results in a mathematical expression may not quite adequately represent them. Usually it represents a system simplified in order that the corresponding mathematical expressions may not be

beyond our powers of manipulation. If the behavior of the simplified system can be regarded as sufficiently similar to that of the real one which we wish to study, we have secured a powerful aid in seeking the implications of the facts which are at our disposal. Whenever any of the natural sciences has been able so to formulate its observed facts, progress in that branch has been accelerated in a manner so remarkable as to leave no room for doubt of the efficacy of the mathematical method. The greatest need of the biological sciences today is to discover amongst the wealth of empirical facts those relationships which are most significant; those whose recognition will result in the transfer of long lists of particular items to a few general classes, thus simplifying our views regarding them.

The physiologist deals with those aspects of life which are subject to observation and measurement by the same instruments and methods that are used in the study of non-living matter. If there be vital phenomena of a transcendent character, these do not concern the physiologist as a physiologist. So soon as the physical and chemical data, including the structural relationships, have been established with reasonable certainty, the work of the mathematician may begin. There is still much uncertainty about some of the data and their improvement by critical study is perhaps the first requisite. In this we shall need some mathematical assistance also. So soon as it becomes possible to construct biological theories cast in mathematical form we may look for rapid progress. Experiment will then be guided along those lines where it is most apt to yield new knowledge of importance. This has been the history of physics and chemistry. The evolution of chemistry into an exact science has been so recent that many of us have actually witnessed the events. We can hardly doubt that a similar outcome for the biological sciences will one day become a matter of history also.

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ON A GENERALIZATION OF THE SECULAR EQUATION*

BY JAMES PIERPONT

1 *Introduction.* The equation we wish to consider is

$$H_n(x) = \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix} = 0. \tag{1}$$

Here $a_{ij}=a_{ji}$, when $i\neq j$; while

$$\begin{aligned} a_{ii} &= \alpha_{ii} - x, \quad \text{for } i = 1,2, \cdots ,r, \\ &= \alpha_{ii} + x, \quad \text{for } i = r+1, r+2, \cdots ,n. \end{aligned}$$

The a_{ij} and α_{ii} are real, and $H(0)\neq 0$. If $r=n$, (1) is the secular equation which it will be convenient to denote by $L_n(x)=0$. When $r=n-1$ the equation (1) plays a fundamental role in classifying quadric surfaces in n-way hyperbolic space. Let us set $n-r=s$ and call $\sigma=|r-s|$ the *signature* of (1). We have then the

THEOREM I. *The number of real roots of $H_n(x)=0$, counting their multiplicity, is not less than its signature.*

This is a corollary of a theorem to which F. Klein calls especial attention (Mathematische Annalen, vol. 23 (1884), p. 562). The proof there given rests on the theory of elementary divisors.† We give here a very simple proof which is a modification of H. Weber's proof that the roots of the secular equation $L_n(x)=0$ are all real.‡ Weber's proof as we shall see, is complicated by his belief that it is necessary to show that

$$L_n'(x) = -\sum_i \frac{\partial L_n}{\partial a_{ii}}, \qquad (i = 1,2, \cdots ,n).$$

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* Presented to the Society, December 29, 1926.

† See T. J. Bromwich, *Quadratic Forms*, Cambridge Tracts, No. 3 (1906), p. 69.

‡ H. Weber, *Algebra*, vol. 1, 1898, pp. 307–310.

2. *Proof of the Theorem.* We turn now to the proof of the above Theorem I. Consider the sequence

$$H_n, H_{n-1}, H_{n-2}, \cdots, H_1, H_0 = 1, \tag{2}$$

where H_k is the determinant of degree k in x obtained by deleting the last $n-k$ rows and columns of (1). For the moment we suppose that no two of the H's vanish for the same x. They are connected by the relations

$$\begin{aligned} H_n H_{n-2} &= H_{n-1}\phi_{n-1} - \psi_n^2{}_{-1}, \\ H_{n-1} H_{n-3} &= H_{n-2}\phi_{n-2} - \psi_n^2{}_{-2}, \\ &\cdots\cdots\cdots\cdots \\ H_2 H_0 &= H_1\phi_1 - \psi_1^2, \end{aligned} \tag{3}$$

where ϕ, ψ are polynomials in x.

Merely for completeness let us show how these relations are obtained, the first for example. Let A_{ij} be the minor of a_{ij} in (1); set $\nu = n-1$. Then

$$B = \begin{vmatrix} A_{\nu\nu} & A_{\nu n} \\ A_{n\nu} & A_{nn} \end{vmatrix} = H_{n-1} A_{\nu\nu} - A_{n\nu}^2 . \tag{4}$$

But $H_n \cdot B = H_{n-2} \cdot H_n^2$. Hence, if $H_n \neq 0$, $B = H_{n-2} \cdot H_n$. This with (4) gives

$$H_n H_{n-2} = H_{n-1}\phi_{n-1} - \psi_n^2{}_{-1}.$$

This relation holds also if $H_n = 0$, as continuity considerations show.*

The equations (3) show that when $H_k = 0$, H_{k+1}, H_{k-1} have opposite signs.

We now consider the signs of the sequence (2). Suppose $H_n = 0$ for $x = a$. Then in a sufficiently small interval δ about $x = a$, H_n changes its sign, while none of the other terms in (2) do. Thus as x passes through δ the sequence (2) gains or loses one variation of sign. On the other hand when x passes through a root of $H_{n-1}, H_{n-2}, \cdots$ no variation is gained or lost as (3) show. For $x = +\infty$ there are r

* See Weber, loc. cit., p. 113; or Kowalewski, *Determinantentheorie*, p. 83.

variations of sign in (2); for $x=-\infty$ there are s, thus $H_n(x)=0$ has at least σ real roots.

We now consider the general case that the sequence (2) has common roots. With Weber we may dispose of this case as follows. Suppose e.g. that H_k, H_{k-1} have common roots. We vary the terms a_{ij} of H_k not in H_{k-1} by small amounts numerically less than some η, so that H_k, H_{k-1} do not have common roots.

In this way we may replace (2) by another sequence

$$K_n, K_{n-1}, K_{n-2}, \cdots, K_1, K_0 = 1 \tag{5}$$

no two of which have a common root. The roots of $K_n=0$ differ from those of $H_n=0$ by an amount as small as we please, for sufficiently small η, moreover the signs of corresponding elements of the sequences (2), (5) are the same for an x for which no element of (2) vanishes. As Theorem I holds for (5), it must hold for (2).

THEOREM II. *The roots of the secular equations are all real.*

For in this equation $s=0$; hence $\sigma=n$.

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A GENERALIZED TWO-DIMENSIONAL POTENTIAL PROBLEM

BY J. R. CARSON

It may be shown that the solution of the problem of electromagnetic wave propagation along a system of straight parallel conductors can be reduced to the solution* of two subsidiary problems: (1) a well known problem in two-dimensional potential theory; and (2) a generalization of the two-dimensional potential problem which is believed to be novel. The generalized problem is believed to possess suffi-

* Subject to certain restrictions to be discussed in a forthcoming paper.

cient mathematical interest, in addition to its technical importance, to justify its publication in this note.

The two-dimensional potential problem will first be stated in order to exhibit its relation to the generalized problem:

Let there be n closed curves in the xy-plane and let a function $\Phi(x, y)$ satisfy Laplace's equation in two dimensions everywhere outside these curves; thus

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi = 0. \tag{1}$$

On the jth curve ($j=1, 2, \cdots, n$), let

$$\frac{\partial}{\partial \tau}\Phi = 0, \quad \int \frac{\partial}{\partial n}\Phi d\tau = Q_j, \quad (j = 1,2, \cdots, n). \tag{2}$$

In equations (2), n and τ are the vectors normal and tangential to the jth curve respectively and Q_j is a specified real constant.

The determination of the potential function Φ in terms of $Q_1, \cdots, Q_n$ and the geometry of the field is a well known problem in the theory of the potential for which very general methods of solution are available. The solution is of the form

$$\Phi = \phi_1(x, y)Q_1 + \cdots + \phi_n(x, y)Q_n, \tag{3}$$

the coefficients $\phi_1, \cdots, \phi_n$, themselves two-dimensional potentials, being determined by the geometry of the field.

The generalized problem, which is the subject of this note, may be stated as follows.

Let there be n closed curves in the xy-plane, and let a function F satisfy the equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)F = 0 \tag{4}$$

everywhere outside these curves.

Inside the jth curve let a function E_j satisfy the equation

$$(5) \qquad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)E_j = \nu_j^2 E_j, \qquad (j = 1,2, \cdots, n),$$

where ν_j^2 is a given parameter, in general complex.

On the jth curve let F and E_j be connected by the equations

$$(6) \quad \frac{\partial}{\partial \tau}F = -\frac{\partial}{\partial \tau}E_j, \quad \frac{\partial}{\partial n}F = -\mu_j\frac{\partial}{\partial n}E_j, \ (j = 1,2, \cdots, n).$$

Here, as above, n and τ are the vectors normal and tangential to the jth curve respectively, and μ_j is a given real parameter.

Finally

$$(7) \qquad \int \frac{\partial}{\partial n}E_j d\tau = -\int \frac{\partial}{\partial n}F d\tau = J_j, \qquad (j = 1,2, \cdots, n),$$

where J_j is a specified constant, in general complex. The integration is carried around the contour of the jth curve.

We require the determination of F outside the curves and E_j inside the jth curve in terms of $\lambda_j, \mu_j, \nu_j, J_j$ and the geometry of the field, in the form

$$(8) \qquad \begin{aligned} F &= f_1(x,y)J_1 + \cdots + f_n(x,y)J_n, \\ E_j &= e_{j1}(x,y)J_1 + \cdots + e_{jn}(x,y)J_n. \end{aligned}$$

This problem has been solved for the practically important case where all the curves are circles by expanding E_j as a Fourier-Bessel series and identifying the constants of the expansion by a process of successive approximations. As yet, however, no serious study has been made of the general problem. Its analogy with the potential problem suggests, however, that extensions of the well known methods there available should be successful in the solution of the generalized problem.

American Telegraph and Telephone Company

A GENERALIZATION OF WARING'S THEOREM ON NINE CUBES*

BY L. E. DICKSON

THEOREM. *Every positive integer p can be expressed as a sum of seven cubes and the double of a cube, the cubes being positive or zero integers.*

On the basis of known tables, this theorem holds for $p \leqq 40{,}000$ as shown by the writer in the American Mathematical Monthly for April, 1927. The further empirical theorems of that paper will not be discussed here.

We shall here prove the above theorem for all sufficiently large integers p. The proof is analogous to that employed by Landau† in proving his theorem that every sufficiently large integer is a sum of at most eight positive integral cubes.

Let r be the real ninth root of $4/3$. The number of the primes $\equiv 2 \pmod 3$ which exceed x and are $\leqq rx$ is known to increase indefinitely with x. As x we choose the first radical in (1). Hence for all sufficiently large integers n, there exist at least ten primes p such that

$$(1) \qquad (n/12)^{1/9} < p \leqq (n/9)^{1/9},$$

$$(2) \qquad p \equiv 2 \pmod 3.$$

The product of the ten primes exceeds $(n/12)^{10/9}$ and hence exceeds n if $n > 12^{10}$. Hence not all of the ten are divisors of n.

To give a numerical illustration, take $n = 9m^9$. Then (1) becomes $m/r < p \leqq m$. For $m = 6000$, $m/r = 5811.2$, and the primes p satisfying also (2) are

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* Presented to the Society, April 15, 1927.

† Mathematische Annalen, vol. 66 (1909), pp. 102–5. Reproduced in his *Verteilung der Primzahlen*, vol. 1, 1909, pp. 555–59.

$$5813,\quad 5843,\quad 5849,\quad 5861,\quad 5867,\quad 5879,$$
$$5897,\quad 5903,\quad 5927,\quad 5939,\quad 5981,\quad 5987.$$

The third prime just exceeds m'/r when $m'=6038$. Hence the last ten primes of our list serve for every m between 6000 and 6038 inclusive.

Let therefore p be a prime not dividing n such that (1) and (2) hold. Then

$$9p^9 \leqq n < 12p^9.$$

A slight modification of Landau's proof of his first lemma shows that if p is an odd prime satisfying (2), every integer not divisible by p is congruent modulo p^3 to the double of a cube. Hence there are integers δ and M satisfying

$$n - 2\delta^3 = p^3M, \quad 0 < \delta < p^3.$$

Then

$$9p^9 - 2p^9 \leqq n - 2p^9 < n - 2\delta^3 = p^3M, \quad p^3M < n < 12p^9.$$

Cancelling the factors p^3, we have

$$7p^6 < M < 12p^6.$$

The further discussion by Landau* applies here unchanged and shows that n is the sum of $2\delta^3$ and seven integral cubes $\geqq 0$. We may replace $2\delta^3$ by $k\delta^3$; $k \geqq 1$.

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* His condition for $M_2>0$ is satisfied if $p\geqq 10$. Since the largest γ is 22, we obtain the milder condition $p\geqq 5$. Hence n exceeds 13 million.

AN ELEMENTARY PROOF BY MATHEMATICAL INDUCTION OF THE EQUIVALENCE OF THE CESÀRO AND HÖLDER SUM FORMULAS*

BY TOMLINSON FORT

For brevity, notation and terminology in this note are generally not explained. It is believed that they will be clear in all instances to any probable reader.

THEOREM. *The Hölder and Cesàro methods of summation are equivalent.*

PROOF. Let $C_n^{(2)}$ represent the Cesàro sum of order r to n terms:

$$(1)\qquad C_n^{(r)} = s_0\frac{r}{r+n} + \cdots + s_k\frac{r(n-k+1)\cdots n}{(r+n-k)\cdots(r+n)} + \cdots + s_n\frac{r(n!)}{r\cdots(r+n)}.$$

We readily verify that $C_n^{(r)}$ satisfies the equation

$$(2)\qquad (n+r+1)\,C_n^{(r+1)} - nC_{(n-1)}^{(r+1)} = (r+1)C_n^{(r)},$$

which may be written in the form

$$\Delta\left\{nC_{(n-1)}^{(r+1)}\right\} + r\,C_n^{(r+1)} = (r+1)C_n^{(r)},$$

or

$$(3)\qquad (n+1)C_n^{(r+1)} + r\sum_{n=0}^{n} C_n^{(r+1)} = (r+1)\sum_{n=0}^{n} C_n^{(r)}.$$

By solving (2) we get the following, as is easily verified:

$$C_n^{(r+1)} = \frac{n!}{(r+2)\cdots(r+n+1)}\sum_{n=0}^{n}\frac{(r+1)\cdots(r+n)}{n!}C_n^{(r)}$$

$$(4)\qquad = \frac{(r+1)!}{(n+1)\cdots(n+r+1)}\sum_{n=0}^{n}\frac{(n+1)\cdots(n+r)}{r!}C_n^{(r)}.$$

*Presented to the Society, December 29, 1926.

Let $H_n^{(r)}$ represent the Hölder sum to n terms of order r. We establish by mathematical induction the following fundamental formula.

$$H_n^{(r)} = k_0^{(r)} C_n^{(r)} + k_1^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)}$$

$$+ k_2^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)} + \cdots \tag{5}$$

$$+ k_{r-1}^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} \cdots \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)},$$

where

$$k_0^{(r)} + k_1^{(r)} + \cdots + k_{r-1}^{(r)} = 1.$$

If we assume (5), we find

$$H_n^{(r+1)} = \frac{1}{n+1} \sum_{n=0}^{n} H_n^{(r)} = k_0^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)}$$

$$+ k_1^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)}$$

$$+ k_2^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)} \tag{6}$$

$$+ \cdots + k_{r-1}^{(r)} \frac{1}{r+1} \sum_{n=0}^{n} \frac{1}{n+1} \cdots \sum_{n=0}^{n} C_n^{(r)}.$$

Substitute for

$$\frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)}$$

in each sum of (6), its value from (3). Take for example

$$k_1^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)} = k_1^{(r)} \frac{1}{r+1} \cdot \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r+1)}$$

$$+ k_1^{(r)} \frac{r}{n+1} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r+1)}. \tag{7}$$

We notice that

$$k_1^{(r)} \cdot \frac{1}{r+1} + k_1^{(r)} \frac{r}{r+1} = k_1^{(r)}.$$

We then notice that we have an expression of the same form as (5), but with r replaced by $(r+1)$. As $H_n^{(1)} = C_n^{(1)}$, proof of the formula follows by induction.

We next prove in a similar manner the formula

$$(8)\quad C_n^{(r)} = h_0^{(r)} H_n^{(r)}$$

$$+ h_1^{(r)} \frac{r!}{(n+1)\cdots(n+r)} \sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-1)}{(r-1)!} H_n^{(r)}$$

$$+ h_2^{(r)} \frac{r!}{(n+1)\cdots(n+r)} \sum_{n=0}^{n} \sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-2)}{(r-2)!} H_n^{(r)}$$

$$+ \cdots + h_{r-1}^{(r)} \frac{r!}{(n+1)\cdots(n+r)} \sum_{n=0}^{n} \cdots \sum_{n=0}^{n} (n+1) H_n^{(r)},$$

where

$$h_0^{(r)} + h_1^{(r)} + \cdots + h_{r-1} = 1.$$

To prove this formula, substitute in (4), and then make the substitution

$$(9)\quad \sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-2)}{(r-2)!} H_n^{(r)}$$

$$= \frac{(n+1)\cdots(n+r-1)}{(r-2)!} \frac{1}{n+1} \sum_{n=0}^{n} H_n^{(r)}$$

$$- \sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-2)}{(r-3)!} \cdot \frac{1}{n+1} \sum_{n=0}^{n} H_n^{(r)}$$

$$= (r-1) \frac{(n+1)\cdots(n+r-1)}{(r-1)!} H_n^{(r+1)}$$

$$- (r-2) \sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-2)}{(r-2)!} H_n^{(r+1)},$$

and similarly for other sums. To prove (9), etc., sum by parts once, using the formula

$$\sum_{n=0}^{n} u(n)\Delta v(n) = u(n)v(n)\Big]_0^{n+1} - \sum_{n=0}^{n} v(n+1)\Delta u(n).$$

Notice that the coefficients in the last member of (9), namely $(r-1)$ and $-(r-2)$, add to unity, and then by mathematical induction formula (8) is proved.

The conclusions of the theorem are readily drawn from equations (5) and (8). Consider first that $C_n^{(r)} \to s$. Then by a repetition of the arithmetic mean theorem each sum in (5) approaches s and hence

$$H_n^{(r)} \to (k_0^{(r)} + k_1^{(r)} + \cdots + k_{r-1}^{(r)})s = s.$$

Next suppose that $H_n^{(r)} \to s$. We do not have a theorem so well known as the arithmetic mean theorem to refer to but, if we notice that when $H_n^{(r)}$ is replaced by a constant s, each sum in (8), as

$$\frac{r!}{(n+1)\cdots(n+r)} \sum_{n=0}^{n}\sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-2)}{(r-2)!}s,$$

equals s, a little simple epsilon work gives the desired result that $C_n^{(r)} \to s$.

Difference equations from which the coefficients $k_j^{(r)}$ and $h_l^{(r)}$ can be calculated might be written down but are omitted as they are not necessary for the proof of the theorem.

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SOME PROPERTIES OF CONTINUOUS CURVES*

BY G. T. WHYBURN

The points A and B of a continuum M are said to be separated in M by a point X of M if $M-X$ is the sum of two mutually separated sets S_1 and S_2 containing A and B respectively. The point P of a continuum M is a cut point of M if and only if the set of points $M-P$ is not connected, i.e., is the sum of two mutually separated point sets.

A continuous curve M will be said to be *cyclicly connected* provided that every two points of M lie together on some simple closed curve which is contained in M. In this paper use will be made of the following fundamental theorem.

THEOREM A. *In order that the continuous curve M should be cyclicly connected it is necessary and sufficient that M should have no cut point.*

A proof for Theorem A will be found in my paper *Cyclicly connected continuous curves*, which will appear soon.

THEOREM I. *If A and B are any two points of a continuous curve M and if K denotes the set of all those points of M which separate A from B in M, then $K+A+B$ is a closed set of points.*

PROOF. The curve M contains a simple continuous arc t from A to B. Clearly K must be a subset of t. Let P be any point of $t-(K+A+B)$. Since P does not belong to K, A and B must both belong to some connected subset† of $M-P$, and by a theorem of R. L. Moore's‡ it follows that $M-P$ contains an arc t' from A to B. On the arcs PA

* Presented to the Society, December 31, 1926.

† See an abstract of a paper by R. L. Wilder, *A characterization of continuous curves by a property of their open subsets*, this Bulletin, vol. 32 (1926), pp. 217–218.

‡ *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 255.

and PB of t, in the order from P to A and from P to B respectively, let X and Y respectively denote the first points belonging to t'. Then no point of the segment XPY of t can belong to $K+A+B$. Since thus every point of $t-(K+A+B)$ belongs to some segment of t which contains no point of $K+A+B$, the set $K+A+B$ is closed.

DEFINITION. A cyclicly connected continuous curve C is said to be a *maximal cyclic curve* of a continuous curve M if C is a subset of M and is not a proper subset of any other cyclicly connected continuous curve belonging to M.

THEOREM II. *If A and B are any two points of a continuous curve M, t is any arc of M from A to B, K denotes the set of all those points of M which separate A from B in M, and S is any maximal segment of $t-(K+A+B)$, then M contains a maximal cyclic curve which contains S.*

PROOF. Let E and F denote the end points of S. Let G denote the collection of all the maximal connected subsets of $M-(E+F)$. At least one element of G, namely, the one which contains S, must have both of the points E and F for limit points. Only a finite number of elements of G can have this property. Let H denote the point set obtained by adding together all the elements of G which do have this property, and let N denote the point set $H+E+F$. Then N is closed and connected. The continuum N is also connected im kleinen. For since M is connected im kleinen, no point of H is a limit point of $M-H$. Hence H is connected im kleinen. Then from a theorem of R. L. Moore's* it easily follows that $H+E+F=N$ must also be connected im kleinen. Thus we see that N is a continuous curve.

Now if N has no cut point, then by Theorem A, N must be cyclicly connected. If N has any cut points, then for every cut point X of N, let H_x denote the maximal connected subset of $N-X$ which contains $(S+X)-X$, and let N_x denote the point set H_x+X. That H_x exists in case X is

* *A report on continuous curves from the viewpoint of analysis situs*, this Bulletin, vol. 29 (1923), pp. 296–297.

not on S is obvious. If X belongs to S, then since X does not belong to $K+A+B$, $M-X$ contains an arc* t' from A to B. The arc t' must contain the points E and F, for E and F belong to $K+A+B$. The arc EF of t' must belong to N, hence also to $N-X$. Therefore, $S-X$, being the sum of the segments EX and EF of t, must lie in some connected subset of $N-X$. Thus, in any case, H_x exists. Now let L denote the set of all points which are common to all the point sets N_x. Clearly L exists, is a closed point set, and contains S. Let C denote the maximal connected subset of L which contains S. I shall show that C is a maximal cyclic curve of M.

Clearly C is closed and connected. I shall first show that if P and Q are any two points of C and POQ is any arc of N from P to Q, then every point of POQ must belong to C. This must be true, for if X is any cut point of N not on POQ, then H_x contains every point of POQ because it contains P and Q. And if X is any cut point of N on POQ, then H_x contains $POQ-X$, because $POQ-X$ is either a single connected set containing a point of H_x or the sum of two connected sets each of which contains a point of H_x. Hence, in any case, POQ belongs to every set N_x and therefore belongs to L and to C.

The continuum C is a continuous curve. For let P be any point of C and ϵ any positive number. Then since N is connected im kleinen, there exists a positive number δ_ϵ such that every point of N whose distance from P is less than δ_ϵ can be joined in N to P by an arc which is of diameter less than ϵ. Let X be any point of C whose distance from P is less than δ_ϵ. Then N contains an arc a from X to P of diameter less than ϵ. But as was shown above, a must belong to C. Hence C is connected im kleinen at every one of its points and is therefore a continuous curve.

The curve C has no cut point. For suppose, on the contrary, that C has a cut point X. Then $C-X=S_1+S_2$,

* See R. L. Wilder, loc. cit., and R. L. Moore, *Concerning continuous curves in the plane*, loc. cit.

where S_1 and S_2 are mutually separated point sets. Let P_1 and P_2 be points of S_1 and S_2 respectively. Now if X is a cut point of N, then since H_x contains P_1 and P_2, it follows* that $N-X$ contains an arc s from P_1 to P_2. And if X is not a cut point of N, again it follows that $N-X$ contains an arc s from P_1 to P_2. And in either case, as was shown above, the arc s must belong to C. Hence S_1 and S_2 are not mutually separated, contrary to supposition. Therefore C has no cut point, and by Theorem A it follows that C is a cyclicly connected continuous curve. That C is a maximal cyclic curve of M follows immediately from the facts (1) that every arc of M joining two points of N must belong wholly to N and (2) that every arc of N joining two points of C must belong wholly to C. This completes the proof.

THEOREM III. *If A and B are any two points of a continuous curve M and if K denotes the set of all those points of M which separate A from B in M, then M contains two simple continuous arcs t_1 and t_2 from A to B whose common part is $K+A+B$.*

PROOF. By Theorem I, $K+A+B$ is a closed set of points. Hence if t is any arc of M from A to B, t must contain K, and $t-(K+A+B)$ is the sum of a countable number of non-overlapping segments $S_1, S_2, S_3, \cdots$. By Theorem II it follows that for every positive integer i, M contains a maximal cyclic curve C_i which contains S_i. For each i, let the end points of S_i be denoted by A_i and B_i. Since C_i is cyclicly connected, then for each i, C_i contains two arcs t_{1i} and t_{2i} from A_i to B_i whose common part is only the points A_i and B_i. Let

$$t_1 = K + A + B + \sum_{i=1,2,3,\cdots} t_{1i}, \qquad t_2 = K + A + B + \sum_{i=1,2,3,\cdots} t_{2i}.$$

Then t_1 and t_2 are simple continuous arcs of M from A to B whose common part is $K+A+B$.

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* See R. L. Moore, loc. cit.

THE DUAL OF A LOGICAL EXPRESSION*

BY B. A. BERNSTEIN

The Peirce-Schröder law of duality in Boolean logic is that for every proposition in the logic there exists another proposition, obtained from the former by interchanging the operations $+$, $\times$ and the elements 0, 1. The rule for obtaining the dual of a logical expression involved in this law is convenient enough when no special form is desired for the dual, but it is generally not at all convenient if the dual is required in the very much desired *normal* form. The main object of this note is to obtain a convenient rule for writing down the dual of an expression in the normal form.

Let $a, b, \cdots, l$ be the *discriminants* of a logical function $f(x, y, \cdots, t)$. Then f, developed normally with respect to its arguments, is given by

$$(1)\quad f(x,y,\cdots,t) = axy\cdots t + bxy\cdots t' + \cdots + lx'y'\cdots t',$$

where the primes indicate negation. If f_1 denote the dual of f, we have by the rule of Peirce and Schröder

$$f_1 = (a_1 + x + y + \cdots + t)(b_1 + x + y + \cdots + t')\cdots (l_1 + x' + y' + \cdots + t'),$$

where $a_1, b_1, \cdots, l_1$ are the respective duals of $a, b, \cdots, l$.† Or, developed normally with respect to $x, y, \cdots, t$,

$$(2)\qquad f_1 = l_1xy\cdots t + \cdots + b_1x'y'\cdots t + a_1x'y'\cdots t'.$$

Let us call two discriminants of a function *conjugate* when the arguments associated with one are the respective negatives of those associated with the other. Then (2) tells us

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* Presented to the Society, San Francisco Section, October 30, 1926.

† The discriminants $a, b, \cdots, l$ may be 0, 1, or functions of other elements, $\alpha, \beta, \cdots, \mu$; so that $a_1, b_1, \cdots, l_1$ will in general be different from $a, b, \cdots, l$ respectively.

that *any discriminant of the dual of a function is the dual of the conjugate of the corresponding* discriminant of the function.* We then have the following rule.

RULE. *To obtain the dual of a logical expression, develop the expression normally with respect to any of its elements and change each discriminant to the dual of its conjugate.*

This is the desired rule. Thus, the dual of

$$axy + abx'y + x'y'$$

is

$$0 \cdot xy + (a + b)xy' + x'y + ax'y'.$$

Our rule brings to light some interesting facts about duals. We have at once that *a function is self-dual if each discriminant is the dual of its conjugate.*

Further, the negative of f given by (1) is

$$(3) \quad f' = a'xy \cdots t + b'xy \cdots t' + \cdots + l'x'y' \cdots t'.$$

So that

$$(f')_1 = (l')_1 xy \cdots t + \cdots + (b')_1 x'y' \cdots t + (a')_1 x'y' \cdots t'$$

and

$$(f_1)' = (l_1)'xy \cdots t + \cdots + (b_1)'x'y' \cdots t + (a_1)'x'y' \cdots t'.$$

But

$$(k')_1 = (k_1)' \quad \text{for} \quad k = 0, 1, \alpha + \beta, \alpha\beta.$$

Hence

$$(4) \qquad (f')_1 = (f_1)' = f_1'.$$

That is, *the dual of the negative of a function is the negative of the dual of the function.*†

With the aid of (4), we get

$$(5) \qquad (((f')_1)')_1 = (((f_1)')_1)' = \cdots = f_{11}'' = f.$$

And so, *a function is left unaltered if operated on in any order by duals and negatives each taken an even number of times.*

* By *corresponding* discriminants of two functions we mean discriminants associated with identical products of the arguments.

† Sheffer's operation $a|b$ is seen to be the dual-negative of ab.

Again, generally $f_1 \neq f'$. Under what condition will $f_1 = f'$? From (2) and (3) we see that *the dual of a function is the same as the negative of the function if each discriminant is the dual-negative of its conjugate.*

Finally, if we have a relation $f = 0$, then in general $f_1 \neq 1$, though always $f' = 1$. What is the condition that $f_1 = 1$ when $f = 0$? By means of (2) and (3) we find this condition to be *the same as the condition that* $f_1 = f'$.

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THE HEAVISIDE OPERATIONAL CALCULUS*

BY H. W. MARCH

In a number of recent papers, Carson† has made a definite advance in the study of the Heaviside operational calculus by showing that the solution of an operational equation of the type in question can be obtained from an integral equation. Having done this, he was able to discuss Heaviside's three principal rules and to derive a number of important theorems by the use of which it is possible to solve by operational methods, problems to which Heaviside's rules are not directly applicable.

Somewhat earlier Bromwich‡ and Wagner§ solved, by the use of contour integrals in the complex plane, problems to which one of Heaviside's rules is applicable. They noted that the corresponding rule of Heaviside, the expansion theorem, follows at once from a calculation of the residues at the poles of the integrand in the case of a suitably restricted

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* Presented to the Society, December 31, 1926.

† J. R. Carson, Bell Technical Journal, vol. 4 (1925), pp. 685–761; vol. 5 (1926), pp. 50–95, 336–384; this Bulletin, vol. 22 (1926), pp. 43–68. See also P. Levy, Bulletin des Sciences Mathématiques, vol. 50 (1926), pp. 174–192.

‡ T. J. I'A. Bromwich, Proceedings of the London Society, (2), vol. 15 (1916), pp. 401–448; see also H. S. Carslaw, Philosophical Magazine, vol. 39 (1920), pp. 603–611.

§ K. W. Wagner, Archiv für Elektrotechnik, vol. 4 (1916), pp. 159–193.

operator. In a later paper Bromwich* also obtained from a contour integral the asymptotic solution for a particular problem.

It is the purpose of the present paper to show the connection between the methods of Carson and Bromwich by showing that Bromwich's contour integral is the solution of the integral equation set up by Carson. In the opinion of the writer the two methods supplement one another in a valuable way. To emphasize the importance of the contour integral in obtaining Heaviside's rules rigorously and simply, the derivation of each of the three principal rules is briefly indicated.

Carson showed that the solution $h(t)$ of the operational equation†

$$h(t) = \frac{1}{H(p)} \tag{1}$$

which satisfies the appropriate initial conditions is the solution of the integral equation

$$\frac{1}{pH(p)} = \int_0^\infty h(t)e^{-pt}dt, \tag{2}$$

the real part of p being assumed to be negative.

Bromwich, on the other hand, attacked the problem directly without using the operational calculus, by assuming the solution of the corresponding differential equation or system of differential equations to be expressed by a contour integral in the complex plane. He found that the solution $h(t)$ is given in certain cases by the integral‡

$$h(t) = \frac{1}{2\pi i} \int_C \frac{e^{pt}}{pH(p)}dp, \tag{3}$$

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* Proceedings, Cambridge Philosophical Society, vol. 20 (1920–21), pp. 423–427.

† For notation and details, see Carson, Bell Technical Journal, vol. 4 (1925), pp. 686–705; this Bulletin, vol. 22 (1926), pp. 43–47.

‡ Proceedings of the London Society, (2), vol. 15 (1916), pp. 410–420.

or by the integral

$$h(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{pt\infty}}{pH(p)} dp, \tag{4}$$

where C in (3) is a closed curve surrounding all the poles of the function

$$\frac{1}{pH(p)},$$

no one of which has, by hypothesis, a positive real part, and where the path of integration in (4) is a straight line parallel to the axis of imaginaries, c being any positive real number. The path of integration in (3) results from a deformation of the path in (4).

It will now be shown that the solution* of the integral equation (2) is given by (4). Fourier's theorem can be written in the form†

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{xy} dy \int_0^\infty e^{-yz} f(z) dz, \tag{5}$$

where c and the path of integration are chosen as in (4). If we write

$$g(y) = \int_0^\infty e^{-zy} f(z) dz, \tag{6}$$

it follows from (5) that

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(y) e^{xy} dy. \tag{7}$$

Accordingly (7) furnishes the solution‡ of the integral equation (6) provided that $g(y)$ is subject to suitable restrictions. These must be such that (6) can be shown to be satisfied when $f(z)$ as given by (7) is substituted in it. This

* See the recent paper by J. D. Tamarkin, Transactions of this Society, vol. 28 (1926), p. 417.

† See, e.g., H. M. MacDonald, Proceedings of the London Society, vol. 35 (1902), p. 428. See also B. Riemann, *Gesammelte Werke*, p. 149.

‡ For a proof of the uniqueness of a continuous solution of equation (6), see M. Lerch, Acta Mathematica, vol. 27 (1903), pp. 339–351.

can be done if $g(y)$ is an analytic function of y at all points in the half-plane in which the real part of y is positive and if further, writing $y=\rho e^{i\theta}$, it is true that in this half-plane

$$|g(y)| < \frac{N}{\rho^{\alpha}},$$

for a sufficiently large ρ, N being a sufficiently large fixed positive number, and α being a fixed positive number. In accordance with (7) the function $h(t)$ defined by equation (4) in terms of Bromwich's contour integral is the solution of Carson's integral equation (2).

Heaviside's rules can be derived in a much more direct and rigorous manner from the contour integral than from the integral equation. Indeed the contour integral appears to furnish the key to the whole situation, making it possible to determine whether or not in a given case the Heaviside rule in question is applicable and to discuss in a satisfactory manner the results obtained by applying these rules. If it appears in a given case that Heaviside's rules are not applicable, the result is to be sought by studying the contour integral itself.

We shall now indicate briefly the derivation of Heaviside's three principal rules and give an example in which it can be seen from the contour integral why Heaviside's rule does not lead to a correct result.

(a) *The Power Series Solution.* Assume that the function $H(p)$ is such that the path of integration in (4) can be deformed into a circle C described about the origin as center, and that on C, $1/H(p)$ can be expanded in the convergent series

$$\frac{1}{H(p)} = a_0 + \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_n}{p^n} + \cdots.$$

Also expand e^{pt} in a power series in p. Then

$$h(t) = \frac{1}{2\pi i}\int_C \left(\frac{a_0}{p} + \frac{a_1}{p^2} + \frac{a_2}{p^3}\cdots\right) \cdot\left(1 + pt + \frac{p^2t^2}{2!} + \frac{p^3t^3}{3!} + \cdots\right)dp.$$

On forming the product of these series term by term, and arranging according to negative and positive powers of p, we find the Laurent expansion of the integrand in a circular ring enclosing the origin as center. The value of the integral is $2\pi i$ times the coefficient of the term in $1/p$. Hence we have

$$h(t) = a_0 + a_1 t + \frac{a_2 t^2}{2!} + \cdots + \frac{a_n t^n}{n!} + \cdots .$$

(b) *The Expansion Theorem.** Assume that $H(p)$ is a rational function which vanishes for n distinct values of p, but which does not vanish for $p=0$, and that the path of integration can be deformed into a circle c described about the origin as center and enclosing all the roots of $H(p)=0$. On calculating the residues at the points $0, p_1, p_2, \cdots, p_n$, the poles of the integrand, it is found that

$$h(t) = \frac{1}{H(0)} + \sum_{j=1}^{j=n} \frac{e^{p_j t}}{p_j H'(p_j)} .$$

The same treatment applies when $H(p)=0$ has multiple roots or when $p=0$ is a root. By an appropriate deformation of the path of integration it is possible to extend the method to cases in which $H(p)$ is a transcendental function.

(c) *The Asymptotic Solution.* The contour integral lends itself equally well to the discussion of the third of Heaviside's principal rules, the asymptotic solution, and the one for which Carson's treatment was not entirely satisfactory. The method given below of obtaining the asymptotic solution was developed independently by the writer but its essential features were discovered earlier by Bromwich† in treating a particular case.

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* The proof outlined here was published almost simultaneously by Bromwich and Wagner in 1916. For other interesting derivations of the expansion theorem, see Bromwich, Philosophical Magazine, vol. 37 (1919), pp. 407–19; and F. D. Murnaghan, this Bulletin, vol. 33 (1927), p. 81.

† Proceedings, Cambridge Philosophical Society, vol. 20 (1920–21), pp. 423–427.

Heaviside's rule states that if $1/H(p)$ can be expanded in the form

$$(8)\quad \frac{1}{H(p)} = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n + \cdots + (b_0 + b_1 p + b_2 p^2 + \cdots + b_n p^n + \cdots) p^{1/2},$$

$h(t)$ is obtained by discarding all terms in the first line excepting a_0, by replacing $p^{1/2}$ in the second line by $1/(\pi t)^{1/2}$ and p^n in this line by d^n/dt^n. It results that

$$(9)\quad h(t) = a_0 + \left(b_0 + b_1 \frac{d}{dt} + b_2 \frac{d^2}{dt^2} + \cdots\right) \frac{1}{(\pi t)^{1/2}}$$
$$= a_0 + \frac{1}{(\pi t)^{1/2}} \left(b_0 - b_1 \frac{1}{2t} + b_2 \frac{1 \cdot 3}{(2t)^2} - b_3 \frac{1 \cdot 3 \cdot 5}{(2t)^3} + \cdots\right).$$

Assume that $H(p)$ is such a function that the path of integration of (4) can be deformed into the path $ANPQB$ (see Fig. 1) enclosing the origin. This can be done in all

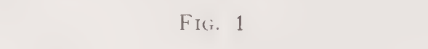

FIG. 1

cases which I have examined in which the formula can be applied and presumably in all such cases.

If $1/H(p)$ can be expanded in the form (8), we note that the integral of each term of the expression

$$\frac{1}{p}(a_1 p + a_2 p^2 + \cdots + a_n p^n + \cdots) e^{pt}$$

along the path $ANPQB$ is zero, while

$$\frac{1}{2\pi i} \int \frac{a_0}{p} e^{pt} dp = a_0,$$

the integral being taken along the same path as before. Accordingly, if we pass over, for the moment, all questions of convergence and make use of a proper determination

of $p^{1/2}$ by writing $p^{1/2} = -i\rho^{1/2}$ along AN and $p^{1/2} = i\rho^{1/2}$ along QB, we find

$$
\begin{aligned}
h(t) &= a_0 + \frac{1}{2\pi i}\int_{AN} \frac{e^{pt}}{p^{1/2}}(b_0 + b_1 p + b_2 p^2 + \cdots)dp \\
&\quad + \frac{1}{2\pi i}\int_{QB} \frac{e^{pt}}{p^{1/2}}(b_0 + b_1 p + b_2 p^2 + \cdots)dp \\
&= a_0 + \frac{1}{\pi}\int_0^\infty (b_0 - b_1\rho + b_2\rho^2 - b_3\rho^3 + \cdots)\frac{e^{-\rho t}}{\rho^{1/2}}d\rho
\end{aligned} \tag{10}
$$

Equation (9) follows immediately on integrating term by term.

This analysis, which is purely formal, has led to the asymptotic solution and gives an intelligible foundation for this rather strange rule of Heaviside's. In certain cases the series in (9) is convergent for all positive values of t, in other cases it is only asymptotically convergent, and in others it is meaningless.* In any case, it is necessary to study the integral and the calculation outlined in obtaining equations (9) and (10). If the coefficients b_0, b_1, $b_2 \cdots$ are such that the series (9) is convergent for positive values of t, the investigation is usually not difficult. If the series is only asymptotically convergent, it will be found in many cases that we can proceed in the following manner. If possible, write the operator in the form

$$
\begin{aligned}
\frac{1}{pH(p)} &= \frac{1}{p}(a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n) \\
&\quad + (b_0 + b_1 p + b_2 p^2 + \cdots + b_n p^n + r_{n+1} p^{n+1})p^{1/2},
\end{aligned}
$$

in which corresponding to a fixed n,

$$
|r_{n+1}| < A,
$$

provided that $|p| < k$, A being a fixed positive number. Let the points corresponding to $p = -k$ be denoted by M and R in Fig. 1. It can usually be shown that t can be chosen so large that the integrals of $e^{pt}/(pH(p))$ from $-\infty$ to M and

* Carson, Bell Technical Journal, vol. 4 (1925), pp. 744 and 749.

from R to $-\infty$, are arbitrarily small in absolute value. Also t can be chosen so large that the integrals from M to N and from Q to R of $r_{n+1}\, p^{n+1/2}\, e^{pt}$ is arbitrarily small. Then for a sufficiently large t, the absolute value of the difference between $h(t)$ and

$$a_0 + \frac{1}{\pi} \int_0^k [b_0 - b_1\rho + b_2\rho^2 - \cdots + b_n(-\rho)^n] \frac{e^{-\rho t}}{\rho^{1/2}} d\rho$$

is an arbitrarily small positive number. After this, the limit k can be replaced by ∞ for a sufficiently large t. Then $h(t)$ is given approximately by

$$h(t) = a_0 + \frac{1}{(\pi t)^{1/2}} \left[b_0 - \frac{b_1}{2t} + b^2 \frac{1 \cdot 3}{(2t)^2} - b_3 \frac{1 \cdot 3 \cdot 5}{(2t)^3} + \cdots + (-1)^n b_n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2t)^n} \right].$$

The importance of the contour integral in determining the applicability of Heaviside's rules is illustrated in the following example. Carson* showed that the application of the method of the asymptotic solution to the operational equation

$$h(t) = \frac{p^{1/2}}{p^2 + \omega^2},$$

led to an incorrect result. The reason that it does so is readily seen from a consideration of the contour integral

$$h(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{p^{1/2}}{p^2 + \omega^2} e^{pt} dp.$$

The path of integration cannot be deformed into the path of Fig. 1, because of the presence of singularities of the integrand at the points $p = \pm i\omega$. It is accordingly not to be expected that the asymptotic solution can be employed.

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* Bell Technical Journal, vol. 4 (1925), p. 749.

EXTENSIONS OF WARING'S THEOREM ON FOURTH POWERS*

BY L. E. DICKSON

1. *Introduction.* In 1770 Waring conjectured that every positive integer p is a sum of nineteen integral biquadrates. It is shown in § 8 that eight of them may be taken equal if $p \leqq 4100$. Again, sixteen of them may be taken equal in pairs if $p \leqq 2400$. All possible similar results are included in Theorem 1 of § 3.

2. *Notations and Definitions.* The form

$$(1) \qquad (a_1, \cdots, a_n) = a_1 x_1^4 + \cdots + a_n x_n^4, \qquad (0 < a_1 \leqq a_2 \leqq a_3 \cdots),$$

is said to be of *order* n and *weight* $a_1+a_2+\cdots+a_n$. Since $ax^4 = x^4+\cdots+x^4$, to a terms, a form of weight w is equal to a sum of w biquadrates. But 79 is not a sum of fewer than nineteen biquadrates. Hence 19 is the minimum weight of a form (1) which represents all positive integers.

Let f be a form (1) which represents p, and let $a_1 = r+s$. The form $g = (r, s, a_2, \cdots, a_n)$ shall be said to be derived from f by the *partition* of a_1 into $r+s$. If we give to the first two variables in g the same value x_1 as was employed in $f = p$, we see that also g represents p. Hence *any form derived from f by partition represents every integer which can be represented by f* (and usually represents further integers).

Write $a = 2^4$, $b = 3^4$, $c = 4^4$, $\cdots$. If a positive integer m can be expressed as a linear combination of $1, a, b, \cdots$, with integral coefficients $\geqq 0$ whose sum is $\leqq 19$, in one and only one way, m shall be called a *simple* number. In case there are exactly two such expressions, m shall be called a *double* number. Similarly for a *triple* or *k-fold* number.

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* Presented to the Society, December 31, 1926.

For example, $19=3+a$ is a double number, while $4+a$ is a simple number.

We write 1_2 for 1, 1; and 1_s for s ones.

3. *Summary of Results.*

THEOREM 1. *If a form of weight* 19 *represents all positive integers, it is* $A=(1122337)$, $B=(1122346)$, $C=(11112238)$, $D=(11112247)$, *or* $E=(11122228)$, *or a form derived from one of the five by partition.*

It is proved* in §§ 6, 7 that both A and B (of the minimum order 7) represent every positive integer $p\leqq 7^4=2401$. A like method was used to verify that D represents every $p\leqq 1300$, and that $F=(1122238)$ represents every $p\leqq 2000$ except the four-fold number

$$443 = 9 + a + 2b + c = 10 + 6a + b + c$$
$$= 6 + 2a + 5b = 7 + 7a + 4b.$$

The partitions $2=1+1$ and $3=1+2$ of F give C and E, respectively, and they represent $1+8+a+2b+d$. Hence C and E both represent every $p\leqq 2000$.

A form is evidently more likely to represent the integers just exceeding a given biquadrate than those just preceding it. As an excellent further check, it was verified that A, B, and F each represent 4000 up to $4096=8^4$. The same is therefore true of C and E.

4. *Conditions that a Form of Weight* 19 *shall Represent all Integers.* Let such a form f have the notation (1). Then

$$a_1 + a_2 + \cdots + a_n = 19. \tag{2}$$

To prove that

$$a_k \leqq 1 + a_1 + a_2 + \cdots + a_{k-1} \qquad (k = 2, \cdots, n), \tag{3}$$

write s for $a_1+\cdots+a_{k-1}$ and suppose that $a_k>1+s$. For $i=1, \cdots, k-1$, partition a_i into as many ones. Thus a

* Mr. K. C. Yang kindly extended my computations from 2000 to 2400 for A and B, and from 1000 to 1300 for D and E.

partition of f is $(1_s, a_k, \cdots, a_n)$, which does not represent $1+s$. The latter is a simple number since $19 \geqq s+a_k > 1+2s$, whence $1+s<10$.

Since f represents 1, $a_1=1$. By (3) with $k=2$, $a_2 \leqq 2$. If $a_2=2$, f would not represent the simple number $110=13+a+b$. Apply (3) with $k=3$. Hence

$$a_1 = 1, \quad a_2 = 1, \quad a_3 \leqq 3. \tag{4}$$

If $a_n \geqq 9$, a partition of f is $(1_{10}, 9)$ which does not represent the simple number $8+3a$. Hence

$$a_n \leqq 8. \tag{5}$$

By (2), (4), (5), we see that $n>4$.

If $a_{n-1}>4$, a partition of f is $(1_9, 5, 5)$, which does not represent the simple number $235=9+4a+2b$, since 9 is not a sum of certain of (1, 1, 1, 5, 5). Hence

$$a_{n-1} \leqq 4. \tag{6}$$

If $a_{n-2}>3$, a partition of f is $G=(1_7, 4, 4, 4)$. But G does not represent the simple number $221=11+3a+2b$, since 11 is not a sum of certain of (1, 1, 4, 4, 4). Hence

$$a_{n-2} \leqq 3. \tag{7}$$

Let $(111a_4 \cdots)$ with $a_4>2$ represent the simple number $15+2a$. Then 15 is a sum of certain of $1, a_4, \cdots, a_n$, contrary to $a_4+\cdots+a_n=19-3=16$, $a_i \geqq 3$.

Let $(113\cdots)$ represent $15+2a$. Then 15 is a sum of certain of $(3, a_4, \cdots, a_n)$, contrary to $a_4+\cdots+a_n=14$, $a_i \geqq 3$.

These two results and (4) give

$$a_3=1 \text{ or } 2. \text{ If } a_3=1, \text{ then } a_4=1 \text{ or } 2. \tag{8}$$

The following forms are excluded:

$$(1_5,3,3,3,5), \quad (1_5,3,3,4,4), \quad (1_4,3_5), \quad (1_2,2_6,5). \tag{9}$$

No one of the first three represents the simple number $207=13+2a+2b$. The fourth does not represent the simple number $141=12+3a+b$.

A partition of $(1_7, 4, 8)$ is G above (7). Partitions of $(1_5, 3, 3, 8)$ and $(1_5, 3, 4, 7)$ give the first and second forms (9) respectively. This proves

(10) f does not end with 4, 8, or 3, 3, 8, or 3, 4, 7.

Let $a_{n-3}>2$. By (7), $a_{n-3}=a_{n-2}=3$. By (6), $a_{n-1}=3$ or 4. In the second case, a partition gives (9_2). If, in the first case, $a_n\geqq 5$, a partition gives (9_1). This proves

(11) If $a_{n-3}>2$, then $a_{n-3}=a_{n-2}=a_{n-1}=3$, $a_n=3$ or 4.

5. *Proof of Theorem* 1. If $n=5$, (4)–(6) give

$$a_1+\cdots+a_5\leqq 1+1+3+4+8=17,$$

contrary to (2). Next, let $n=6$. If $a_3=1$, (4)-(7) give

$$a_1+\cdots+a_6\leqq 1+1+1+3+4+8=18,$$

contrary to (2). Hence $a_3=2$. By (2), (5)-(7),

$$19-4=a_4+a_5+a_6,\quad a_4\leqq 3,\quad a_5\leqq 4,\quad a_6\leqq 8,$$

whence the three signs are all $=$. By (10), the ending 4, 8 is excluded. Hence $n\geqq 7$.

Let $n=7$. By (5)-(7), $a_5\leqq 3$, $a_6\leqq 4$, $a_7\leqq 8$. If $a_3=a_4=1$, (2) gives $f=(1111348)$, which is excluded by (10). Next, let $a_3=1$, $a_4=2$. Then $a_5+a_6+a_7=19-5=14$, whence the sets of values of a_5, a_6, a_7 are 2, 4, 8; 3, 3, 8; 3, 4, 7, all excluded by (10). By (8), there remains only the case $a_3=2$. If $a_4>2$, (11) contradicts (2). Hence $a_4=2$, $a_5+a_6+a_7=13$. If $a_5=3$, f is A or B. If $a_5=2$, f is either (1122247), a partition of which gives the fourth excluded form (9), or else (1122238), the least number not represented by which is the quadruple number

$$443=9+a+2b+c=10+6a+b+c$$
$$=6+2a+5b=7+7a+4b.$$

Let $n=8$. By (5)-(7), $a_6\leqq 3$, $a_7\leqq 4$, $a_8\leqq 8$.

First, let $a_3=1$. Then $a_4\leqq 2$ by (8). If $a_5>2$, (11) gives $a_5+a_6+a_7+a_8=12$ or 13, while $a_1+a_2+a_3+a_4\leqq 5$, contrary to the relation (2). Hence we find $a_5\leqq 2$. If $a_4=2$, then

$a_5=2$, $a_6+a_7+a_8=19-7=12$. The sets of values of a_6, a_7, a_8 are 2, 2, 8; 2, 3, 7; 2, 4, 6; 3, 3, 6; 3, 4, 5. The first gives E. For the second, f is derived from A by the partition $3=1+2$. For the last three sets, f is a partition of B. Finally, let $a_4=1$. If $a_5=2$, the sets of values of a_6, a_7, a_8 are 2, 3, 8; 2, 4, 7; 3, 3, 7; 3, 4, 6. The first two give C and D. The last two yield partitions $2=1+1$ of A and B. If $a_5=1$, the sets are 2, 4, 8; 3, 3, 8; 3, 4, 7, which are excluded by (10).

Second, let $a_3=2$. If $a_4\geqq 3$, f is evidently $(1, 1, 2, 3_5)$, a partition of which is the third excluded form (9). Hence $a_4=2$. If $a_5>2$, f is evidently the partition (11223334) of A. There remains the case $f=(11222a_6a_7a_8)$. The sets of values of a_6, a_7, a_8 are 2, 2, 7; 2, 3, 6; 2, 4, 5; 3, 3, 5; 3, 4, 4. For the first and third sets, the partitions $7=2+5$ and $4=2+2$ respectively yield the fourth excluded form (9). For the second and last sets, f is a partition of B. For the fourth set, f is a partition of A.

Let $n=9$. By (5)-(7), $a_7\leqq 3$, $a_8\leqq 4$, $a_9=8$.

First, let $a_6>2$. By (11), $a_6=a_7=a_8=3$, $a_9=3$ or 4. We get the third excluded form (9) and the partitions $(1_4, 2, 3_3, 4)$ and $(1_3, 2_2, 3_4)$ of A.

Second, let $a_6=1$. Since the ending 4, 8 is excluded, f is evidently one of the partitions $(1_6, 2, 3, 8)$, $(1_6, 2, 4, 7)$, $(1_6, 3, 3, 7)$, $(1_6, 3, 4, 6)$ of C, D, A, B, respectively.

Finally, let $a_6=2$. If $a_5=1$, the symbol for f contains 1_5 and 2, followed by 2, 2, 8; 2, 3, 7; 2, 4, 6; 3, 3, 6; or 3, 4, 5. Thus f is a partition of C, A, B, A, or B, respectively. For $a_5=2$, $a_4=1$, f contains 1_4 and 2_2, followed by 2, 2, 7; 2, 3, 6; 2, 4, 5; 3, 3, 5; or 3, 4, 4, and is a partition of A or B. For $a_5=a_4=2$, $a_3=1$, f contains 1_3 and 2_3, followed by 2, 2, 6; 2, 3, 5; 2, 4, 4; or 3, 3, 4, and is a partition of B. There remains only the case in which f contains 1_2 and 2_4, followed by 2, 2, 5; 2, 3, 4; or 3, 3, 3. The first yields the fourth excluded form (9). The last two yield partitions of B.

Let $n\geqq 10$. The forms which satisfy (2), (5)-(7), and $a_1=a_2=1$ are all partitions of A, B, C or D. We subdivide cases according to the number of 1's in the symbol. For

example if $n=10$, the first case yields only $(1_8, 3, 8)$ and $(1_8, 4, 7)$.

6. *Forms related to B.* The tables employed extend to $7^4=2401$. To the sums of two biquadrates we add the double of each such sum, and obtain a table α of all integers <2401 which are represented by (1122). To each number in α we add the triples of biquadrates and obtain a table β of all integers <2401 which are represented by (11223). To each number in β we add the quadruples of biquadrates and obtain the following table γ of all integers <2401 which are represented by (112234):

0–13, 16–28, 32–43, 48–58, 64–73, 80–93, 96–106, 108, 112–23, 128–38, 144–53, 160–73, 176–88, 192–203, 208–18, 225–33, 241–53, 256–68, 272–83, 288–98, 304–13, 320–33, 336–48, 352–63, 368–78, 384–93, 400–13, 416–28, 432–43, 448–58, 465–73, 481–93, 498–508, 512–23, 528–38, 544–53, 560–73, 576–88, 592–603, 608–18, 624–37, 639–53, 656–68, 672–83, 688–98, 705–15, 717, 721–33, 737–48, 753–63, 768–78, 784–97, 800–13, 816–28, 832–43, 848–58, 864–77, 880–93, 896–908, 912–23, 928–38, 945–57, 961–70, 972–3, 977–88, 993–1003, 1009–18, 1024–37, 1040–50, 1052–3, 1056–68, 1072–83, 1088–98, 1104–17, 1120–30, 1132, 1136–48, 1152–63, 1168–78, 1185–97, 1201–10, 1212, 1217–28, 1233–43, 1249–61, 1265–77, 1280–92, 1296–1308, 1312–23, 1328–41, 1344–57, 1360–72, 1376–88, 1392–1403, 1408–21, 1424–33, 1435–7, 1440–8, 1450–52, 1456–68, 1472–83, 1488–1501, 1505–17, 1521–32, 1536–48, 1552–63, 1568–81, 1584–97, 1600–12, 1616–28, 1632–43, 1648–61, 1664–76, 1680–88, 1690–1, 1696–1708, 1713–23, 1728–41, 1745–56, 1761–72, 1777–88, 1792–1803, 1808–21, 1824–36, 1840–51, 1856–68, 1872–85, 1888–1901, 1904–16, 1920–32, 1934, 1936–48, 1952–65, 1968–81, 1985–96, 1998, 2000–11, 2017–28, 2033–45, 2048–61, 2064–76, 2080–92, 2096–2106, 2108, 2112–25, 2128–41, 2144–56, 2160–72, 2176–88, 2192–2205, 2208–20, 2225–36, 2241–52, 2257–66, 2268, 2273–85, 2290–2300, 2304–16, 2320–32, 2336–46, 2352–65, 2368–80, 2384–95, 2400.

THEOREM 2. *$B_m=(112234m)$ represents all positive integers $\leqq 2400$ if and only if $m=6, 7, 8, 9$.*

The numbers in table γ together with the numbers obtained by adding 6 to them give all numbers $\leqq 2400$ except $p=240$, 480, and $q=1455$. Each $p-6\cdot 2^4$ and $q-6\cdot 3^4$ is in γ. This proves Theorem 2 when $m=6$.

For $m=7$, 8, or 9, the numbers in γ together with the numbers obtained by adding m to them give all numbers $\leqq 2400$.

The least positive integer not represented by B_m is 234, 74, 59, 44, 29, 14 when $m=10$, 11, 12, 13, 14, $m>14$, respectively.

7. *Forms related to A.* To each number in table β(§ 6) we add the triples of biquadrates and obtain the following table δ of all integers <2401 which are represented by (112233):

0–12, 16–27, 32–42, 48–57, 64–72, 80–92, 96–105, 112–22, 128–37, 144–52, 160–72, 176–87, 192–202, 209–17, 225–32, 241–52, 256–67, 272, 274–82, 288–97, 304–12, 320–32, 336–47, 352–62, 368–77, 384–92, 400–2, 404–12, 416–27, 432–42, 449–57, 465–72, 482, 484–92, 497–507, 512–22, 528–37, 544–52, 560–72, 576–87, 592–602, 608–12, 614–7, 624–36, 640–52, 656–67, 672–82, 689–97, 705–14, 716, 721–7, 729–32, 737–47, 753–62, 768–77, 784–96, 800–12, 816–27, 832–42, 848–57, 864–76, 880–9, 891–2, 896–907, 912–22, 929–37, 945–56, 961–8, 971–2, 977–87, 993–1002, 1009–17, 1024–36, 1040–9, 1051, 1056–63, 1065–7, 1072–82, 1088–97, 1104–16, 1120–9, 1131, 1136–47, 1152–62, 1169–77, 1185–96, 1201–8, 1211, 1217–24, 1226, 1233–7, 1240–2, 1249–60, 1265–76, 1280–91, 1296–1307, 1312–22, 1328–40, 1344–56, 1360–7, 1369–71, 1376–87, 1392–1401, 1408–20, 1424–32, 1434–6, 1440–8, 1450–1, 1456–67, 1472–82, 1489–1500, 1505–16, 1521–31, 1536–47, 1552–62, 1568–80, 1584–93, 1595, 1600–7, 1609–10, 1616–27, 1632–42, 1648–60, 1664–5, 1667–72, 1674–5, 1680–7, 1690–1, 1697–1707, 1712–22, 1729–40, 1745–7, 1749–55, 1761–70, 1778–87, 1792–1802,

1808–20, 1824–7, 1829–33, 1835, 1840–7, 1849–50, 1856–67, 1872–84, 1888–1900, 1904–15, 1920–31, 1934, 1936–47, 1952–64, 1969–77, 1979–80, 1982, 1985–95, 2001–10, 2017–25, 2027, 2033–44, 2048–60, 2064–75, 2080–91, 2096–2107, 2112–24, 2128–39, 2144–55, 2160–71, 2176–7, 2179–85, 2187, 2192–2204, 2209–19, 2225–35, 2241–51, 2257–65, 2274–84, 2290–9, 2304–14, 2320–31, 2336–45, 2352, 2354–64, 2368–79, 2384–5, 2387–94, 2400.

We may now readily verify that $A=(1122337)$ represents all positive integers <2401. The numbers in δ together with the numbers obtained by adding 7 to them give all integers <2401 except

$$p = 240,\ 403,\ 480\text{–}1,\ 483,\ 613,\ 976,\ 1216,\ 1232,\ 1238\text{–}9,\ 1673,\ 1696,\ 1748,\ 1828,\ 2273,\ 2353;$$

$$q = 620,\ 735,\ 1071,\ 1245\text{–}6,\ 1375,\ 1615,\ 1695,\ 1855.$$

Each $p-7\cdot 2^4$ and $q-7\cdot 3^4$ is in δ.

Finally, $(112233m)$ fails to represent 621, 622, 73, 58, 43, 13 when $m=8, 9, 10, 11, 12, m\geqq 13$, respectively.

THEOREM 3. *The form* $(112233m)$ *represents all positive integers* <2401 *if and only if* $m=7$.

8. THEOREM 4. *Every positive integer* $\leqq 4100$ *can be expressed in the form* $S+8x^4$, *where* S *is a sum of eleven integral biquadrates, and* $x=0, 1,$ *or* 2.

At the suggestion of Jacobi, Bretschneider* gave a Table 1 of all those decompositions of $1, \cdots, 4100$ into biquadrates >0 for which the number of the latter is a minimum. From it† we conclude that all numbers $\leqq 4100$ are sums of eleven biquadrates $\geqq 0$, except

(12) 12–15, 27–31, 42–47, 57–63, $\cdots$, 4091–5.

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* Journal für Mathematik, vol. 46 (1853), pp. 1–23.

† Or by taking the aggregate of the numbers in Parts XI–XVIII of his Table 2, which lists the integers which are sums of a prescribed number of biquadrates >0.

If p occurs in the set (12), $p-8$ is not in it except for $p=232, 240, 472, 480$. For these four, $p-8\cdot 2^4$ is not in the set (12). This proves Theorem 4.

9. Corresponding results for cubes are obtained in the writer's paper in the American Mathematical Monthly for April, 1927. Assistance has been provided by the Carnegie Institution for the more elaborate investigation of fifth and higher powers.

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TESTS FOR PRIMALITY BY THE CONVERSE OF FERMAT'S THEOREM*

BY D. H. LEHMER

There are, generally speaking, two distinct methods for determining the primality of a large integer without trying possible divisors. Up to this time the method which goes by the name of Lucas' test† has yielded the most results. It is particularly well adapted to the investigation of Mersenne numbers and has consequently led to the identification of the three largest primes heretofore known, namely, $2^{89}-1$, $2^{107}-1$ and $2^{127}-1$. The other method is based on the converse of Fermat's theorem. It is the purpose of this paper to discuss certain improvements in this method, and to apply it to some numbers of the form $10^n\pm 1$.

It has long been known that the simple converse of Fermat's theorem, namely: *If* $a^x\equiv 1 \pmod N$ *for* $x=N-1$, *then* N *is a prime*, is *not* true, as is shown by the simple example: $4^{14}\equiv 1 \pmod{15}$. A true converse of this theorem was first given by Lucas‡ in 1876: *If* $a^x\equiv 1 \pmod N$ *for* $x=N-1$, *but not for* $x<N-1$, *then* N *is a prime*. In 1891 he proved the following theorem.§

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* Presented to the Society, San Francisco Section, April 2, 1927.

† American Journal, vol. 1 (1878), pp. 184–220.

‡ Lucas, loc. cit., p. 302.

§ *Théorie des Nombres*, 1891, pp. 423, 441.

THEOREM 1. *If* $a^x \equiv 1 \pmod{N}$ *for* $x = N-1$, *but not for* x *a proper divisor of* $N-1$, *then* N *is a prime.*

When applied to a particular N, this theorem exhibits three defects. In the first place, the complete factorization of $N-1$ must be known. Secondly, the number of values of x which must be tried in order to show that the second part of the hypothesis is fulfilled, may be impossibly large. Thirdly, the condition for primality is sufficient but not necessary. If, however, N is of the form 2^n+1, the first two defects vanish; for in this case all the divisors of $N-1$ are powers of 2, so that in testing for the first part of the hypothesis, the second part is automatically taken care of in the successive squarings of the residues modulo N. Unfortunately the only numbers of the form 2^n+1 that have any chance to be primes are the Fermat numbers in which n is a power of 2. The numbers $2^{128}+1$ and $2^{256}+1$ have been tested in this way by Morehead and A. E. Western,* and both numbers were found to be composite. The next such number awaiting investigation is $2^{1024}+1$, a number of 309 digits. A skillful computer could test this number in about ten years. As far as is known to the present author, no prime above the range of ordinary methods of factorization has ever been identified by the converse of Fermat's theorem.† It is clear that Theorem 1 must be improved before further results are possible.

The first defect has to do with the factorization of $N-1$, and is difficult to overcome in case N is a number of unknown form. In many cases it is no easier to factor $N-1$ completely than to factor N. If, however, N is the maximum algebraically prime factor of a number of the form $y^n \pm 1$, it is usually

* This Bulletin, vol. 11 (1905), pp. 543; and vol. 16 (1909), pp. 1–6.

† An attempt to establish the primality of $2^{61}-1$ was made by Seelhoff (Zeitschrift für Mathematik und Physik, vol. 31 (1886), pp. 174–178) using methods depending on the converse of Fermat's theorem. The proof is invalid, as was pointed out by Cole. It depends upon the false proposition that if a divides b and also c, then either b divides c or else c divides b. Despite this obvious error, the proof of the primality of $2^{61}-1$ is invariably attributed to Seelhoff.

possible to decompose $N-1$ into algebraic factors as is shown in the following examples:

$$\frac{y^n \pm 1}{y \pm 1} - 1 = \frac{y}{y \pm 1}(y^{n-1} - 1),$$

$$\frac{y^{4n+2} \pm 1}{y^2 \pm 1} - 1 = \frac{y^2}{y^2 \pm 1}(y^{4n} - 1),$$

$$\frac{y^{4n} \pm 1}{y^4 \pm 1} - 1 = \frac{y^4}{y^4 \pm 1}(y^{4n-4} - 1),$$

$$\frac{y^{8n} \pm 1}{y^8 \pm 1} - 1 = \frac{y^8}{y^8 \pm 1}(y^{8n-8} - 1),$$

$$\frac{y^{n^2} \pm 1}{y^n \pm 1} - 1 = \frac{y^n}{y^n \pm 1}(y^{n^2-n} - 1),$$

. . . . . .

The large factors on the right are differences of squares. Thus the factors of $N-1$ are made to depend upon factorizations of $y^n \pm 1$ for much smaller values of n. These may be taken from tables* previously computed. In case $N-1$ cannot be expressed in some such form as the above, one can always be assured of having n as a factor of $N-1$. If a small factor of N is known (as it often is from congruence tables) and if N' is the residuary factor, the factorization of $N'-1$ is again very difficult. In this case, however, we have n as a factor, but it is of little use except in reducing the size of the number to be factored. We shall see later that it is not necessary to know the complete factorization of $N-1$.

The next improvement in Theorem 1 that suggests itself is the reduction of the number of values of x to be tried. In the following theorem this number is reduced from the number of divisors of $N-1$ to the number of its prime factors.

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* Cunningham and Woodall, *Factorization of* $y^n \pm 1$, London, 1925.

THEOREM 2. *If* $a^x \equiv 1 \pmod{N}$ *for* $x = N-1$, *but not for* x *a quotient of* $N-1$ *on division by any of its prime factors, then* N *is a prime.*

Let

$$N - 1 = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t} = m_i p_i, \qquad (i = 1,2,3, \cdots, t).$$

Then we have for an hypothesis:

$$a^{N-1} \equiv 1 \pmod{N} \tag{1}$$

$$a^{m_i} \not\equiv 1 \pmod{N}, \qquad (i = 1,2,3, \cdots, t). \tag{2}$$

Now let ϵ be the smallest value of x satisfying $a^x \equiv 1 \pmod{N}$. Then ϵ divides $N-1$. For, if not, let $N-1 = n\epsilon + \delta$. Then since

$$a^{N-1} = a^{n\epsilon+\delta} \equiv 1 \pmod{N},$$

it follows that $a^\delta \equiv 1 \pmod{N}$, which is impossible, since $\delta < \epsilon$. But ϵ does not divide m_i, for if it did, we would have $a^{m_i} \equiv 1 \pmod{N}$. Now let us write

$$\epsilon = n p_1^{\beta_1} p_2^{\beta_2} \cdots p_t^{\beta_t},$$

where β_i may be zero and n is free of p_i. Consider the quotient

$$\frac{N-1}{\epsilon} = \frac{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}}{n p_1^{\beta_1} p_2^{\beta_2} \cdots p_t^{\beta_t}}.$$

Since $N-1$ is divisible by ϵ, we see that $n = 1$, and further,

$$\beta_1 \leqq \alpha_1, \ \beta_2 \leqq \alpha_2, \ \cdots, \ \beta_t \leqq \alpha_t.$$

Now the quotient

$$\frac{m_1}{\epsilon} = \frac{p_1^{\alpha_1 - 1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}}{p_1^{\beta_1} p_2^{\beta_2} \cdots p_t^{\beta_t}}$$

is not an integer, but it is identical with the preceding quotient except for the p_1's, so that we can write $\beta_1 > \alpha_1 - 1$, and in general $\beta_i > \alpha_i - 1$. But we have seen that $\beta_i \leqq \alpha_i$. Hence $\beta_i = \alpha_i$ and $\epsilon = N-1$. Finally let $\phi(N)$ be the totient of N. Then $a^{\phi(N)} \equiv 1 \pmod{N}$. Now if N were composite,

we would have $\phi(N) < N-1$. But this is impossible since $N-1=\epsilon$, and this is the smallest value of x satisfying $a^x \equiv 1 \pmod{N}$. Hence N is a prime.

Although this theorem is an improvement on Theorem 1, it still has the third defect of that theorem; that is, it tells us nothing of the character of a number N which satisfies the first part of the hypothesis but not the second. We have this situation, for example, whenever a is a quadratic residue of the prime N. It is clear that this defect cannot be eliminated by any theorem having the general form of Theorems 1 and 2. With this in mind we change our point of attack, and, by introducing a new element in the hypothesis, arrive at the following theorem.

THEOREM 3. *If* $a^x \equiv 1 \pmod{N}$ *for* $x = N-1$ *and if* $a^x \equiv r > 1$ *for* $x = (N-1)/p$, *and if* $r-1$ *is prime to* N, *then all the prime factors of* N *are of the form* $np^\alpha + 1$, *where* α *is the highest power to which the prime* p *occurs as a divisor of* $N-1$.

Let $N-1 = qp^\alpha = mp$. Then we have for an hypothesis:

$$a^{N-1} \equiv 1 \pmod{N}, \tag{1}$$

$$a^m \equiv r \pmod{N}, \tag{2}$$

$$r - 1 \text{ is prime to } N. \tag{3}$$

Now let κ be any prime factor of N and let ξ be the smallest value of x such that $a^x \equiv 1 \pmod{\kappa}$. Then since $a^{N-1} \equiv 1 \pmod{\kappa}$, we can show as in the preceding theorem that ξ divides $N-1$. Also by Fermat's theorem $a^{\kappa-1} \equiv 1 \pmod{\kappa}$. Hence ξ divides $\kappa - 1$. But ξ does not divide m, for if it did we would have $a^m \equiv 1 \pmod{\kappa}$, which would imply that $r-1$ is divisible by κ. But $r-1$ is prime to N, and hence is prime to κ. Let us write $\xi = n_1 p^\beta$, where n_1 is prime to p, and β may be zero. Consider the quotient

$$\frac{N-1}{\xi} = \frac{qp^\alpha}{n_1 p^\beta}.$$

Since q is also prime to p, we have $\beta \leqq \alpha$. Now the quotient

$$\frac{m}{\xi} = \frac{q p^{\alpha-1}}{n_1 p^{\beta}}$$

is not an integer, since m is not divisible by ξ. Hence $\beta > \alpha - 1$. That is, $\beta = \alpha$ and $\xi = n_1 p^{\alpha}$. But since ξ divides $\kappa - 1$, we have $\kappa - 1 = n_2 \xi = n_1 n_2 p^{\alpha}$. Hence $\kappa = n p^{\alpha} + 1$.

This theorem is the basis of a method given by Pocklington,* the importance of which seems to have escaped attention. In order to remove the final defect of Theorem 1, we must discuss the cases in which the various parts of the hypothesis are not fulfilled. If $a^{N-1} \equiv 1 \pmod{N}$, then N is composite, and its factors are not obtainable by this method. If $a^{m} \equiv 1 \pmod{N}$, all the factors of N divide the number $a^{m} - 1$, whose primitive factors are of the form $2nm + 1$. It is usually too troublesome, however, to try to show that N is a primitive factor. In practice, it is better to take m/p_2 or m/p_1^2 for a new value of m. If then $a^{m} \not\equiv 1$, all the factors of N are of the form $n p_2^{\alpha_2} + 1$ or $n p_1^{\alpha_1 - 1} + 1$, respectively. If $p_1 = 2$, this last case may be investigated without further calculation. However, this failure of the second part of the hypothesis is very rare, especially when care is taken in choosing the base a. Finally, if $r - 1$ is not prime to N, their G.C.D. will be a factor of N.

Suppose that we have applied Theorem 3 to a particular N and we find that $\kappa = n p_1^{\alpha_2} + 1$. Then N is a prime if $p^{\alpha_1} > N^{1/2}$. If on the other hand $p_1^{\alpha_1}$ is too small, other factors of $N - 1$ may be dealt with in the same manner, so that we obtain $\kappa = n p_1^{\alpha_1} p_2^{\alpha_2} \cdots + 1$, a form sufficiently exclusive to meet our needs. Suppose we have found in this way that $\kappa = nP + 1$, where P is the product of certain odd prime factors of $N - 1$. Our first inclination is to let n take on successive integral values, beginning with unity, and then try as divisors of N the prime values of κ. If $N^{1/2} > 10^{7}$ however, κ will run above the present limit of the list of primes, and we must

* Proceedings of the Cambridge Philosophical Society, vol. 18 (1914–16), p. 29.

try composite values of κ as well. Of course, about 5/6 of these are easily seen to be multiples of small primes; as for the others, it is easier actually to try them as divisors. All this may entail considerable labor, which may, for the most part, be obviated by seeking to express N as the difference of two squares. If every factor of $N=a^2-b^2$ is of the form $nP+1$, where P is odd, it follows that*

$$a^2 \equiv N \pmod{P^2}, \tag{1}$$

$$a \equiv 1 \pmod{P}. \tag{2}$$

One of the two values of a involved in (1) is eliminated by (2) so that a is restricted to one case in P^2. In fact, if

$$N \equiv 1 + kP \pmod{P^2},$$

then

$$a \equiv 1 + \frac{k}{2} P \pmod{P^2},$$

where, if k is odd, $k/2$ is taken modulo P. If the factors of N are, at the same time, of the form $n2^\lambda+1$, a similar restriction may be set up modulo $2^{2\lambda-1}$, which may be useful when λ is large. Suppose it is known that no factor of N is less than some limit W. Then a has the following range of values:

$$N^{1/2} < a < \frac{1}{2}\left(W + \frac{N}{W}\right).$$

In our case W is at least as large as $2P$, so that we have

$$N^{1/2} < a < \left(P + \frac{N}{4P}\right).$$

Although a has a greater range than κ, the possible values of a are more restricted especially when P is large. When P is small, so that there are a great number of values of either a or κ to be tried, use can be made of the "movable strip" method† of combining linear forms, which again favors the

* Lawerence, Quarterly Journal, vol. 28 (1896), pp. 285–311. Kraitchik, *Théorie des Nombres*, Paris, 1922, p. 146.

† Kraitchik, loc. cit., Chapter 2; also see Lawerence, loc. cit.

search for a. It is sometimes advantageous to use both methods. The direct method may be used to increase the limit W, thus decreasing the range for a, so that the work may be easily finished by the difference of squares.

Of the numbers $y^n \pm 1$, those for which $y=2$ have been most extensively investigated. Perhaps $y=10$ comes second in importance. If the results with 10^n-1 are comparatively few, it is not to be attributed to lack of interest, but rather to the lack of adequate methods with which to deal with these larger numbers. The preceding theory is most easily applied to divisors of $10^n \pm 1$, because of the ease with which the calculations may be performed.

Instead of employing the modulus N in our work, we may use any multiple of N; and, having found the residue of a^{N-1} modulo kN, we have only to divide the result by N to obtain the desired result. If N is a divisor of $10^n \pm 1$, we may choose k such that $10^n \pm 1 = kN$. The advantage in using the modulus $10^n \pm 1$ is that the division by this modulus may be performed by a single subtraction or addition. To cast 10^n+1 out of a number of $2n$ digits, we separate it into periods of n digits each, beginning at the right, and subtract the second period from the first. For 10^n-1, we simply add the two periods. An added advantage in using a composite modulus is that it provides an easy method of checking the work. We may select some small divisor of k, such as δ, and then construct an auxiliary table of $a^x \pmod{\delta}$, containing at most $\phi(\delta)$ entries. This table enables us to predict what any particular residue of $a^x \pmod{kN}$ should be congruent to $\pmod{\delta}$, by finding what x is congruent to $\pmod{\phi(\delta)}$. The calculation of the residue of a^{N-1} may be most easily performed as follows. We first make a table of exponents beginning with $n-1$, in which each entry is obtained by writing in the greatest integer in half the entry just above. This table has roughly $3 \log_{10} N$ entries, the final one being unity. Starting now at the bottom, we make a table of powers of a, each line being obtained by taking the square of the preceding entry, modulo $10^n \pm 1$, and multiplying this result

by a, whenever the desired entry corresponds to an odd exponent. On arriving at the top of the table, we cast N out of the final result and obtain $a^{N-1} \pmod N$. Of course the same procedure applies to the calculation of the residue of a^m.

Below are the results of applying Theorem 3 to seven divisors of $10^n \pm 1$ for $n = 19$, 20, 23, 24, 27, and 31. Besides the check already mentioned, every step in the following work has been verified by casting out multiples of 1001. It is confidently believed that, in those cases where $a^{N-1} \not\equiv 1$, this result is not due to a slip in the computation. However, the actual residues are given in hopes that they may prove useful to future workers who may wish to verify the results obtained.

1. *Example* 1. $N = 440,334,654,777,631$.

This number is the residuary factor of $10^{27}-1$. In fact

$$10^{27}-1 = 3^5 \cdot 37 \cdot 757 \cdot 333667 \cdot N.$$

The discovery of the factors 3 and 757 by congruence tables makes the factorization of $N-1$ a rather serious problem. At first sight we have

$$N-1 = 2 \cdot 3^3 \cdot 5 \cdot 1,630,869,091,769.$$

The first two factors are of no use, since we know in advance that every factor of N is of the form $54n+1$. After examining the large factor for divisors less than 1000 without result, 5 was chosen as p^α; it was found that

$$3^{(N-1)/5} = 3^m \equiv 31343325933897 = r \pmod N,$$

and further that

$$r^5 \equiv 3^{N-1} \equiv 1 \pmod N,$$

$r-1$ being prime to N. Every factor of N is then of the form $5n+1$. It was next decided to undertake the arduous though not impossible task of seeking to represent N as the difference of two squares, taking for W the limit 120,000 set by congruence tables. Before actually starting the work, however, a further attempt to factor the number 1,630,869,091,769

was made by means of the "factor stencils" of D. N. Lehmer,* which, though still incomplete, are fortunately available to the present author. Although this number of unknown form is nearly 1000 times larger than the intended limit of the stencils, it was found without great difficulty that

$$1,630,869,091,769 = 31249 \cdot 52,189,481.$$

Choosing 52189481 as a new value of p^α, it was easily found that

$$3^m \equiv 78533825886276 = r \pmod{N},$$

$r-1$ being prime to N. The factors of N are then of the form $52189481n+1$. But $N^{1/2}$ is less than 20984153. Hence N is a prime and we have the complete factorization

$$10^{27} - 1 = 3^5 \cdot 37 \cdot 757 \cdot 333667 \cdot 440334654777631.$$

2. *Example* 2. $N = 9,999,000,099,990,001.$

This number is $(10^{20}+1)/(10^4+1.)$ It was found that

$$3^{N-1} \equiv 3703264653988674 \pmod{N}.$$

Hence N is composite. This number has been examined for factors less than its cube root without success. It is therefore the product of two primes.

3. *Example* 3. $N = 9,999,999,900,000,001.$

This number is $(10^{24}+1)/(10^8+1)$. In this case we have

$$N - 1 = \frac{10^{24}+1}{10^8+1} - 1 = \frac{10^8}{10^8+1}(10^{16} - 1)$$
$$= 2^8 \cdot 5^8 \cdot 3^2 \cdot 11 \cdot 73 \cdot 101 \cdot 137.$$

The divisors 2^8 and 5^8 were chosen as values of p^α and tested in one operation as follows. It was first found that

$$7^{(N-1)/10} \equiv 7128121476353673 = r \pmod{N}.$$

Then

$$r^2 \equiv 7^{(N-1)/5} \equiv 428233546143224 \pmod{N},$$

* Proceedings of the National Academy, vol. 11 (1925), p. 97.

and finally

$$r^5 \equiv -1 \quad \text{and} \quad r^{10} \equiv 1 \pmod{N}.$$

Since r^2-1 is prime to N, it follows that every factor of N is of the form 5^8n+1 and also of the form 2^8n+1. But $N^{1/2}<10^8$, so that N is a prime and we have the complete factorization:

$$10^{24}+1 = 17 \cdot 5882353 \cdot 9999999900000001.$$

4. *Example* 4. $N = 909,090,909,090,909,091.$

This number is $(10^{19}+1)/11$. It was found that

$$3^{N-1} \equiv 1 \pmod{N}.$$

Now we have

$$N-1 = \frac{10^{19}+1}{10+1} - 1 = \frac{10}{11}(10^{18}-1)$$
$$= 2 \cdot 3^4 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 52579 \cdot 333667.$$

Choosing as p the number 333667, we obtain

$$3^m \equiv 314776999832050172 = r \pmod{N},$$

$r-1$ being prime to N. Every factor of N is then of the form $333667n+1$, and also $19n+1$; or, in other words, $6339673n+1$. Using this restriction, we might test directly for the factors of N with n ranging from 1 to 150. About 125 of these values of κ might be ruled out as being multiples of small primes. The remaining 25 would have to be tested. Instead, let us write $N=a^2-b^2$. Then since

$$N \equiv 1 + 56743 \cdot 6339673 \pmod{(6339673)^2},$$

it follows that

$$a \equiv 1+3202708 \cdot 6339673 \pmod{(6339673)^2},$$

or

$$a = 4019453746929n + 20304121434485. \tag{1}$$

Taking $2P$ or 12679346 as a value of W, we find that a has the following range:

$$933462589 < a < 35855622005. \tag{2}$$

The smallest value of a allowed by (1) is more than 500 times larger than the largest value allowed by (2). Hence a does not exist and N is a prime. We then have the complete factorization

$$10^{19} + 1 = 11 \cdot 909090909090909091.$$

5. *Example* 5. $N = 999,999,999,000,000,001.$

This number is $(10^{27}-1)/(10^9-1)$. In this case it was found that

$$3^{N-1} \equiv 437299180785434949 \pmod{N};$$

N is therefore composite. This number and the third one examined above fill in two entries in the following rather curious table:

91	Composite
9901	Prime
999001	Composite
99990001	Prime
9999900001	Composite
999999000001	Prime
99999990000001	Composite
9999999900000001	Prime
999999999000000001	Composite

Unfortunately, the next entry in this table is composite.

6. *Example* 6. $N = 11,111,111,111,111,111,111,111.$

This number is $(10^{23}-1)/9$. It was found that

$$3^{N-1} \equiv 1268486354649455149380 \pmod{N}.$$

Hence we know that N is composite. This number is listed as a prime in a table of divisors of $10^n \pm 1$ given by Loof.* In a French edition† of this table is given the curious entry: $11111 \cdot 11111$? Bickmore‡ marks this number as a prime

* Archiv der Mathematik und Physik, vol. 16 (1851), pp. 54–57.

† Nouvelles Annales, vol. 14 (1855), pp. 115–117. Compare Messenger of Mathematics, vol. 33 (1903–04), p. 96.

‡ Nouvelles Annales, (3), vol. 15 (1896), p. 222.

on the authority of Loof. The character of $(10^{23}-1)/9$ as revealed for the first time by the above result will be settled once and for all only when the factors are discovered. The numbers made up entirely of the digit 1 are of interest on account of the rarity of primes among them. Only those numbers that have a prime number of digits have any chance to be primes, and this set may be said to correspond to the Mersenne numbers. Besides the obvious cases 1 and 11, this former set contains but one other prime less than the limit $(10^{37}-1)/9$. This prime is $(10^{19}-1)/9$ and was examined independently by Hoppe and Kraitchik by the difference of two squares. By way of comparison with the Mersenne numbers, it is found that there are no less than 9 primes less than $2^{37}-1$. The numbers $10^{2^n}+1$, which correspond to the Fermat numbers, also show a scarcity of primes. The only primes that have been found in this set are 11 and 101, up to the limit $10^{64}+1$.

7. *Example* 7. $N = 909,090,909,090,909,090,909,090,909,091$.

This number is $(10^{31}+1)/11$. In this case it was found that

$$3^{N-1} \equiv 1 \pmod{N}.$$

Now we have

$$N - 1 = \frac{10^{31} + 1}{10 + 1} - 1 = \frac{10}{11}(10^{30} - 1)$$

$$= 2 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 31 \cdot 37 \cdot 41 \cdot 211 \cdot 241 \cdot 271 \cdot 2161 \cdot 9091 \cdot 2906161.$$

Here $N-1$ breaks up into so many small factors that no single one used for p gives sufficient restriction. Selecting $p_1 = 2906161$ and $p_2 = 9091$, it was found that

$$3^{m_1} \equiv 40421597451230127586416772 1473 = r_1 \pmod{N},$$

and

$$3^{m_2} \equiv 55433011245022437225001034 8117 = r_2 \pmod{N},$$

r_1-1 and r_2-1 being prime to N. Hence the factors of N belong to the forms

$$\left.\begin{matrix} 31n+1 \\ 9091n+1 \\ 2906161n+1 \end{matrix}\right\} \text{ or } 819017199181n+1.$$

Now $N^{1/2}=953462589245592$. To test directly for the factors of N would require 1164 values of n. Instead, we find that

$$N \equiv 1 + 645945952735 \cdot 819017199181 \pmod{(819017199181)^2},$$

so that

$$a \equiv 1 + 732481575958 \cdot 819017199181 \pmod{(819017199181)^2}$$

or

$$a = 670789172554289827070761n + 599915008792806066890399. \tag{1}$$

To find the range for a, we let $W=2P=1638034398362$ and obtain

$$953462589245592 < a < 554988900110999445. \tag{2}$$

We see that the smallest value of a in (1) is more than a million times larger than the largest value in (2). Hence a does not exist and N is a prime. We then have the complete factorization

$$10^{31}+1 = 11 \cdot 909090909090909090909090909091.$$

The author's acknowledgements are due to Miss E. M. Trotskaia for help in carrying out the above calculations.

The University of California

ON ANALYTIC SOLUTIONS OF DIFFERENTIAL EQUATIONS IN THE NEIGHBORHOOD OF NON-ANALYTIC SINGULAR POINTS*

BY B. O. KOOPMAN†

We shall consider the system

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n} \tag{1}$$

in the complex neighborhood of a point, say $(0, 0, \cdots, 0)$, at which the X's vanish simultaneously. We shall suppose throughout that

$$X_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + \phi_i(x_1, \cdots, x_n), \qquad (i = 1, 2, \cdots, n),$$

and that the general solutions of the related linear system

$$\frac{dx_1}{a_{11}x_1 + \cdots + a_{1n}x_n} = \cdots = \frac{dx_n}{a_{n1}x_1 + \cdots + a_{nn}x_n} \tag{2}$$

approach zero, without being identically zero.‡ Consider a solution of the equations (2). When will there exist an analytic solution of the equations (1) which behaves essentially like this near $(0, 0, \cdots, 0)$? How many such solutions will there be? Our object is to answer these questions without restricting the functions ϕ_i, except as regards their behavior in a certain region of the complex $x_1x_2 \cdots x_n$-space abutting on the origin, in which the given solution of (2) is embedded. We shall thus obtain a treatment less restricted than those of Briot and Bouquet, Poincaré, Picard, Dulac, Horn, etc., who assume the ϕ's to be convergent power series of terms of degree higher than the first.§

* Presented to the Society, October 30, 1926.

† National Research Fellow in Mathematics.

‡ See, however, the concluding note of this paper.

§ For full references, see J. Malmquist, *Sur les points singuliers des équations différentielles*, Arkiv för Matematik, Astronomi och Fysik, vol. 15, No. 3, (1920), pp. 1–5.

It is necessary first of all to reduce the system to a standard form. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the roots of the equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0,$$

and $r_1, r_2 \cdots, r_\nu$ their respective multiplicities $(r_1+\cdots+r_\nu=n)$. Then we may, according to the classical theory,* transform the equations (1) by means of a linear substitution of the x's so that the expressions for the new X's in terms of the new x's become

$$(3)\quad \begin{cases} \lambda_1 x_1+\phi_1,\ \lambda_1 x_2+x_1+\phi_2,\ \cdots,\lambda_1 x_{r_1}+x_{r_1-1}+\phi_{r_1}, \\ \lambda_2 x_{r_1+1}+\phi_{r_1+1},\ \lambda_2 x_{r_1+2}+x_{r_1+1}+\phi_{r_1+2},\ \cdots, \\ \qquad\qquad \lambda_2 x_{r_1+r_2}+x_{r_1+r_2-1}+\phi_{r_1+r_2}, \\ \cdots\cdots\cdots\

Here, dt is the common ratio in (2). With the aid of these expressions it is easy to prove that, in the case we are considering, where the general solution of (2) approaches $(0, 0, \cdots, 0)$, the points λ_i all lie on the side away from the origin of a straight line, l: $\arg \lambda = c$, drawn in the λ-plane, not through the origin. Now the transformation of the x's which leads to the expressions (3) may be so conducted that $\lambda_1' = 1$, while the real parts of $\lambda_2, \cdots, \lambda_\nu$ are not less than 1. This is accomplished by dividing the X's through by the appropriate $\lambda' e^{c-\pi/2}$, say $\lambda_1' e^{c-\pi/2}$, at a certain stage of the transformation. Furthermore, the effect of performing the substitution

$$x_{r_1+1} = x_1 x_{r_1}'{}_{+1}, \quad x_{r_1+2} = x_1 x_{r_1}'{}_{+2}, \cdots, \quad x_{r_1+r_2} = x_1 x_{r_1}'{}_{+r_2},$$

is to replace the values in (3) by similar expressions in which λ_2 is replaced by $\lambda_2 - \lambda_1$. Thus we may assume that the real parts of $\lambda_2, \cdots, \lambda_\nu$ are all less than 2.

Consider then the system

$$(1') \qquad \frac{dx_i}{dt} = X_i, \qquad (i = 1, 2, \cdots, n)$$

where the X's have the values (3), and where

$$1 \leqq \lambda_i' < 2, \quad (\lambda_i = \lambda_i' + (-1)^{1/2}\lambda_i''), \quad (i = 1, 2, 3, \cdots, \nu).$$

Let there be given a special solution:

$$(4') \qquad x_i = f_i(t), \qquad (i = 1, 2, \cdots, n)$$

of the corresponding linear equations (2); i. e., let known values of the constants C in equations (4) be given. Let T be the semi-infinite band in the t-plane defined by the inequalities $T: -\infty < t' \leqq 0$, $-h < t'' < h$, where $t = t' + (-1)^{1/2}t''$, and h is some positive constant. By means of the equations (4') the strip T goes over into a two-dimensional figure in the complex $2n$-dimensional $x_1 x_2 \cdots x_n$-space; as the point t moves out to infinity in T, the corresponding point $(x_1, x_2, \cdots, x_n)$ approaches $(0, 0, \cdots, 0)$. Now consider the $2n$-dimensional region R of the complex $x_1 x_2 \cdots x_n$-

space corresponding to the given positive constants ρ, $\delta(<2-\lambda_i')$:

$$R: |x_i| < \rho\,; \quad |x_i - f_i(t)| < e^{(2-\delta)t'}\,; \quad (i = 1, 2, \cdots, n).$$

Here t is supposed to remain in T. Our hypotheses with regard to the functions ϕ_i are as follows:

(*a*) *They are single-valued and analytic in the region R.*

(*b*) *All their first partial derivatives approach zero as* $(x_1, x_2, \cdots, x_n)$ *approaches* $(0, 0, \cdots, 0)$ *through points of R.*

(*c*) *The* ϕ*'s approach zero as the square of the distance of the point* $(x_1, x_2, \cdots, x_n)$ *from* $(0, 0, \cdots, 0)$, *regarded as points in 2n-dimensional space. Analytically, this means that there exists a constant K such that*

$$|\phi_i(x_1, \cdots, x_n)| < K(|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2), \quad (i = 1, 2, \cdots, n), \tag{5}$$

for all $(x_1, x_2, \cdots, x_n)$ *in R.*

Under these hypotheses we shall prove that *the equations* (1′) *admit the solutions* $x_i = F_i(t)$, $(i = 1, 2, \cdots, n)$, *single-valued and analytic in a certain sub-region* T' *of* T, $T' : -\infty < t' < \Gamma \leqq 0$, $-h < t'' < h$, *and approaching zero as* $t' = -\infty$. *Furthermore, that there is a positive constant B and an arbitrarily small constant* γ *such that* $|F_i(t) - f_i(t)| < Be^{(2-\gamma)t'}$, $(i = 1, 2, \cdots, n)$; *and that* $F_i(t)$ *is the only solution having these properties.*

If the ϕ's were known functions of the time, the knowledge of the solutions (4) would lead to the explicit solution of (1′) in terms of definite integrals by means of the classical methods. In the actual case, the formal process will yield integral equations which are equivalent to the equations (1′). These will be solved by the method of successive approximations.*

* The idea of extending a solution of equations like (2) to the equations (1) by the method of successive approximations has been made use of by I. Bendixon (Acta Mathematica, vol. 24 (1900)) and, in a form quite

The hypothesis (b) implies that the functions ϕ_i satisfy a Lipschitz condition in the region R:

$$(6)\qquad \begin{aligned} &\left|\phi_i(x_1^1, \cdots, x_n^1) - \phi_i(x_1^2, \cdots, x_n^2)\right| \\ &\qquad < A(\left|x_1^1 - x_1^2\right| + \cdots + \left|x_n^1 - x_n^2\right|), \end{aligned}$$

and that *the constant A can be taken as small as we like provided we take ρ sufficiently small.* In the usual proof of (6) a "convex" region is assumed, in which the points (x^1) and (x^2) can always be joined by a straight line segment lying wholly in its interior.* It will be possible to apply the method to our case if we can show that the given points may always be joined by a regular curve lying in R, the ratio of the length of which to the distance between the points being less than a fixed constant independent of (x^1) and (x^2). A little computation, with the aid of the expressions (4), will show that, if ρ be sufficiently small, the curve defined below has the requisite properties. Suppose that

$$\begin{aligned} &|x_1^1|^2 + |x_2^1|^2 + \cdots + |x_n^1|^2 \\ &\qquad \leqq |x_1^2|^2 + |x_2^2|^2 + \cdots + |x_n^2|^2; \end{aligned}$$

through the point (x^1) pass the curve obtained by setting $t'' = 0$ in equations (4) (this involves making a proper choice of the C's). Let $(\bar{x}^2)$ be its point of intersection with the hypersphere through (x^2):

$$\begin{aligned} &|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 \\ &\qquad = |x_1^2|^2 + |x_2^2|^2 + \cdots + |x_n^2|^2. \end{aligned}$$

Now the curve we want is composed of the straight line joining (x^2) with $(\bar{x}^2)$, and of the arc of the preceding curve intercepted by (x^1) and $(\bar{x}^2)$. The remainder of the proof

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similar to our own, by O. Perron (Mathematische Zeitschrift, vol. 14, (1922)). These authors were studying the geometric nature of the integral curves at a singular point in the real xy-plane, and were not concerned with the complex domain.

* See E. Goursat, loc. cit., p. 380.

consists in examining a line integral taken along this curve, and applying the hypothesis (b).*

We turn now to the proof of the existence theorem. We shall give the details of the proof for a system of the second order, leaving the obvious extension to the reader. The first typical case is that of the system†

$$(1'') \qquad \frac{dx}{dt} = x + \phi(x,y), \qquad \frac{dy}{dt} = \lambda y + \psi(x,y),$$

where $1 \leqq \lambda' < 2$, and where ϕ and ψ verify the hypotheses (a), (b), (c) with respect to the solution

$$(4'') \qquad x = ae^t, \qquad y = be^{\lambda t}.$$

The corresponding integral equations are found to be

$$(7) \qquad \begin{aligned} x &= ae^t + \int_{-\infty}^{t} \phi(x,y)e^{t-\xi}d\xi, \\ y &= be^{\lambda t} + \int_{-\infty}^{t} \psi(x,y)e^{\lambda(t-\xi)}d\xi, \end{aligned}$$

where ξ is a variable of integration: $x = x(\xi)$, $y = y(\xi)$ in the integrand. We have chosen $-\infty$ as the lower limit, since we are interested in solutions which differ from $(4'')$ by an amount which vanishes in comparison with e^t as $t' = -\infty$. The path of integration could be any curve running from the given value t out to infinity in the strip T; but it is desirable to fix it definitely, and we shall take it to be the horizontal ray running from t to the left indefinitely, i. e., $-\infty < \xi' \leqq t$, $\xi'' = t''$.

Let the successive approximations be defined by recurrence as follows:

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* See Goursat, loc. cit., p. 380. We use the finiteness of the logarithmic spiral, and such curves.

† For simplicity in writing, we are taking the imaginary part of the coefficient of x in $(1'')$ zero; but the proof in the general case is quite similar.

$$x_0 = ae^t, \qquad x_{k+1} = ae^t + \int_{-\infty}^{t} \phi(x_k, y_k)e^{t-\xi}d\xi\ ;$$

$$y_0 = be^{\lambda t}, \qquad y_{k+1} = be^{\lambda t} + \int_{-\infty}^{t} \psi(x_k, y_k)e^{\lambda(t-\xi)}d\xi\ ;$$

$$(k = 0,1,2,3, \cdots).$$

We have to show that the functions x_k and y_k all exist and are analytic, and approach limits which are the required solutions as $k = \infty$. Writing as usual

$$u_k = x_k - x_{k-1}, \qquad v_k = y_k - y_{k-1}, \quad (k = 1,2,3, \cdots),$$

we have

$$x_k = x_0 + u_1 + u_2 + \cdots + u_k,$$

$$y_k = y_0 + v_1 + v_2 + \cdots + v_k,$$

and we must examine the series

$$(8) \qquad \begin{cases} u_1 + u_2 + u_3 + \cdots \\ v_1 + v_2 + v_3 + \cdots \end{cases}$$

for convergence.

From the expression

$$u_1 = \int_{-\infty}^{t} \phi(x_0,y_0)e^{t-\xi}d\xi\ ,$$

and the hypothesis (c), we have

$$|u_1| \leqq \int_{-\infty}^{t'} |\phi(x_0, y_0)|e^{t'-\xi'}d\xi', \qquad (\xi = \xi' + (-1)^{1/2}\xi''),$$

$$< K\int_{-\infty}^{t'} (|x_0|^2 + |y_0|^2)e^{t'-\xi'}d\xi'$$

$$< Ke^{t'} \int_{-\infty}^{t'} (|a|^2 e^{\xi'} + |b|^2 e^{(2\lambda'-1)\xi'-2\lambda''t''})d\xi'$$

$$< K\left(|a|^2 + \frac{|b|^2}{2\lambda' - 1} e^{2(\lambda'-1)t'-2\lambda''t''}\right)e^{2t'}.$$

Similarly,

$$(9) \qquad |v_1| < K\left(\frac{|a|^2}{2-\lambda'} + \frac{|b|^2}{\lambda'} e^{2(\lambda'-1)t'-2\lambda''t''}\right) e^{2t'}.$$

Hence it is possible to find a positive constant M such that

$$|u_1| < Me^{2t'}, \qquad |v_1| < Me^{2t'}.$$

Moreover, the preceding computation establishes the existence and analyticity of the functions u_1, v_1 in T, as it follows from the elementary considerations of the uniform convergence of definite integrals.

For the further u's and v's we have the equations

$$u_{k+1} = \int_{-\infty}^{t} [\phi(x_k, y_k) - \phi(x_{k-1}, y_{k-1})] e^{t-\xi} d\xi ,$$

$$v_{k+1} = \int_{-\infty}^{t} [\psi(x_k, y_k) - \psi(x_{k-1}, y_{k-1})] e^{\lambda(t-\xi)} d\xi ,$$

$$(k = 1, 2, 3, \cdots).$$

Suppose that all the functions up to and including x_k, y_k exist and are analytic in the region T, that they map T in the interior of R, and, finally, that there exist a positive constant H_k for which $|u_k| < H_k e^{2t'}$, $|v_k| < H_k e^{2t'}$. It will then follow from hypothesis (b) that

$$|u_{k+1}| \leqq \int_{-\infty}^{t'} |\phi(x_k, y_k) - \phi(x_{k-1}, y_{k-1})| e^{t'-\xi'} d\xi'$$

$$< A \int_{-\infty}^{t'} (|u_k| + |v_k|) e^{t'-\xi'} d\xi'$$

$$< 2AH_k e^{t'} \int_{-\infty}^{t'} e^{\xi'} d\xi' = 2AH_k e^{2t'}. \qquad \text{[See (6).]}$$

Similarly,

$$(10) \qquad |v_{k+1}| < \frac{2AH_k}{2-\lambda'} e^{2t'}.$$

Thus, if G denote the greater of the two quantities 2 and $2/(2-\lambda')$, we have

$$|u_{k+1}| < AG \cdot H_k e^{2t'}, \qquad |v_{k+1}| < AG \cdot H_k e^{2t'}.$$

Moreover, u_{k+1} and v_{k+1} exist and are analytic in T, and likewise for x_{k+1} and y_{k+1}.

We wish to know when the locus of the corresponding point, (x_{k+1}, y_{k+1}), will lie inside R. We have

$$\begin{aligned} |x_{k+1} - ae^t| &\leqq |u_1| + |u_2| + \cdots + |u_{k+1}| \\ &< (1 + AG + A^2G^2 + \cdots + A^kG^k)Me^{2t'}, \end{aligned}$$

the last inequality resulting from the repeated application of the results of the two preceding paragraphs. Now consider the geometric series $1+AG+A^2G^2+A^3G^3+\cdots$. *We shall suppose that* ρ *is taken so small that* $AG<1$; this is permissible in view of what has been deduced from hypothesis (b); and in order to be sure that the loci of $(x_0, y_0), \cdots, (x_k, y_k)$ lie in this region, it is only necessary to confine t to a suitable region T': $t'<\Gamma$. Then the geometric series will converge, and have the sum $1/(1-AG)$. Hence

$$(11) \qquad |x_{k+1} - ae^t| < \frac{M}{1-AG} e^{2t'}, \quad |y_{k+1} - be^{\lambda t}| < \frac{M}{1-AG} e^{2t'}.$$

A comparison of these inequalities with those defining R shows that if Γ in the definition of T' be given a sufficiently small value (which will be independent of k), and if t be confined to T', the locus of (x_{k+1}, y_{k+1}) will lie wholly within R.

Thus, our points (x_i, y_i) will never leave R as long as t is in T', our terms will all be determined by recurrence, and the above inequalities will apply to them. They will be single-valued and analytic for t in T', and the series will be dominated by the convergent geometric series. It follows by the usual reasoning that (x_i, y_i) converges to (x, y), the required solutions of (7) and hence of $(1'')$.

The inequalities (11) evidently yield the result

$$|x - ae^t| < Be^{2t'}, \qquad |y - be^{\lambda t}| < Be^{2t'}.$$

Now there can not be more than one solution which is thus related to $(ae^t, be^{\lambda t})$. For if $\bar{x}$, $\bar{y}$ were a second solution, it would satisfy (7), and it would follow by previous formulas that

$$\bar{x} - x_{k+1} = \int_{-\infty}^{t} [\phi(\bar{x}, \bar{y}) - \phi(x_k, y_k)]e^{t-\xi}d\xi, \text{ etc.}$$

Hence

$$|\bar{x} - x_{k+1}| < A \int_{-\infty}^{t'} (|\bar{x} - x_k| + |\bar{y} - y_k|)e^{t'-\xi'}d\xi', \text{ etc.}$$

It follows by recurrence, as before, that

$$|\bar{x} - x_k| < A^k G^k \cdot Be^{2t'}, \text{ etc.};$$

hence we have*

$$x \equiv \bar{x}, \qquad y \equiv \bar{y}, \qquad \text{(q.e.d.)}.$$

The form of the uniqueness theorem originally stated is proved in the same manner.

We turn now to the second typical case for the system of the second order. It is that of the equations

$$\frac{dx}{dt} = x + \phi(x, y), \qquad \frac{dy}{dt} = x + y + \psi(x,y).$$

The solution of the corresponding linear system is

$$x = ae^t, \qquad y = (at + b)e^t,$$

and the equivalent integral equations are

$$\begin{cases} x = ae^t + \displaystyle\int_{-\infty}^{t} \phi(x,y)e^{t-\xi}d\xi, \\ y = (a + bt)e^t + \displaystyle\int_{-\infty}^{t} [\psi(x,y) + (t - \xi)\phi(x,y)]e^{t-\xi}d\xi. \end{cases}$$

The solution by successive approximations is just the same as before, and the computation is performed on parallel lines. The inequality (9) is replaced by the following:

* See Goursat, loc. cit., p. 377.

$$|v_1| \leqq e^{t'} \int_{-\infty}^{t'} (|\psi_0| + |t-\xi| \cdot |\phi_0|) e^{-\xi'} d\xi'$$

$$< Ke^{t'} \int_{-\infty}^{t'} (|x_0|^2 + |y_0|^2)(1 + |t-\xi|) e^{-\xi'} d\xi'$$

$$< Ke^{t'} \int_{-\infty}^{t'} (|a|^2 + |a\xi + b|^2)(1 + |t-\xi|) e^{\xi'} d\xi'.$$

Hence, if we take t in a sufficiently restricted region T', we have

$$|v_1| < M' e^{t'} \int_{-\infty}^{t'} e^{\xi' - \gamma} d\xi' = M e^{(2-\gamma)t},$$

where γ is a positive constant which we may take as small as we wish, provided Γ be taken sufficiently small. Likewise, to prove the inequality corresponding with (10) we have

$$|u_k| < H_k e^{(2-\gamma)t'}, \qquad |v_k| < H_k e^{(2-\gamma)t'},$$

whence

$$|v_{k+1}| < e^{t'} \int_{-\infty}^{t'} A(|u_k| + |v_k|)(1 + |t-\xi|) e^{-\xi'} d\xi'$$

$$< AH_k e^{t'} \int_{-\infty}^{t'} (1 + |t-\xi|) e^{(1-\gamma)\xi'} d\xi'.$$

Here again, if t is taken in a region T', with Γ sufficiently small, we shall have

$$|v_{k+1}| < H_k \cdot AGe^{(2-\gamma)t'}.$$

We leave it to the reader to complete the proof.

Note: Although we have supposed throughout that the general solution of the equations (2) approaches zero as $t = \infty$, it will be seen that even in the contrary case, if a particular solution of (2) approaches zero, it is possible to frame a theorem, like the one which has been given, which establishes the extension of the particular solution to the given differential equations (1).

PRINCETON UNIVERSITY

CORRADO SEGRE

BY J. L. COOLIDGE

There is a pronounced rise and fall in the tide of mathematical interest in different countries and at different times. This is well shown in the case of geometry. In Great Britain and Ireland, in the Victorian era, Hamilton and Clifford, Salmon and Casey, Cayley and Sylvester, constituted a veritable English-Irish geometric school. With the deaths of these men, and the decline in the fashion for algebraic invariants, all of this efflorescence seemed to wither away, till the University of Oxford appointed to the Savillian professorship of geometry a man who had attained high distinction in the analytical theory of numbers. In France, the name of Chasles exercised, at one time, a fascination which we perhaps do not quite understand today. Darboux, with his masterly *Théorie Générale des Surfaces*, raised to himself a monument more enduring than bronze, but no geometric school arose therefrom, and the center of gravity of mathematical interest in France today is certainly not in the field of geometry. In Germany, the fluctuation has been distinctly less. In Italy, the interest in geometry has produced notable results. The classical differential geometry owes an immense debt to Beltrami and Bianchi; the Riemannian geometers take as their starting point the absolute calculus of Ricci and Levi-Civita; as for the modern birational geometry, that is almost a monopoly of Italian mathematicians and a few others like Macaulay and Snyder, who have been strongly influenced by Italian thought.

What is the reason for this state of affairs? It would be equally wrong to ascribe it entirely to chance or to seek a recondite explanation in the psychology of peoples. The real ground is to be found in the presence or absence of great teachers of geometry. Salmon was a great teacher indeed, but Dublin was not a good center for distributing the pure milk of the word. Darboux was a marvellous lecturer, but he lacked the personal qualities necessary to inspire youth. It is of Jules Tannery, not of Darboux, that the French mathematicians of today say: "C'était notre père à nous tous." In Germany, on the other hand, Klein's extraordinary inspiration was felt quite as much in geometry as elsewhere; Reye created an enthusiasm for pure projective geometry that went almost to excess; the present geometric activity in Hamburg and Bonn is directly traceable to the influence of Study. The geometric "risorgimento" in Italy is the result of the efforts of certain great teachers such as Cremona, Battaglini, D'Ovidio, and the subject of the present sketch.

Corrado Segre was born in Saluzzo, August 20, 1863*, and received his laureate at Turin when less than twenty years of age. He never left

* In preparing the present account, I have consulted, in addition to Segre's own writings, Castelnuovo, Rendiconti dei Lincei, (5a), vol. 33[2]

the Piedmontese capital thereafter, except for short periods; his active life was passed there as assistant and as professor. At some period during the early part of his career, he had a severe attack of brain fever brought on by overwork. He complained sadly that his capacity for labor was never the same thereafter. When one reflects that the number of his published titles is 128, it fatigues the imagination to speculate as to what he might have accomplished had he retained his early strength. He died unexpectedly, after an operation, on May 18, 1924.

The connecting thread which runs through Segre's work is the projective geometry of n dimensions. In the hands of a man of small grasp, this might easily degenerate into barren formalism. Not so in his case. He never lost sight of the relations between a general formulation and concrete problems, and never failed to familiarize himself with other methods of treating the question in hand, besides those which he personally preferred. Let us trace the line of his thought. The simplest figures in any space are the linear sub-spaces. Suppose first that we have a finite number of such. The obvious problems are those of incidence. The beginner has little idea of the complications which quickly arise in such questions. Is it at all obvious that in four-space exactly five planes will intersect six given lines in general position? Segre treats them in two papers separated by a considerable interval of time.* Next to linear spaces we have quadratic varieties. Segre's thesis is devoted to a study of the hyperquadric in n-space.† A single hyperquadric is of slight mathematical interest compared with the study of two such figures and the classification of their various types of intersection. The beautiful plane curve which Darboux has called a cyclic and the English and Irish have described as a bicircular quartic, is the stereographic projection on the projective plane of the intersection of the Gauss sphere with a general quadric surface. Ascending one dimension, the general cyclide or quartic surface with a double conic may be reached in a similar way by a projection in four-space. Segre devoted one of his earliest and largest papers to the classification of such surfaces.‡ Going up two more dimensions, the line-geometry of Plücker and Klein consists in identifying the lines of a projective three-space, with the points of a hyperquadric in five-space. The intersections of two hyperquadrics will correspond to the lines of a quadratic complex. Segre studies the general quadratic complex and some particular varieties, with special reference to the singular elements, most fruitfully by this method.§

(1924); Baker, Journal of the London Society, vol. 1 (1926); and especially Loria, Annali di Matematica, (4), vol. II (1925). Here will be found a complete bibliography of Segre's work.

* Rendiconti di Palermo, vol. 2 (1888); Rendiconti dei Lincei, (5), vol. 9^2 (1900).

† Torino Memorie, (2), vol. 36 (1883).

‡ Mathematische Annalen, vol. 34 (1884).

§ There are a number of papers on this topic. See Loria, loc. cit., pp. 14–15.

The problem of classifying the intersections of two hyperquadrics is best solved by the use of the elementary divisors of Weierstrass. An almost identical algebraic analysis will lead to the classification of linear transformations and their fixed elements. This problem also received Segre's attention.*

Reverting to linear spaces, Segre had the curious but fruitful idea of studying those systems of points which are in one to one correspondence with groups of points, one in each of a number of given linear spaces. The simplest example of this is the perfect correspondence between the individual points of a general quadric and pairs of points of two given lines. The case which particularly interested Segre was that where there were two conjugate imaginary linear varieties.† These considerations led him to another field to which, for a few years, he gave considerable attention. If a complex space of any number of dimensions be given, there are two questions which immediately suggest themselves. How can the points of this space, and of various loci therein, be represented by real geometrical figures? What are the geometrical characteristics of the loci of points whose coordinates are functions of a certain number of *real* parameters? For instance, how shall we best represent all the complex points of a plane? Given a system of points in a plane whose coordinates depend analytically on two real parameters, how shall we determine whether they generate a curve? Segre laid the foundation for the future study of the second of these problems and added considerably to our knowledge of the first.‡ He thereby established contact with Study, whom he subsequently met at the Heidelberg Congress in 1904, and with whom he retained cordial relations for many years. It must be confessed, however, that few mathematicians have been found who cared to enter this which Segre calls "Un nuovo campo di ricerche geometriche." Even so competent a critic as Baker§ fails completely to grasp the real significance of this part of Segre's work. Perhaps he felt some regret that so little was done to extend geometrical science in these directions. The present writer feels that regret keenly.

Let us return to linear spaces and suppose that we have not a finite but an infinite number of them. We are led, as the case may be, to problems of algebraic or of differential geometry. Suppose that we have a one-parameter system of lines in our space. If it be an algebraic system, it will correspond to a curve in 5-space, and so have a definite genus. But these lines will generate an algebraic surface with various genera, arithmetic, geometric, etc. On the other hand if the system be not algebraic, but still analytic, the lines will generate a surface. Is this surface de-

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* Rendiconti dei Lincei, (3), vol. 19 (1884); Torino Memorie, (2), vol. 37 (1885); Journal für Mathematik, vol. 100 (1886).

† Rendiconti di Palermo, vol. 5 (1891).

‡ Torino Atti, vol. 25 (1889), vol. 26 (1890); Mathematische Annalen, vol. 40 (1891).

§ Loc. cit., p. 270.

velopable, and, if not, what can be said of the other system of asymptotic lines besides that formed by the rectilinear generators? Segre studied questions of each of these types, the algebraic ones in his earlier years, the differential ones later on.* In fact the subject of projective differential geometry was that to which he gave most attention during the later years of his life. He made the acquaintance of Wilczynski at Heidelberg in 1904 and thoroughly appreciated the importance of his work. Yet an outsider cannot help feeling some regret that there still appears a considerable gulf between the Italian doctrine of projective differential geometry, developed by Segre, Bompiani, Fubini and others, and the American contributions of Wilczynski and Gabriel Green, of whom, by the way, no Italian seems to have heard the name.

It is time to turn from Segre's scientific writings to five important articles of an expository character which he produced at various times. The longest, and by far the most important of these was his article in the Mathematical Encyclopedia on the geometry of n-space.† The thought that he must one day complete this, depressed his spirits at times for a good many years, for he was one who took his responsibilities seriously, and he felt in honor bound to put the thing through. Complete it he finally did, thus earning the gratitude of geometers for many years to come. To quote Baker‡

"For completeness of detail, breadth of view, and generous recognition of the work of a host of other writers, this must remain for many years a monument of the comprehensiveness of the man."

In 1891, while still a very young teacher, he wrote for the benefit of his pupils "Su alcuni indirizzi nelle investigazioni geometriche"§ which was reprinted many years later in this Bulletin‖ under the title *On some tendencies in geometrical investigations*. It is a cheering message on the whole, almost as helpful to the young mathematicians of our time and country, as to those of the epoch and place for which it was originally intended. The point which Segre enforces most strongly is that at all costs the beginner must learn to choose really significant subjects. To acquire the instinct to do this, he should study the mathematical classics. He must especially avoid trivial questions and perfectly obvious extensions or generalizations. The young geometer should also be familiar with the leading methods and results of modern analysis, and welcome help from every quarter, regardless of whether his own preference is for synthetic, algebraic, arithmetical or transcendental methods. With regard to rigor, Segre recommends the geometer to be as rigorous as he can, and to own up like a man when he makes use of methods of doubtful parentage.

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* See especially Rendiconti di Palermo, vol. 30 (1910).

† Encyklopädie der mathematischen Wissenschaften, III C 7, pp. 769–972.

‡ Loc. cit., p. 271.

§ Rivista di Matematica, vol. 1 (1891).

‖ Vol. 10 (1903–04).

At the same time he does not consider that geometry is necessarily a succession of δ, ϵ processes, and is not shocked at the judicious use of the convenient phrase "in general".

Very close in spirit to this is his address delivered many years later at the Heidelberg Congress *La geometria d'oggidi e i suoi legami coll' analisi.** His thesis is that geometry and analysis deal with what is essentially the same subject matter, but with different emphasis, terminology and esthetic aim, a view which probably is even more acceptable today than it was when first expressed. He lays special stress on various branches of algebraic geometry, particularly those properties of figures which are unaltered by birational transformation. Near the close, in speaking of the classic memoir of Brill and Noether† he remarks "Tutta una scuola di geometri italiani riconosce nella Memoria di Brill e Noether il suo punto di partenza." This statement falls short of the truth today, through excess of modesty. The present tendency is to take as point of departure two remarkable, and mutually complementary articles which appeared a dozen years before Segre's address, and which deal with the birationally invariant properties of curves.‡ The first by Bertini follows classical algebraic methods, which are set forth with astonishing clearness and elegance. The second by Segre, which shows a remarkable range of mathematical reading from Kronecker to Castelnuovo, follows whenever possible the fundamental methods of the projective geometry of n-space. What has n-space to do with algebraic plane curves? A very great deal. Suppose that we have on a plane curve, not one point, nor a group of points, but a system of such groups, each group containing n points, while the system depends linearly on r independent parameters. These numbers will be birationally invariant. Let us suppose that there are no fixed points common to all of the groups, and that the system is not an involution, i. e., every group containing a general point does not automatically contain certain others. Lastly we may assume that the system is cut by $r+1$ linearly independent adjoint curves $f_0, f_1, \cdots, f_r$. If we write

$$\rho X_i = f_i(x_1, x_2, x_3),$$

we shall have a transformation which carries our plane curve into a space curve of order n in a space of r dimensions, and the groups of our system into the intersections of our new curve with the hyperplanes of its space. For instance, if we have a plane quartic curve with two double points, the conics through these points will meet the curve in a three-parameter system of groups of four points, and will enable us to transform the curve birationally into the elliptic quartic of three-space. This is another aspect of the stereographic projection of a space quartic into a bicircular plane quartic already discussed.

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* Published in volume 2 of the Transactions of that Congress; and also in Rendiconti di Palermo, vol. 19 (1905).

† *Über die algebraischen Funktionen und ihre Anwendung in der Geometrie*, Mathematische Annalen, vol. 7 (1873).

‡ Annali di Matematica, (2), vol. 22 (1893).

We have just used the words "adjoint curves." An adjoint to a given algebraic plane curve is a second which has the multiplicity $s_i - 1$ or more, wherever the original curve has the multiplicity s_i. When the singular points of the original curve are "ordinary", that is to say, when the tangents are all distinct, this definition is perfectly satisfactory. But when the tangents fall together in groups, the singular point may become a very ugly customer, and call for extremely careful treatment in determining the deficiency of the curve and the conditions imposed on adjoints. The problem of determining the number of intersections of two curves absorbed by a common singularity has attracted the attention of geometers from H. J. S. Smith to Enriques. It is the subject of Segre's last long expository paper,* the method of treatment being a careful study of the structure of the resolvent.

Behind Segre the geometer and the expositor, there remained always Segre the teacher and friend. An interesting figure he was in the lecture room. Of medium height and frail, half seated on the end of the table, gesticulating rapidly with his left hand, while his right whirled his watch-chain about with astonishing angular velocity, his whole being was thrown into the task of driving home the essentials of what he had in mind. The subject matter of his course was new every year. There was no limit to the amount of care and patience which he would bestow on one of his pupils. To one who apologized for taking so much of his valuable time with trivial difficulties, he wrote: "Surtout, il ne faut jamais hésiter à me consulter quand vous trouverez une difficulté. Il n'y a que les ignorants et les paresseux qui n'ont jamais de difficultés."

He had a sympathetic, sensitive nature, a spiritual quality, which impressed all who had the privilege of his friendship. Most truly Castelnuovo wrote of him†:

"Corrado Segre ebbe una virtù preziosa concessa a pochissimi eletti: la virtù di rendere migliori tutti coloro che lo avvicinavano. Chi ha potuto vivere qualche tempo nella sua intimità ne ha sentito influenza benefica per tutta la vita."

HARVARD UNIVERSITY

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* Giornale di Matematiche, vol. 36 (1897).

† Loc. cit., p. 359.

OSGOOD ON ANALYTIC FUNCTIONS OF SEVERAL VARIABLES

Lehrbuch der Funktionentheorie. By W. F. Osgood. Part I. Leipzig, B. G. Teubner, 1924. v+242 pp.

It may appear strange to give a review of a part of a volume, an "*Erste Lieferung*." There is, however, good justification for this. The present 242 pages make a substantial unit by themselves. They comprise the general concepts and theorems which form the basis for a general theory of analytic functions of two or more variables. Subsequent chapters to be added later will be devoted to special branches from the main trunk, such as the theory of $2n$-ply periodic functions of n variables, theta-functions, or automorphic functions in more than one variable; and so on. Since also some considerable time will elapse before the writing and publication of these chapters, it seems both suitable and desirable to review the present brochure.

The need of an up-to-date coordination and systematization of material relating to analytic functions of several variables is almost too apparent to enlarge on. While many of the results and methods for analytic functions of one variable can be carried over without difficulty to functions of many variables, the difficulties at other points are so great, the pitfalls so hidden, the progress so slow, as to make it almost legitimate to say that a profound gulf exists between the theory of analytic functions for a single variable and that for several variables.

The literature pertaining to analytic functions of several variables is rather scattered, and little in the nature of systematic exposition is to be found. Chapter XVII in Goursat's *Cours d'Analyse* and Chapter IX in the second volume of Picard's *Traité d'Analyse* are excellent so far as they go, but are confined to the elementary aspects of the subject. Then there are the Calcutta lectures of Forsyth upon *Functions of Two Complex Variables* (1914), but so different are they in character, scope, and content from the work under review that they increase the desirability and usefulness of the latter. Synoptic reports on analytic functions of many variables are to be found in fourteen pages of the section written by Osgood a quarter-century ago for the Encyklopaedie der Mathematischen Wissenschaften, in Hartogs' Beright in the Jahresbericht der deutschen Mathematiker-Vereinigung (vol. 16 (1907)), and finally in Bieberbach's recent encyclopaedia article (1921), giving the *Neuere Untersuchungen über Funktionen von komplexen Variablen*, of which, however, only a tenth is devoted to analytic functions of several variables. Lastly, I may cite as a forerunner of the present work the Madison Colloquium Lectures of Osgood (1914), treating *Topics in the Theory of Functions of Several Complex Variables*. Naturally, these lectures have also largely the nature of encyclopaedic sketches or skeleton outlines without the connective tissues.

Such a meager amount of literature reveals how strangely little the analytic function-theory has been in fashion mathematically, although its

fundamental importance is undeniable. One of the reasons for this is doubtless its difficulty. The task before Osgood was no easy one: to find out where the known stops and where the unknown begins, to observe carefully the limitation upon the theorems for the case of n variables, to ascertain and weigh the value and significance of the newer methods which have been beaten out where the older methods collapsed. I have often wished for a place in mathematical literature where one could find a comprehensive, connected, and properly restricted modern exposition of analytic functions of several variables, a spot upon which one could plant his feet as on a rock. At last we have it!

Osgood's material is divided into three chapters: Chapter 1. *Integraldarstellungen und mehrfache Reihen. Die erweiterten Räume.* Chapter 2. *Implizite Funktionen. Teilbarkeit.* Chapter 3. *Singuläre Stellen und analytische Fortsetzung. Rationale Funktionen.*

The first chapter, for the most part, contains the theorems which can be obtained by ready extension of the theory for one variable. For this purpose two standard methods are available, the one generalizing Cauchy's integral, and the other using the power series of Weierstrass. Correspondingly, the phrases "Integraldarstellungen" and "Mehrfache Reihen" appear in the title of the chapter. Consistently with volume I, an analytic function $f(z_1, \cdots, z_n)$ is defined as a function which is one-valued at every inner point of a region T and has throughout the region a partial derivative with respect to each of the n variables. The continuity of the function is rightly assumed at the start. In the third chapter, as a much later development, it is shown how this assumption of the continuity of $f(z_1, \cdots, z_n)$ can be removed. The removal was first effected by Osgood when the function is bounded, and later Hartogs succeeded in removing even this restriction.

To treat analytic functions *im Grossen* their behavior at infinity must be considered. Accordingly, it is necessary to explain how the n-fold complex space of the variables z_i or the equivalent real space $(x_1, y_1, \cdots, x_n, y_n)$ of $2n$ dimensions is to be extended by the invention of points and configurations at infinity ("Die erweiterten Räume"). One of the admirable features of the book is the treatment of this matter. The reader may recall how Bôcher (this Bulletin, vol. 20 (1914), p. 185) has emphasized that there were other ways than the conventional one of extending space and that the object in view must determine the character of the space-completion "at infinity." Osgood re-emphasizes this view by indicating several possible extensions for the space of the n variables of our analytic functions. The usual extension gives what the author fittingly calls the "space of analysis," and is obtained by adding a single point in each of the n complex planes of our n variables $z_1, \cdots, z_n$. In consequence, the *infinite region* consists of all points $(z_1, \cdots, z_n)$ for which one or more of the n coordinates is infinite. Another kind of space, which I shall refer to as *complex projective space*, results by basing the extension upon the familiar linear projective group in n variables (allowed to be complex), and this has been fruitfully employed by Osgood himself.

The second chapter is rooted in Weierstrass's so-called preparation theorem, with which it opens. Upon this as a foundation is built the theory of implicit analytic functions, and another application is to the concept and theory of reducibility (Teilbarkeit). The former of these two topics is treated with great fullness, as we would wish, since the field of implicit functions has been especially studied in America by Osgood and his former pupils, Dederick, Clements, etc., and also by Bliss. Systems of simultaneous analytic equations and parameter representation *im Kleinen* are also considered.

In the third chapter the subject of singular points and configurations is first introduced, and it is devoted in large part to an exposition of the theorems of Hartogs, Levi, Cousin, etc., which treat of analytic behavior, meromorphic behavior, etc., *im Grossen*. It has the special interest of bringing to the reader the modern methods devised to establish recalcitrant theorems which, though perhaps long surmised, have occasioned great difficulty and ofttimes have been demonstrated either restrictedly or inadequately.

This division of contents is very natural, and altogether logical. But there is so much material to pack away as to give trouble in sectional subdivision—at least, it seems so to the reviewer. One finds a certain lack of swing and natural evolution, particularly in the first two chapters. This seems to me to be the one and only marked deficiency of the book. To apply a trite phase, the trees in the wood are so thick that the map is easily lost. The reader will, I think, find it helpful to keep on hand Osgood's Colloquium Lectures, and use it as an aid in getting a bird's eye view over the whole. No one, however, can fail to feel the difficulty of organizing so much material nor be unaware that the parts fit together, like the stones of a grand mosaic, to make one great design.

The richness of the material is a constant delight. It is impossible to read without perceiving the tremendous amount of study which must have been spent in collating and correlating, and without prizing the thoroughness and completeness with which this has been done. Further, the character of the analytic function theory for two or more variables is such as to demand scrupulous care, and we are glad that the task has fallen to an Osgood. We congratulate ourselves that he did not conventionally stop with the analytic theory of a single variable.

The successive topics are developed very fully. In Chapter I there is frequently a rapid change from one topic to another having little relation to it. To cite only a single instance, the section on homogeneous coordinates is succeeded by one on Laurent's series, and this is followed by one on analytic continuation. The section on Laurent's series is thus sawed off from other work on series. The matter is, however, very closely interwoven. If one attempts to select and study some particular topic here or there, he will usually find it imperative to go back and digest something that has gone before, and from this he may have to turn to something still further back. There is also continual reference by chapter and section numbers to volume I. Osgood presupposes justly that the reader should already have a firm grasp on the function theory for one

variable. The book is not one to be used to get an understanding of function theory for one variable from that for n variables by putting $n=1$. One must be content to proceed slowly and to think and ponder as he proceeds. When he has so done, he will be well satisfied with the grip obtained.

I turn now to a more detailed notice of the features of Chapter I. After a variety of topics, such as the Differential, the Mittelwerthsatz, Liouville's Theorem, and certain work on implicit functions, he paves the way for the all important Cauchy integral theorems. In §8 we find the needed theorems regarding the analytic character of the integral

$$\int_{C_1} dt_1 \int_{C_2} dt_2 \cdots \int_{C_m} f(t_1, t_2, \cdots, t_m; z_1, \cdots, z_n) dt_m,$$

where f is continuous in its $n+m$ arguments for values of the t_i on "regular" curves C_i in the t_i-planes and for values of the z_i in n regions of the complex plane of these n variables, and furthermore is analytic in the z_i for these values of the arguments. In §9 the form of the region is specified to which the work is subsequently largely confined. This is what he terms a *cylindrical* region (Zylinderbereich) in the space S_{2n} of $2n$ dimensions which is obtained by associating arbitrarily the values $z_i \equiv x_i + iy_i$ belonging to the inner points of n independent regions (not necessarily simply connected) in the n-planes of the separate variables. The care of the author in specializing the regions he uses and thus preventing his reader from mentally extending conclusions to any old kind of region is a laudable characteristic of the book. After the introduction of Cauchy's integral formula for $f(z_1, \cdots, z_n)$ and the deduction from it of various important consequences, the scene shifts to a rather uninteresting analogue of the mean value theorem of the differential calculus and then to a nineteen page discussion of "Mehrfache unendliche Reihen," which by the definition are absolutely convergent. For power series $P(z_1-a_1, \cdots, z_n-a_n)$, it is noted that the region of absolute convergence may consist not merely of the simply connected region of $2n$ dimensions usually considered, but also, possibly, of regions of $2k$ dimensions ($k<n$) in the coordinate planes. I mention this casually to illustrate thereby the variety of interesting and minor features which arise from the author's care. The sections on series culminate, of course, in the familiar Cauchy-Taylor power expansions in n variables. In the sections next following, the space for the analytic functions is extended appropriately to infinity, first without the introduction of homogeneous coordinates. Then, after their introduction, he goes on to establish that in the resulting $2(n+1)$-fold space of the new homogeneous variables corresponding to the space S_{2n} of the original variables z_i, the function will be analytic or meromorphic in terms of the new $n+1$ homogeneous variables if it is so in the corresponding space of the original n variables. We are next surprised by the remarkable theorem (p. 57) that if $f(w,z)$ is analytic and finite for the region $|w|+|z|>G$, where G denotes a fixed number, then f is a constant. This theorem is one of few which is without an analogue for the analytic function of one variable. The subject of analytic continuation of functions by successive overlapping regions is, of course, conventional, but in considering analytic extension along a regular curve in

S_{2n} the author takes the pains to introduce a specific form of region both for extended and unextended space to take the place of the circle of convergence used in analytic extension along a curve in the complex plane of a single variable, and this is done both for the space of analysis and for complex projective space.

The first three sections of Chapter II relating to the so-called preparation theorem of Weierstrass are particularly good, just as we would wish at so important a place. For the sake of what is attached to it, I repeat the theorem here: If an analytic function $F(w, z_1, \cdots, z_n)$ vanishes at a point $(b, a_1, \cdots, a_n)$—for convenience taken to be $(0, a_1, \cdots, a_n)$—but without vanishing identically around it, then in an appropriate vicinity, $|w|<r$, $|z_j-a_j|<h$, $(j=1, \cdots, n)$, we can express F in the form

$$F(w,z_1, \cdots, z_n) = (w^m + A_1 w^{m-1} + \cdots + A_m)\ \Omega\ (w, z_1, \cdots, z_n),$$

where the A_k are functions of $z_1, \cdots, z_n$, which are analytic and vanish at the point $(a_1, \cdots, a_n)$, while Ω is a function which does not vanish in the vicinity specified. The case in which $F(w, a_1, \cdots, a_n)$ vanishes identically is first discharged in the manner of Weierstrass by an appropriate linear transformation in the $n+1$ variables $w, z_1, \cdots, z_n$, and then Osgood goes on to prove that the form above given to the preparation theorem will hold also in that case for $n=1$, if we insert before w^m in the above formula a coefficient $A_0(z)$ which vanishes at $z=0$. By means of a special example Osgood shows further that a like form will not hold for $n>1$. This failure to hold is just one of many instances showing how extremely careful one must be in generalizing the ordinary analytic theory for one independent variable.

The first factor in the above preparation theorem brings before the reader an instance of what the author calls a pseudo-polynomial, that is, an expression $a_0 w^m + a_1 w^{m-1} + \cdots + a_m$ in which the a_k denote functions $a_k(z_1, \cdots, z_n)$ which are all analytic at some point $(c_1, \cdots, c_n)$ under consideration, called the *Spitze* of the pseudo-polynomial. The existence of the pseudo-polynomial in the above preparation theorem suggests the possibility of further decomposition of the analytic function into pseudo-polynomial factors. We are thus led to the theories of divisibility, reducibility, resolution into prime factors, etc., which follow. It may be remarked that the theories are almost altogether "im Kleinen." By equating the pseudo-polynomial to zero we are led to implicit function theory and also to a study of functions on the resulting pseudo-algebraic "Gebilde." A considerable part of this second chapter is rather specialized work on implicit function theory, the Jacobian, Riemannsche Räume generalizing Riemannian surfaces, systems of pseudo-algebraic equations, etc. For a first reading, one can well pass over a large portion of this chapter, confining his attention after §10 to the parts relating to functions on a pseudo-algebraic Gebilde and to parameter representation, and proceed directly to Chapter III. Indeed, we rather regret that some of the material was not postponed to a Chapter IV, so that first essentials could be more rapidly reached and picked out.

Chapter III compels one's interest. In the opening section, the author first discusses the topic of the singularity of the analytic function. The

chapter deals chiefly with the unessential singular point (ausserwesentliche singuläre Stelle), in the vicinity of which the function can be expressed in the form $G(z_1, \cdots, z_n)/H(z_1, \cdots, z_n)$, where G and H are both analytic and the latter vanishes at the point. When the numerator does not vanish, we get an unessential singularity of the first kind, which is the analogue of the familiar pole, and when it vanishes, an unessential singularity of the second kind, which much resembles the essential singularity of an analytic function of the single variable. When all the singularities in a given region are unessential, the function is, of course, called meromorphic.

Some of the greatest difficulties in the study of analytic functions of many variables arise from the fact that the zeros and singularities are never isolated, but form a continuous configuration. Thus, unessential singularities of the first kind fill the $(2n-2)$-fold manifold $H(z_1, \cdots, z_n) = 0$, and those of the second kind (in whose vicinities the function is unbounded) fill a $(2n-4)$-fold manifold. Not only are these (obviously) excluded in the two remarkable theorems due to Kistler, which the author goes to prove, but also all other singularities. The first of these theorems is as follows. If a function $F(z_1, \cdots, z_n)$ is analytic in a $2n$-dimensional region T except possibly for the points of a $(2n-2)$ fold manifold and is furthermore limited, then the function can be continued analytically over the *entire* region T, so that the points of the exceptional manifold, if singular, are removable discontinuities. The second theorem is like the first except that the possible exceptions make at most a $(2n-4)$-fold manifold, while the restriction that the function shall be limited is altogether omitted. The method of proof of these theorems and of others consists mainly of adroit use of Cauchy's integral formula. The same tool gives a number of very interesting theorems, due for the most part to Hartogs, which continue a function analytically beyond the region of its analytic definition. These culminate in Hartogs' fine result that a function $f(z_1, \cdots, z_n)$ which is one-valued and analytic on the boundary of a "regular" cylindrical region permits of analytic extension over the whole interior of the region. This conclusion is noteworthy inasmuch as the corresponding statement is not true in the case of a function of a single variable. After the theorems of Hartogs come the noteworthy theorem of E. E. Levi relating to meromorphic continuation of an analytic function. In particular, the theorem of Hartogs last cited will continue to hold if the word "analytic" is replaced by "meromorphic."

These results can only hint at the wealth of interesting theorems in this third chapter of Osgood's work. We find in accessible form Cousin's extensions of the well known theorems of Weierstrass for the representation of an entire function as a product of factors exhibiting zeros, and of Mittag-Leffler for the representation of a meromorphic function with arbitrarily distributed poles. The extension of the former given by Cousin contained a vitiating error, in that Cousin did not properly delimit the connectivity of the region. Gronwall, who started on the consideration of analytic functions of several variables in his Uppsala thesis, later rendered an

important service in detecting the hidden error and properly restricting the result.

Finally, we reach the theorem that a function meromorphic in a simply connected region (which may be the entire finite space) can be expressed as the quotient of two functions analytic over the region. This is established by Osgood without the use of the harmonic functions of many variables employed by Poincaré in the first demonstration given, the theory of which is even today without systematic development. Then follows the Weierstrass-Hurwitz theorem that a function meromorphic over the entire plane is a rational function. Special attention may be called to the remarkable fact that the theorem is not merely valid for "the space of analysis", but was proved by Osgood to hold also for complex projective space (and even for space extended in a mixed manner to infinity (Jackson)). I can not forbear to mention the simple proof that every one-to-one analytic* transformation of the extended space of analysis into itself has the form

$$z_i' = \frac{a_k z_k + b_k}{c_k z_k + d_k}, \qquad (i = 1, \cdots, n),$$

where the n values of k appearing in the right-hand members are the numbers $1, 2, \cdots, n$ in arbitrary order. This theorem is well known for the case of a single variable z, though even for this case the reader may experience trouble in ascertaining where a demonstration is to be found. References to literature are abundantly given by Osgood, but at this point, as at some others, we find none at all. We have the suspicion, therefore, that the general proof here given is the authors' own, as well as a corresponding theorem for complex projective space.

The volume is by no means easy to read, but grows simple if one delves and ponders. It shows the same thorough, massive scholarship, extending to all details, which I commented upon so favorably in my review of volume I. If we deplore again that the book could not have been written in English, we recognize a certain fitness that it appears in German, for it displays the inspiration of a German graduate training, as well as the persistent, accurate scholarship so characteristic of German work.

The author wishes me to call attention to the following corrections. Page 129, line 4 from the top. Reference should be to the Beispiel von §2 (not §22). Also in the middle of the page read $2(y/x)$ in place of $2(y/z)$. Page 33, line 3 from bottom. Instead of $a_k \neq 0$, $k = 1, \cdots, n$, read $\sum |a_k| > 0$, $(k = 1, \cdots, n)$.

E. B. VAN VLECK

* Of course, with proper modification in the infinite region.

THE NEW RIEMANN-WEBER

Die Differential- und Integralgleichungen der Mechanik und Physik, the 7th Edition of Riemann-Weber's *Partiellen Differentialgleichungen der mathematischen Physik*, edited by P. Frank and R. von Mises. *Erster, mathematischer Teil*, edited by R. von Mises with the assistance of L. Bieberbach, C. Carathéodory, R. Courant, K. Loewner, H. Rademacher, E. Rothe, C. Szegö. Braunschweig, Vieweg, 1925. 687 pp.+76 figures. Paper, 40 Rentenmark; bound, 44 Rentenmark.

As a compliment to the memory of Riemann and H. Weber, the present work is nominally issued as the seventh edition of Riemann-Weber, *The Partial Differential Equations of Mathematical Physics*. The fourth edition of Riemann's treatise under the above title was Heinrich Weber's first contribution. The fifth edition (Weber's second) was issued in 1910 and the sixth edition was a mere reprint of the fifth. H. Weber before his death had planned an edition which should be entirely rewritten with the cooperation of a group of younger mathematicians and physicists. His chief collaborators were to be his son, R. Weber, and J. Wellstein. The latter two did not survive the elder Weber very long, and the work was then undertaken by R. von Mises with the cooperation of the seven listed above.

In the main the present plan differs from the original in a division into a mathematical part (the present volume under review), and a second, a physical part announced for an early appearance. The second part is to be edited by P. Frank and written with the help of R. Furth, A. Sommerfeld, F. Noether, Th. v. Karman, and E. Trefftz. Though written by eight collaborators, the present work exhibits a degree of unity far beyond what might be expected. To be sure, the field as a whole lends itself by its very nature to systematic treatment and unification. But the skill with which this is accomplished is due chiefly to the labor of the editor and chief contributor, von Mises.

The following is a summary of the table of contents by chapters.

Part I. Fundamental preliminary theory. 1. Real functions. 2. Linear forms. 3. Complex variables. 4. Infinite series and products. 5. Calculus of variations.

Part II. Ordinary differential equations. 1. Problems involving initial conditions. 2. Boundary value problems of the second order. 3. Particular functions corresponding to second order boundary value problems. 4. Development of arbitrary functions. 5. Developments in Bessel's functions.

Part III. Integral equations and potential. 1. Summary of problems and results. 2. Solution of integral equations. 3. Applications of integral equations to boundary value problems. 4. Potential.

Part IV. Partial differential equations. 1. Problems involving initial values. 2. Potential in the plane. 3. Potential in three-space. 4. Boundary value problems for partial differential equations of the second order.

5. Other problems in partial differential equations. 6. Calculus of variations and boundary value problems.

There is some overlapping in the various chapters, but this is of a helpful kind. Beginning with the fundamentals of the theory of functions of a real variable and of a complex variable, through the calculus of variations and vector and tensor analysis, the work proceeds with a definite end in view. All of these matters are to join on to specific topics in mathematical physics.

It is clear to the reviewer that the present plan purposes not merely to rearrange material but to offer a well organized presentation of the mathematical "tool-box" of modern theoretical physics. That the reader is carried well beyond undergraduate courses in calculus and differential equations is attested by such topics as "functions of limited variation," "Stieltjes and Lebesgue integrals," "conformal mapping," "extremals," "canonical coordinates," "contact transformations," "characteristic functions," "Dirichlet's principle," "infinitely many variables," etc. The volume is indeed of value alike to the worker in physics and to the mathematical specialist, even though not all the recent advances in the various fields, especially results of a very particular nature, are included. But in the main, where the method is of importance, a detailed discussion is at hand. For example, for special results, the present volume is not as complete as a book which appeared a year earlier (1924) written by one of the present writers, R. Courant, *Methoden der Mathematischen Physik*, which forms volume XII of *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*. The present book, however, offers a wider perspective. The treatment is not encyclopedic, nor exhaustive, but it lays a broad foundation and provides a sufficient framework around which courses in "pure" or "physical" mathematics may be built.

To give a detailed account of the contents of the book would be to rehearse not only the recent developments of the past thirty years work which may be included under the heading of differential and integral equations, but most of the modern and classical developments of analysis. The names of von Mises, Bieberbach, Carathéodory and Courant, well known both as research workers and excellent expositors, as well as the others, are the best recommendation one can give for the value of the present work. Supplementary references to more extensive and special treatises are to be found at the end of most of the chapters, together with occasional mention of original memoirs.

H. J. ETTLINGER

SHORTER NOTICES

Mécanique Analytique et Théorie des Quanta. By G. Juvet. Paris, Blanchard, 1926. 6+153 pp.

Two-thirds of this book deal with general mechanical principles and perturbation theory. The opening chapters are concerned mainly with the Hamiltonian canonical equations and the Hamilton-Jacobi partial differential equation. The style and method of demonstration are usually varied enough from those given in the standard texts on dynamics so that even the discussions and proofs of the standard theorems do not seem hackneyed, but occasionally one feels that the mathematical framework is a bit more elaborate than necessary for understanding most of the literature of the quantum theory. A happy feature is the emphasis that the canonical equations furnish the "characteristic curves" of the Hamilton-Jacobi partial differential equation, for this observation to a certain degree bridges the gap that usually exists between the physicist's presentation of the Hamiltonian dynamical technique and the analysis of general first-order partial differential equations found in such treatises as those of Goursat, Jordan, etc. In Chapter V the separation of variables is considered at some length. The two following chapters are devoted to perturbation theory and the methods of successive approximation developed by Delaunay, Lindstedt, and Bohlin. No attempt is made to delve into the intricate question of the convergence of the trigonometric series. One admires the candor with which Juvet acknowledges that his treatment in Chapter VII follows closely that in Poincaré's *Méthodes Nouvelles de la Mécanique Céleste.* One might wish that all authors were equally frank, for to quote Juvet, "Nous ne croyons pas qu'il suffice de citer un auteur lorsqu'on lui prend tout ce qu'on dit, il convient de dire que tout lui revient." Nevertheless this chapter is of distinct value because it summarizes and brings together material which is rather widely scattered through the various chapters of volume II of Poincaré's treatise. It is indeed interesting that the perturbation technique of the astronomer has lent itself with so little modification to the Bohr theory of atomic structure.

The last third of the book deals with application of dynamics to the quantum theory of atomic structure. The analysis is, of course, entirely on the basis of the old Bohr theory, as it was written before advent of the new matrix dynamics of Born, Heisenberg, and Jordan, or of the wave mechanics of Schrödinger. Chapter XII presents Burgers' first method of proving the principle of adiabatic invariance, and no mention is made of his second method (summarized in an appendix of Sommerfeld's *Atombau*), which in our opinion is simpler once the transformation theory of dynamics has been established. The last few pages contain an interesting discussion of the relative utility of the various perturbation methods developed in Chapters VI and VII, but these methods are not illustrated

by applications to specific problems such as the anharmonic oscillator, helium atom, or dispersion. The quantum theory is replete with problems of this type which can be handled only by successive approximations, and it seems unfortunate that no space could be found for their inclusion after such careful development of the general perturbation theory. In fact, not even the appropriate references to the literature are given. The separation of variables, however, is illustrated by the conventional examples of the Stark effect and relativity corrections in hydrogen.

Professor Juvet's book covers a field quite distinct from most other treatises, and is written with characteristic French smoothness of style. The general flavor is mathematical rather than physical, for, to quote from the preface, "The aim is simply to put a mathematical instrument at the disposal of those who are interested in physics, and at the same time to acquaint mathematicians with one of the most elegant applications of analytical mechanics."

J. H. Van Vleck

The Quantum Theory of the Atom. By George Birtwistle. Cambridge University Press, 1926. 236 pp.

Just what form the quantum theory of the atom will take by the time this review appears in print it is impossible to say; but whatever the quantum theory may become, its original form will always be a matter of interest. This theory assumes a definite model of the atom—the Rutherford atom—and applies to this the laws of classical dynamics. From the continuous manifold of dynamically possible orbits, the quantum theory allows only a discreet manifold.

Mr. Birtwistle has written an excellent account of the quantum theory based on these ideas. He first shows how the quantum theory was introduced by Planck in obtaining his formula for the distribution of radiant energy; he then describes Bohr's original theory of the hydrogen spectrum, Einstein's photo-electric equation and his deduction of Planck's formula. Then follow a number of chapters on the general dynamical theory, which is needed in the applications of the quantum theory that follow: to the fine structure of spectral lines, to the Stark and the Zeeman effects. A brief account is given of spectral series together with Bohr's theory of the building up of atoms and the theory of band spectra. The classical theory of perturbations of dynamical systems is next given, with an application to the anharmonic oscillator. The last chapter contains a sketch of the dispersion theory of Kramers.

This book was published while the quantum mechanics of Heisenberg was in process of development, and this theory is just mentioned at the end of the volume. At the present time the undulatory mechanics of de Broglie and Schrödinger is attracting much attention. But whatever may be the fate of these recent speculations, there can be no question as to the value of the older form of the quantum theory in view of its achievements; and it would be difficult to find a better account of it than in this book.

E. P. Adams

Leçons sur les Propriétés Extrémales et la Meilleure Approximation des Fonctions Analytiques d'une Variable Réelle. By Serge Bernstein. Paris, Gauthier-Villars, 1926. 10+207 pp.

This book, which is one of the Borel series of monographs, forms an exposition of the course of lectures given by Professor Bernstein at the Sorbonne in May, 1923. The results are for the most part due to the author himself, and the majority of them appear here in print for the first time. The work as a whole is a development of the methods of Tchebycheff in the study of the field indicated by the title. The great variety of interesting and important results that are here contained in some two hundred pages, furnishes the best possible support for the writer's contention that the ideas of Tchebycheff are of fundamental importance in this domain of the theory of analytic functions of a real variable.

The first chapter is concerned with extremal properties on a finite segment of polynomials and other functions depending on a given number of parameters, and develops the algebraic basis for the whole subject. It contains among other important theorems a classical result of the author concerning the best approximation to $|x|$ by polynomials of given degree. We also find here his fundamental theorem with regard to the relationship between the maximum modulus of a polynomial and that of its derivative.

In the second chapter a study is made of extremal properties on the entire axis of reals of algebraic functions and of certain types of integral functions. In connection with the application of these properties to polynomial approximation there is introduced the notion of a "function of comparison", $\phi(x)$, and the extremal properties of the expression

$$\epsilon_n = \left| \frac{f(x) - P_n(x)}{\phi(x)} \right|,$$

where f is a given function and P_n an arbitrary polynomial of degree n, are studied. It will be seen that this use of a function of comparison is analogous to the notion of relatively uniform convergence introduced by E. H. Moore. We find also in this chapter the interesting theorem that any continuous function of a real variable, $f(x)$, such that $\lim_{x\to+\infty} f(x) = A$, can be uniformly approached on the whole axis of reals by means of rational fractions possessing given poles $\alpha_n \pm \beta_n$, provided the series $\Sigma|\beta_n|/(\alpha_n^2+\beta_n^2)$ diverges. On the other hand, the uniform approach to $f(x)$ by rational fractions, possessing poles such that the above series is convergent, implies the fact that $f(x)$ is an analytic function having no singularities in the finite plane other than poles at $\alpha_n \pm \beta_n$. In the latter part of the chapter the fundamental results regarding the relationship between the maximum of the modulus of a polynomial and that of its derivative, referred to in connection with Chapter II, are generalized to the case of integral functions.

The third chapter deals with the question of the best approximation to analytic functions possessing given singularities. As that topic had previously been treated by de la Vallée Poussin in his *Leçons sur l'Approximation des Fonctions d'une Variable Réelle*, also of the Borel series, Pro-

fessor Bernstein places the emphasis on those problems that were not discussed in the above mentioned monograph. He studies in particular the determination of the successive terms in the asymptotic development of the best approximation and the problem of the best approximation to a function having an essentially singular point. This latter is a problem of unusual difficulty, and it is noteworthy that in a wide variety of cases a solution of the problem can be obtained by the use of the same methods as in the case of polar singularities.

The book closes with two notes. The first one, which is rather extensive, develops certain interesting relationships between the problem of analytic extension of analytic functions of real variables and that of polynomial approximation to such functions. The second note, which is brief, deals with a property of analytic functions of genre zero.

As a whole the book has to a very great extent that admirable characteristic of the Borel series: namely, that its interest lies not only in the topics specifically treated, but also in the variety of problems for further research that are either definitely indicated or incidentally suggested.

C. N. Moore

Potentialtheorie. By Dr. W. Sternberg. Two volumes. I. *Die Elemente der Potentialtheorie,* 136 pages, 1925. II. *Die Randwertaufgaben der Potentialtheorie,* 133 pages, 1926. Volumes 901 and 944 of the Sammlung Göschen. Berlin, Walter de Gruyter and Co.

The two volumes give a brief introduction to the theory of the various potentials in two and three dimensional space and to the two boundary value problems. The theorems and methods are in general well known. The author has attempted to reduce the necessary preparation of the reader to a minimum. A knowledge of the calculus is sufficient except in the case of the chapter on the Fredholm equation. After a short *mengentheoretische* introduction, the first volume takes up the definition and some properties of volume, surface, and double layer potentials. It is shown that the definition of the potential as a solution of Laplace's equation is coextensive with the usual integral definition. This new definition necessitates a new definition of mass and in this the author follows Plemelj. After a chapter on the necessary variations of Stokes' Theorem, the volume closes with a discussion of the continuity properties of the potentials and their derivatives. In the first part of the second volume the author confines himself chiefly to a statement of the Dirichlet and Neumann problems, their solution for the circle and sphere and a discussion of the Green's function. No mention is made of the method of Neumann or of the method of *balayage* of Poincaré. The remainder of the volume is given over to the theory of Fredholm's equation and its application to the boundary value problems. The chapter on the Fredholm equation is rather too brief to be useful to one who knows nothing of the theory. The work is carefully written and though very concise, is readable and is to be recommended.

C. A. Shook

Die Grundlagen der ägyptischen Bruchrechnung. By Dr. O. Neugebauer. Berlin, Springer, 1926. 45 pp., and six tables.

This monograph is on the philosophy of Egyptian mathematics. The author advances the thesis that "Egyptian mathematics rests exclusively upon the fundamental operation of addition," that it is wrong to attribute to the Egyptians a knowledge of the four fundamental operations of arithmetic. According to him, Eisenlohr, Peet, Cantor, Hultsch and others have read into Egyptian mathematics processes which are not really there. It seems to the reviewer that Neugebauer establishes the possibility of explaining the arithmetical operations of the Egyptians on the additive principle alone, but that he has not demonstrated that the Egyptians actually limited themselves to that principle. In general, it seems improbable that Egyptian mathematicians able to compute correctly the volume of the frustrum of a pyramid, operated with an arithmetic that did not involve the idea of multiplication. However, Neugebauer's claim is worthy of careful study. An interesting process in Egyptian mathematics, emphasized by our author, is the equivalent of the modern so-called Russian process of multiplication. To multiply 7 by 5, put down

/1	7
2	14
/4	28

and add the numbers marked by a stroke.

Florian Cajori

Mathematische Instrumente. By F. A. Willers. Sammlung Göschen. Berlin und Leipzig, Walter de Gruyter, 1926. 144 pp.

This little monograph, written by Dr. Willers of the Technische Hochschule of Charlottenburg, is No. 922 of the well known Göschen collection, and contains a clear and adequate description of the most important and useful mathematical instruments and machines.

In the introduction, Dr. Willers stresses the necessity of familiarizing oneself with the mechanical construction and with the theory of such instruments, in order to be able to handle them intelligently and rapidly. Only in this manner may one gain an enormous advantage over the drudgery of many mental calculations.

The booklet is written especially with this purpose in view, and is divided into eight chapters, dealing with continuous calculating apparatus, like slide rules, calculating machines, instruments for graphing and measuring curves, planimeters, harmonic analysers, mechanical solution of boundary problems, geometrical apparatus, like affinographs, pantographs, and perspectivographs.

Altogether, Dr. Willers' little treatise will be a welcome addition to the literature for all those who do not want to make a special study of mathematical instruments but who wish to be able to use such auxiliaries, when necessary or desirable, for the solution of some particular problem requiring arduous numerical work.

Arnold Emch

Analyse Fonctionnelle. By Paul Lévy. Mémorial des Sciences Mathématiques, fasc. V. Paris, Gauthier-Villars, 1925. 56 pp.

The author gives in this brief discussion an exposition of what is necessarily only a part of the subject of functional analysis, but in such a way as to impress the reader with the existence of new ideas under the sun. The condensation naturally enforces slow reading.

Chapter I gives the fundamental notions of function space and the generalized definition of distance, in particular the space Ω, where the square of the distance between two functions is the integral of the square of their difference, and the space ω, in which distance may be defined as the maximum of the numerical value of the difference. The linear functional in the two spaces takes on different forms, in ω being given by a Stieltjes integral and in Ω by a Lebesgue integral; in Ω, orthogonal axes may be regarded as defined by a set of an infinite number of orthogonal functions $f_n(x)$; and a function $\Sigma a_n f_n(x)$ as having coordinates $a_1, \cdots, a_n, \cdots$. Consideration of the vectors in this space yields various trigonometric formulas. Consideration of the linear transformation in the two spaces leads to different types of integral equations.

Chapter II gives an introduction to equations in functional derivatives. The reader will probably turn to references where he can find more detail, but he will be able here to get a notion of complete integrability and of characteristics.

In Chapter III is developed the idea of integration in function space. On account of the fact that the ratio of the volumes of two similar figures becomes infinite or zero as the number of dimensions becomes infinite, the idea of volume must be replaced by that of average. Here again, however, we meet the same difficulty, in that the neighborhood of one position is apt to predominate to an infinite extent in the determination of the average. (This aspect of infinity will be more or less familiar to the reader as the characteristic of one of the well known methods of arriving at Maxwell's law of distribution for the velocities in a gas of a large number of molecules.) It becomes desirable therefore to consider other kinds of weighted means.

The author flirts throughout in engaging fashion with applications to harmonic functions, considering $\partial^2 g(M_1, M_2)/\partial n_1 \partial n_2$ as a function of curves, the equation in functional derivatives of the Dirichlet problem and the nature of harmonic functions in space of an infinite number of dimensions.

The theories expounded owe, of course, to the founders of the subject, and much also to Lévy himself and to Gateaux, Wiener, Fréchet and Daniell.

G. C. Evans

NOTES

The opening number of volume 29 of the Transactions of this Society (January, 1927) contains the following papers: *Sets of independent postulates for the arithmetic mean, the geometric mean, the harmonic mean, and the root-mean-square*, by E. V. Huntington; *Irregular differential systems of order two and the related expansion problems*, by M. H. Stone; *Some problems in the theory of interpolation by Sturm-Liouville functions*, by C. M. Jensen; *Singular ruled surfaces in space of five dimensions*, by E. B. Stouffer; *On a type of completeness characterizing the general laws for separation of point-pairs*, by C. H. Langford; *Integers and basis of a number field*, by N. R. Wilson; *Implicit functions and their differentials in general analysis*, by T. H. Hildebrandt and L. M. Graves; *Application of the theory of relative cyclic fields to both cases of Fermat's last theorem* (second paper), by H. S. Vandiver; *Riemann integration and Taylor's theorem in general analysis*, by L. M. Graves; *Alternatives to Zermelo's assumption*, by A. Church; *On a general theorem concerning the distribution of the residues and non-residues of powers*, by J. M. Vinogradov; *On the bound of the least non-residue of nth powers*, by J. M. Vinogradov; *On convergence factors in multiple series*, by C. N. Moore.

The Executive Committee of the coming International Congress of Mathematicians, of which Professor S. Pincherle is chairman, has selected the dates September 1–10, 1928, for this meeting. American mathematicians will be gratified to know that these particular dates were selected in order to make possible attendance from America. A preliminary announcement will be issued in the near future.

At the Annual Meeting to be held in conjunction with the meetings of the American Association for the Advancement of Science at Nashville, Tennessee, December 27–28, 1927, Professor James Pierpont will represent this Society at the usual joint session in which several organizations participate; he will deliver an address on *mathematical rigor, past and present*.

The office of the Society in New York has been moved to Room 1324 of the newly erected Physics Building of Columbia University. Mail addressed to the old street number will continue to reach the office properly; but telegrams and express packages should be addressed to the new office.

At its annual meeting of 1926, the American Society of Mechanical Engineers considered the question of organizing a professional division of mechanics, physics, and applied mathematics. A committee consisting of S. Timoshenko (chairman), A. L. Kimball, and H. A. S. Howarth was appointed to formulate a definite plan for this division.

The Ackermann-Teubner memorial prize for 1926 has been awarded to Professor Wilhelm Blaschke, of the University of Hamburg, for his book *Kreis und Kugel* (Leipzig, Veit, 1916).

The Ernst Abbe memorial prize for physics has been awarded to Professor Wilhelm Wien, of the University of Munich.

The Italian Society of Sciences (the XL) has awarded its mathematics prize for 1926 to Professor A. Comessatti, of the University of Padua, for his researches in algebraic geometry.

The gold medal of the Royal Astronomical Society has been awarded to Professor Frank Schlesinger, of Yale University, for his work in stellar parallax and astronomical photography.

Dr. Elihu Thomson, of the General Electric Company, has been awarded the Faraday medal of the British Institution of Electrical Engineers for 1927.

The Astronomical Society of the Pacific has awarded its Bruce gold medal for "distinguished services to astronomy" to Professor H. H. Turner, of Oxford University.

The Collingwood prize of the American Society of Civil Engineers has been awarded to C. V. von Abo, of Johannesburg, South Africa, for his paper on *Secondary stresses in bridges*.

Dr. C. E. Guye, professor of physics at the University of Geneva, has been elected a corresponding member of the Paris Academy of Sciences.

Professor Max Planck, of the University of Berlin, has been made an honorary doctor of engineering of the Berlin Technical School.

At the University of Paris, Dr. Eugène Bloch, maître de conférences, has been promoted to the chair of theoretical and celestial physics.

Dr. P. Cousin, professor of the differential and integral calculus at the University of Bordeaux, has been made dean of the University.

Professor F. Cecioni, of the University of Catania, has been transferred to the University of Pisa. He will teach algebraic analysis.

Professor O. Chisini, of the University of Milan, has been promoted to a full professorship of analytic geometry.

Dr. E. Fermi has been appointed to an associate professorship of theoretical physics at the University of Rome.

Dr. E. Persico and Dr. G. Sansone have been appointed to associate professorships at the University of Florence. The former will teach theoretical physics and the latter the calculus.

Dr. R. Serini has been appointed to an associate professorship at the University of Pavia. He will teach theoretical physics.

The following 44 doctorates with mathematics or mathematical physics as major subject were conferred by American universities during 1926; the university, month in which the degree was conferred,

minor subject (other than mathematics), and title of dissertation are given in each case if available.

Evelyn F. Aylesworth, California, May, *The dielectric constant of atomic hydrogen and the Stark effect.*

Wealthy Babcock, Kansas, June, physics, *On the geometry associated with certain determinants with linear elements.*

Mr. H. W. Bailey, Illinois, May, astronomy, *The summability of single and multiple Fourier series.*

R. W. Barnard, Chicago, December, *The Fredholm theory of linear integral equations in general analysis for quaternionic-valued functions.*

Martha H. Barton, Johns Hopkins, June, Physics, *Some applications of the generalized Kronecker symbol.*

Florence Black, Kansas, June, physics, *A reduced system of differential equations for the invariants of ternary forms.*

H. L. Black, Illinois, May, physics, *A Cremona group isomorphic with the group of the twenty-seven lines on a cubic surface.*

E. T. Browne, Chicago, September, *Involutions that belong to a linear class.*

N. B. Conkwright, Illinois, May, astronomy, *The summability of Birkhoff series.*

A. E. Cooper, Chicago, June, *A topical history of the theory of quadratic residues.*

A. H. Copeland, Harvard, *Studies on the gyroscope.*

C. M. Cramlet, Washington, June, physics and astronomy, *Invariant tensors and their application to the study of determinants and allied tensor functions.*

H. T. Davis, Wisconsin, June, physics, *An existence theorem for the characteristic numbers of a certain boundary value problem.*

M. S. Demos, Harvard, *The group characteristics of the general and special quaternary linear homogeneous congruence groups.*

R. D. Doner, Illinois, May, physics, *The determination of the Peirce and Scheffers algebras of order eight.*

Fay Farnum, Cornell, June, physics, *Triadic Cremona nets of plane curves.*

Orrin Frink, Columbia, December, *The operations of boolean algebras.*

R. J. Garver, Chicago, September, I: *On Tschirnhaus transformations*; II: *Division algebras of order sixteen.*

J. S. Georges, Chicago, September, *Associativity conditions for division algebras corresponding to any abelian group.*

R. F. Graesser, Illinois, May, statistics and physics, *A certain general type of Neumann expansions and expansions in confluent hypergeometric functions.*

Marian C. Gray, Bryn Mawr, June, physics, *Theory of singular ordinary differential equations of the second order.*

L. S. Hill, Yale, June, *Aggregate functions and an application in analysis situs.*

H. M. Hosford, Illinois, May, physics, *On the summability of Fourier-Bessel and Dini expansions.*

C. M. Huber, Illinois, May, theoretical physics, *On complete systems of irrational invariants of associated point sets.*

F. E. Johnston, Illinois, May, astronomy, *Transitive substitution groups containing a regular subgroup of lower degree.*

B. F. Kimball, Cornell, June, physics, *Geodesics on a toroid.*

F. W. Kokomoor, Michigan, June, physics, *The teaching of elementary geometry in the seventeenth century.*

B. O. Koopman, Harvard, *On rejection to infinity and exterior motion in the restricted problem of three bodies.*

B. C. Patterson, Johns Hopkins, June, geophysics, *The algebraic and differential invariants of inversive geometry.*

H. R. Phalen, Chicago, June, *Metric properties of the quadric of Moutard.*

I. R. Pounder, Chicago, September, *A method of successive approximations for a partial equation of hyperbolic type.*

H. H. Pryde, New York University, June, *On a certain quintic curve with a triple point.*

D. E. Richmond, Cornell, June, physics, *Geodesics on surfaces of genus zero with knobs.*

C. F. Roos, Rice Institute, June, theoretical economics, *Generalized Lagrange problems in the calculus of variations.*

P. D. Schwartz, Yale, June, *Studies in non-euclidean geometry.*

M. M. Slotnick, Harvard, *Fundamental transformations of surfaces.*

H. L. Smith, Chicago, September, *The Minkowski linear measure for a simple rectifiable curve.*

P. A. Smith, Princeton, *Approximation of curves and surfaces by algebraic curves and surfaces.*

Marion E. Stark, Chicago, September, *A self-adjoint boundary value problem associated with a problem of the calculus of variations.*

Guy Stevenson, Illinois, May, physics, *Expansions of the Neumann type in terms of products of Bessel functions.*

M. H. Stone, Harvard, *Ordinary linear homogeneous differential equations of order n and the related expansion problems.*

Takashi Terami, California, May, physics, *The solution of the differential equation of a vibrating membrane by successive approximations.*

E. T. Virata, Johns Hopkins, June, geophysics, *W-surfaces which have an isometric spherical representation of their lines of curvature.*

P. S. Wagner, Johns Hopkins, June, geophysics, *An extension of Clifford's chain.*

The following graduate courses in mathematics are announced for the summer of 1927:

Brown University, June 20 to July 30.—By Professor R. E. Langer: Theory of integral equations.—By Professor M. H. Ingraham: Finite groups and the Galois theory of equations.

University of Chicago, first term, June 20 to July 27; second term, July 27 to September 2.—By Professor G. A. Bliss: Calculus of variations;

Thesis work in analysis.—By Professor H. E. Slaught: Differential equations; Theory of definite integrals.—By Professor E. T. Bell: Theory of functions of a complex variable; Theory of modular systems; Reading and research in the theory of numbers.—By Professor H. H. Mitchell: Introduction to higher algebra; Analytic theory of numbers; Reading and research in algebra and the theory of numbers.—By Professor W. C. Graustein: Analytic projective geometry; Metric differential geometry; Reading and research in differential geometry.—By Professor Mayme I. Logsdon: Algebraic geometry of hyper-spaces; Reading and research in algebraic geometry.—By Professor Warren Weaver: Electro-dynamics; Reading and research in electro-dynamics.—By Professor Walter Bartky: Celestial mechanics; descriptive astronomy.

Columbia University, July 11 to August 19.—By Professor E. R. Hedrick: Theory of functions of a real variable; Fundamental concepts of mathematics.—By Professor W. B. Fite: Higher algebra.—By Professor G. A. Pfeiffer: Projective geometry.—By Professor K. W. Lamson: Differential equations.

University of Illinois, June 20 to August 13.—By Professor J. B. Shaw: Synoptic course in higher mathematics.—By Professor R. D. Carmichael: Relativity.—By Professor Arnold Emch: Geometric transformations; Mathematics of statistics.— By Dr. M. G. Carman: Advanced calculus.—By Dr. F. E. Johnston: Theory of equations.

University of Iowa, first term, June 13 to July 22; second term, July 25 to August 26.—First term: Dr. M. A. Nordgaard: Subject matter and teaching of mathematics.—By Dr. L. E. Ward: Ordinary differential equations; Differential geometry.—By Professor C. C. Wylie: Astronomy; Mathematics of finance; Celestial mechanics.—By Professor Roscoe Woods: Projective geometry.—Second term: By Dr. M. A. Nordgaard; History of mathematics.—By Dr. N. B. Conkwright: Differential equations.—By Professor E. W. Chittenden: Matrices and determinants; Introduction to the analysis of continua.

Johns Hopkins University, June 28 to August 5.—By Professor J. R. Musselman: Projective geometry.

University of Michigan, June 27 to August 19.—By Professor W. B. Ford: Advanced calculus; Infinite series with special reference to Fourier series.—By Professor L. C. Karpinski: Teaching of geometry; History of mathematics.—By Professor Peter Field: Vector analysis.—By Professor T. R. Running: Empirical formulas.—By Professor T. H. Hildebrandt: Theory of functions of a complex variable.—By Professor H. C. Carver: Theory of probability; Advanced mathematical theory of statistics.—By Professor L. A. Hopkins: Elements of mechanics.—By Professor V. C. Poor: Differential equations.—By Professor C. J. Coe: Analytic mechanics; Integral equations.—By Professor Norman Anning: Solid analytic geometry; Finite differences.—By Professor J. A. Nys-

wander: Higher algebra.—By Professor G. Y. Rainich: Quadratic forms and quadratic numbers.—By Professor R. L. Wilder: Differential equations; Analysis situs.—By Mr. S. E. Field: Projective geometry.

University of Minnesota, first term, June 17 to July 29; second term, July 30 to September 3.—First term: By Professor DunhamJackson: History of ancient and modern mathematics.—By Professor W. L. Hart: Differential equations.—By Professors Hart and Jackson: Reading in advanced mathematics.—Second term: Professor R. W. Brink: Selected topics in advanced mathematics.

Ohio State University, first term, June 18 to July 23; second term, July 25 to August 31.—By Professor Henry Blumberg: Introduction to the theory of functions of a complex variable; Problems in analysis; Reading and research in analysis.—By Professor H. W. Kuhn: Analytic projective geometry; Finite groups; Reading and research in group theory.

University of Pennsylvania, July 5 to August 13.—By Professor M. J. Babb: Theory of numbers.—By Professor J. R. Kline: Theory of functions of a real variable.

Stanford University.—By Professor W. A. Manning: Modern algebra; Groups of finite order.—By Dr. Harold Hotelling: Analysis situs; Probability and statistics.—By Dr. H. W. Brinkmann: Differential equations; Theory of functions of a real variable.—By Miss Weiss: Theory of functions of a complex variable.

University of Texas, first term, June 8 to July 20; second term, July 20 to August 31.—First term: By Dean H. Y. Benedict: Advanced calculus.—By Professor E. L. Dodd: Functions of real variables; Probability.—By Professor R. L. Moore: Theory of sets; Foundations of geometry.—By Professor H. J. Ettlinger: Calculus of variations; Mathematical physics.—By Professor C. D. Rice: Advanced calculus.—By Professor P. M. Batchelder: Theory of equations.—Second term: By Professor A. C. Lunn: Relativity; Crystallography.—Professor H. J. Ettlinger: Calculus of variations; Mathematical physics.—By Professor H. S. Vandiver: Finite groups; Definite integrals.—By Professor Goldie Horton: Advanced calculus.

University of Wisconsin, general session, June 25 to August 5.— By Professor L. W. Dowling: Projective geometry; Analytic geometry.— By Professor J. H. Taylor: Differential equations; Theory of equations.— By Professor R. W. Babcock: Analytical mechanics; Vector analysis.— Special nine weeks course for graduates, June 25 to August 26: Professor E. B. Skinner: Linear algebras; Theory of finite groups.

The Massachusetts Institute of Technology has received a grant of $230,000 from the Guggenheim Fund for the Promotion of Aeronautics; the gift will provide a building and equipment.

The University of Pittsburgh has appointed a number of engineers of the Westinghouse Company as part-time lecturers at the University in electrical and mechanical engineering, physics, and engineering mathematics; among these lecturers is Dr. Joseph Slepian.

Professor S. D. Wiksell, of Lund University, has been invited to lecture at the University of Michigan, during the coming academic year, on mathematical statistics.

Mr. S. F. Bibb, of the University of North Dakota, has been appointed assistant professor of mathematics at the Armour Institute of Technology.

Professor A. B. Coble of the University of Illinois has been appointed to a full professorship of mathematics in The Johns Hopkins University, and will assume his new duties in September, 1927.

Professor Arnold Dresden of the University of Wisconsin has been appointed to a full professorship of mathematics at Swarthmore College, and will assume his new duties in September, 1927.

Associate Professor C. B. Upton, of Teachers College, Columbia University, has been promoted to a full professorship of mathematics.

Haton de la Goupillière, formerly director of the Ecole Supérieure des Mines, and member of the Paris Academy of Sciences, has died, at the age of ninety-four.

Professor Carl Runge, of the University of Göttingen, died January 3, 1927, at the age of seventy-one.

Sir George Greenhill died February 17, 1927. He had been a member of the American Mathematical Society since 1897.

Mr. L. A. Anderson, formerly actuary of the Central Life Assurance Society, Des Moines, died January 20, 1927; he was a member of this Society.

Professor S. R. Cruse, of the University of Arizona, died February 8, 1926; he was a member of this Society.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

BACHMANN (P.). Allgemeine Arithmetik der Zahlen-Körper. (Zahlentheorie, Band V.) Leipzig, Teubner, 1926. 548+22 pp.

BAEUMLER (A.) und SCHROETER (M.). Handbuch der Philosophie. 6te und 7te Lieferung: Logik und Systematik der Geisteswissenschaften, von E. Rothacker. München und Berlin, Oldenbourg, 1926.

BIERBERBACH (L.). See ENRIQUES (F.).

BOREL (E.). Space and time. London, Blackie, 1926. 14+234 pp.

BOUTROUX (P.). Das Wissenschaftsideal der Mathematiker. Uebersetzt von H. Pollaczek. (Wissenschaft und Hypothese, Band 28.) Leipzig, Teubner, 1927.

BREUSCH (F.). Was ein mathematisches Gymnasium leisten könnte. Karsruhe, G. Braun, 1926.

CHINI (M.). Esercizi di calcolo infinitesimale. 4a edizione. Livorno, Giusti, 1926. 320 pp.

CREAMGA (S. L.). Directions cyclifiablement conjuguées. Courbes à courbure normale constante. Jassy, Goldner, 1926.

DAKIN (A.). Practical mathematics. Part 2. London, Bell, 1926. 8+257+10 pp.

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ENRIQUES (F.). Zur Geschichte der Logik. Grundlagen und Aufbau der Wissenschaft im Urteil der mathematischen Denker. Deutsch von L. Bieberbach. (Wissenschaft und Hypothese, Band 26.) Leipzig, Teubner, 1927.

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FRAENKEL (A.). Zehn Vorlesungen über die Grundlegung der Mengenlehre. (Wissenschaft und Hypothese, Band 31.) Leipzig, Teubner, 1927.

GRANT (J. D). Notes on the calculus of finite differences. Ann Arbor, Edwards, 1926.

HAERING (T. L.). Ueber Individualität in Natur- und Geisteswelt. Leipzig, Teubner, 1926. 6+114 pp.

HAUSDORFF (F.). Mengenlehre. 2te, neugearbeitete Auflage. Berlin, de Gruyter, 1927. 285 pp.

HEINRICH (M.). Logarithmentafeln. 2te, umgearbeitete Auflage. Berlin, Grotesche Verlagsbuchhandlung, 1926.

HENNIG (P.). Meine Lösung des Fermat-Problems. Berlin, Selbstverlag, 1926.

JALLER (A.). Doppeldenken. Grundlagen einer neuen Weltanschauung. Berlin, Schwetschke, 1926.

JOACHIMI (O.). Mathematische Formelsammlung. Halle, Buchhandlung des Waisenhauses, 1925.

LEHEUX (J. W. N.). Beginselen der nomographie. Deventer, Kluwer, 1926. 63 pp.

LEWIS (G. N.). The anatomy of science. New Haven, Yale University Press, 1926. 10+221 pp.

MEWES (R). Lösung der überviergradigen Gleichungen nebst Anwendung in der Technik. Berlin, Weyland, 1925. 48 pp.

OPPENHEIM (P.). Die natürliche Ordnung der Wissenschaften. Grundgesetze der vergleichenden Wissenschaftslehre. Jena, Fischer, 1926.

PAPELIER (G.). Exercices de géométrie moderne. 8 volumes. Paris, Vuibert, 1925–26. 132+84+88+128+104+136+128+96 pp.

PATERNO (E.). La Società Italiana delle Scienze detta dei XL. Il suo passato ed il suo avvenire. Roma, Bardi, 1926.

POLLACZEK (H.). See BOUTROUX (P.).

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SCHMIDT (H.). Die Kant-Laplacesche Theorie. Ideen zur Weltanschauung von Immanuel Kant und Pierre Laplace. Leipzig, Kröner, 1925. 20+228 pp.

SCHROETER (M.). See BAEUMLER (A.).

SCHUH (F.). Het getalbegrip, in het bijzonder het onmeetbare getal. Groningen, Noordhoff, 1927.

SINGER (C.). See YELDHAM (F. A.).

STEFFENSEN (J. F.). Interpolation. Baltimore, Williams and Wilkins, 1927. 10+248 pp.

STRUIK (R.). See DE VRIES (H.).

DE VRIES (H.). Die vierte Dimension. Nach der zweiten holländischen Ausgabe ins Deutsche übertragen von R. Struik. Leipzig, Teubner, 1926. 10+167 pp.

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WRIGHT (R. M.). See DURELL (C. V.).

YELDHAM (F. A.). The story of reckoning in the middle ages. With an introduction by Charles Singer. London, Harrap, 1926. 96 pp.

PART II. APPLIED MATHEMATICS

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AUERBACH (F). Physik in graphischen Darstellungen. 2te Auflage. Leipzig, Teubner, 1925. 12+286 pp.

BAUDOUIN (M.). La préhistoire par les étoiles. Paris, Maloine, 1926. 40+329 pp.

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Bisconcini (G.). Elementi di matematica finanziaria e attuariale. 3a edizione. Roma, Signorelli, 1927.

Bleich (F.) und Melan (E.). Die gewöhnlichen und partiellen Differenzengleichungen der Baustatik. Berlin, Springer, 1927, 8+350 pp.

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Bowley (A. L.). Elements of statistics. 5th edition, revised. London, P. S. King, 1926. 475 pp.

Tychonis Brahe opera omnia. Tomus 13. Edidit I. L E. Dreyer. Hauniae, Libraria Gyldendaliana, 1926. 4+398 pp.

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Bryant (J. M.) and Correll (J. A.). Alternating current circuits. New York, McGraw-Hill, 1925.

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Dreyer (I. L. E.). See Tycho Brahe.

VON EBERHARD (O.). See CRANZ (C.).

ECKHART (L.). Konstruktive Abbildungsverfahren. Eine Einfürung in die neuren Methoden der darstellenden Geometrie. Wien, Springer, 1926. 4+119 pp.

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GOT (T.). See BOULANGER (A.).

GRAAFLAND (J. C. N.). See SCHUH (F.)

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HEDGES (E. S.) and MYERS (J. E.). The problem of physico-chemical periodicity. London, Arnold, 1926. 95 pp.

HILLS (W. D.). Mechanics and applied mathematics. Part 2: Applied mathematics. London, University of London Press, 1926. 11+248 pp.

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view-point. 2d edition, revised and enlarged. London, Chapman and Hall, 1926. 22+472 pp.

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MELAN (E.). See BLEICH (F.).

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PARVULESCO (C.). Les amas globulaires d'étoiles et leurs relations dans l'espace. Paris, Gauthier-Villars, 1925. 142 pp.

PATTON (A. C.). See CRUM (W. L.).

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RESEARCH DEPARTMENT, WOOLWICH Description and discussion of the air disturbance, round bullets in free flight London, H. M. Stationery Office, 1925. 24 pp.+4 plates.

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TERMAN (F. E.). See FRANKLIN (W. S.).

VERONNET (A.). Constitution et évolution de l'univers. Paris, Doin, 1926. 480 pp.

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THE APRIL MEETING IN CHICAGO

The twenty-seventh Western meeting of the Society was held at the University of Chicago on Friday and Saturday, April 15 and 16, 1927. The attendance at this meeting included about one hundred and twenty persons, among whom were the following seventy-seven members of the Society:

B. M. Armstrong, Frank Ayres, R. P. Baker, G. A. Bliss, Blumberg, Brahana, Brand, O. E. Brown, C. C. Camp, Chittenden, Conkwright, Cope, Curtiss, H. T. Davis, Denton, Dickson, Dresden, Emch, Frink, C. A. Garabedian, Raymond Garver, Gouwens, Griffiths, Harkin, Harshbarger, W. L. Hart, Hickson, Hodge, Hofmann, L. A. Hopkins, Dunham Jackson, R. L. Jackson, B. W. Jones, Krathwohl, LaPaz, Lytle, McFarlan, MacDuffee, W. D. MacMillan, Marquis, T. E. Mason, E. H. Moore, E. J. Moulton, Mullings, Palmer, Parkinson, Pettit, Rainich, C. J. Rees, C. E. Rhodes, D. P. Richardson, H. L. Rietz, P. G. Robinson, Roos, Roth, Schottenfels, Sharpe, J. B. Shaw, Sherer, Shohat, Simmons, W. G. Simon, Sisam, Slaught, A. W. Smith, Virgil Snyder, J. H. Taylor, E. L. Thompson, J. S. Turner, Van Vleck, Vass, Wahlin, L. E. Ward, Warren Weaver, John Williamson, F. E. Wood, J. W. A. Young.

On Friday afternoon Professor E. W. Chittenden gave the symposium address entitled *Some aspects of general topology*. This address will appear in full in an early issue of this Bulletin.

On Friday evening members and guests of the Society gathered at a dinner in the Del Prado Hotel. President Snyder presided and called on Professors Coble, Bliss, Slaught and Dresden, who spoke of various recent developments in the work of the Society.

The papers listed below were presented on Friday and Saturday forenoons. The first session was divided into two sections: the first, on Geometry and Point Sets, at which President Snyder presided, included the papers numbered 1 to and including 11, and 48; before the second section, on Algebra, Theory of Numbers and Applied Mathematics, presided over by Professor D. R. Curtiss, were presented the papers numbered 12 to and including 20, 45 and 50. The remaining papers were read at a long session on Saturday

forenoon, presided over by Professors Rietz, Jackson and Bliss. The papers numbered 7, 8, 9, 10, 11, 18, 21, 23, 27, 34 to and including 42, 45 were read by title. Professor Uspensky was introduced to the Society by Professor Dresden, and Mr. Keller by Professor W. D. MacMillan.

1. Dr. H. R. Brahana: *Regular maps and their groups.*

In this paper we prove that to every group generated by two operators one of which is of order two, there corresponds a regular map. The symmetric groups, the alternating groups, the metacyclic groups and certain classes of their sub-groups, and the sub-groups modulo n of the modular group are in this class. There are just eight regular maps on a surface of genus two; of these the only one which does not contain two regions adjacent to each other along more than one edge is a map of 16 triangles.

2. Professor H. P. Pettit: *A special quartic cyclide.*

The special quartic cyclide is generated by a pencil of planes and a related system of spheres quadratic in the parameter λ. The axis of the pencil of planes is a double line on the quartic. The quartic is tangent to the envelope of the family of spheres along a sextic curve, which also lies on two cubic cyclides. Any sphere cuts the quartic cyclide in the sphero-circle and a finite sphero-sextic which lies on a ruled cubic. The quartic cyclide may be generated in a triply infinite number of ways by a pencil of cubics and a projectively related pencil of concentric spheres. There are 108 circles on the surface not belonging to the infinite system of circular generators. The locus of the centers of the generators is a rational space quartic. There exist eight pairs of minimal lines on the quartic.

3. Professor H. P. Pettit: *A sextic cyclide.*

The sextic cyclide under consideration is generated by a pencil of spheres and a related family of spheres quadratic in the parameter λ. The sphero-circle appears as a triple conic and the base circle of the pencil as a double conic on the sextic. The envelope of the quadratic family of spheres is a quartic cyclide, tangent to the sextic along a space sextic curve. Any sphere cuts the sextic in a sphero-sextic which lies on a ruled cubic. The locus of the centers of the circular generators is a rational septic curve. There exist 9 pairs of minimal lines on the sextic.

4. Professor Solomon Lefschetz: *Correspondences on algebraic curves.*

In the 1926 volume of the Transactions of this Society the author developed a general theory of continuous transformations of manifolds. The object of the present communication is to apply it to correspondences on algebraic curves. The situation being simpler than for general manifolds, many proofs can be greatly simplified. Much of the theory may be developed with a minimum use of analytical machinery.

5. Professor M. I. Logsdon: *A hypersurface in S_4 invariant under the general projective group of points on a line.*

Associating with the G_4 on the x-line given by $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ a point $P(a_0a_1a_2a_3a_4)$ in a space of four dimensions, the triply infinite group of transformations on x will transform P into a variety of 3 dimensions. It is shown that the order of this variety is six, that the system of these varieties of index 1 has as common elements (a) the rational (skew) quartic in S_4; (b) the developable consisting of the lines tangent to this quartic; (c) developables consisting respectively of planes and three-spaces osculating the quartic. Special cases and the dual situation are of interest since they offer a geometric interpretation of well known facts concerning the invariants of the biquadratic form.

6. Professor M. I. Logsdon: *Curves in r-space invariant under a net of homographies containing the identity.*

Enriques (Rendiconti dei Lincei, 1890) studied invariant loci under pencils of homographies in a space of n dimensions; Bonola (Rendiconti del Istituto Lombardo, 1908) studied invariant loci under nets of homographies in three-space when identity is an element of the net. The author, extending the study to nets of general homographies in r-space, finds: (1) to every such net containing the identity homography, there corresponds an invariant curve of order $r(r+1)/2$ and genus $r(r-1)/2$, on which there is an involution, g^1_{r+1}, of groups of $r+1$ points not cut out by hyperplanes; (2) to every curve in r-space of the above properties, there corresponds a pencil of homographies in the space for which the curve is invariant, and whose construction is given; (3) this net is unique. Thus the problem of normalization of a net of homographies depends on the number of invariants of the net. This number is the genus of the curve. A later paper will discuss normal types when identity is not present in the net.

7. Professor R. M. Mathews: *Desmic configurations on pencils of syzygetic cubics.*

The locus of the quadruples of poles of Caporali's fixed line (v) with respect to the cubics of a syzygetic pencil of parameter m is a point quartic Q which passes through the vertices of the four triangles of flex axes of the pencil Q cuts each cubic m in 12 points which form three desmic quadrangles, and (v) is the common satellite line with respect to m of their 16 lines of perspectivity. The 18 sides of these quadrangles cut Q again by threes in 12 points which form the associated desmic configuration on the corresponding cubic m in a derived syzygetic pencil, and the common satellite line for the 16 lines here is the locus of the pole of (v) with respect to the class cubics of the syzygetic pencil dual to pencil m. This line is also the contact tangent of (v) for that cubic of the dual pencil which touches (v). It cuts Q in the four points which are the poles of (v) with respect to the four triangles of the flex axes.

8. Professor R. L. Wilder: *Concerning a theorem of J. R. Kline.*

J. R. Kline proved (see this Bulletin, vol. 23 (1917), pp. 290–292) the following theorem: If, in euclidean space R_n of n dimensions $(n>1)$, M is a domain and $G_1, G_2, G_3, \cdots$ is a countable infinity of nowhere dense closed point sets no one of which disconnects any domain, then $M-(G_1+G_2+G_3+\cdots)\times M$ is a non-vacuous arcwise connected point set. The present paper shows that the condition that the sets G_n should be nowhere dense may be replaced by the much weaker condition that no G_n is identical with R_n. The relations between the thus modified conditions imposed on G_n and the following conditions are then discussed, viz., (a) G_n is closed and punctiform, (b) G_n is closed and of dimension $\leqq n-2$ (in the Menger-Urysohn sense). Either conditions (a) or (b) may be substituted for the modified conditions on G_n and the above theorem is still true.

9. Professor R. L. Wilder: *Concerning zero-dimensional sets in the plane.*

Sierpinski has shown (Fundamenta Mathematicae, vol. 2 (1921), p. 89) that sets which are zero-dimensional in the Menger-Urysohn sense are identical with those sets that are homeomorphic with subsets of the irrational points of the straight line, and hence (Fréchet, Mathematische Annalen, vol. 68 (1910), p. 154) homeomorphic with subsets of the set I of points in the plane both of whose coordinates are irrational. The present paper considers the conditions under which a zero-dimensional set M in the plane is *isotopic* with a subset of I. It is shown that the zero-dimensionality of the set is not sufficient, and that a necessary and sufficient condition is that M be locally separated by simple closed curves. By introducing the notion of accessibility from all sides a characterization of zero-dimensional sets in the plane is obtained, as well as another necessary and sufficient condition for isotopism of a set with a subset of I dependent upon arcwise accessibility from all sides. As a corollary of these results, it is shown that every plane punctiform F_6 is isotopic with a subset of I.

10. Dr. G. T. Whyburn: *On the separation of plane connected point sets.*

The following results are established. (1) If A, B, and C are points of a bounded continuum M such that no pair of these points disconnects M but their sum does separate M, then $M-(A+B+C)$ is the sum of two mutually separated *connected* point sets. (2) If the connected set M is separated into mutually separated sets M_1 and M_2 by the omission of n of its connected subsets $A_1, A_2, \cdots, A_n$ then $M_i(i=1,2)+A_1+A_2+\cdots+A_n$ is the sum of at most n mutually separated connected point sets. Furthermore, if $M_1+A_1+A_2+\cdots+A_n$ is the sum of k such sets, then $M_2+A_1+A_2+\cdots+A_n$ is the sum of at most $n-k+1$ mutually separated connected point sets. (3) A bounded continuum M is disconnected by the omission of two of its non-cut points A and B if and only if there exist two complementary domains of M such that both of the points A and B are ac-

cessible from each of these domains. Interesting generalizations of (1) and (3) are established.

11. Dr. G. T. Whyburn: *Concerning certain subsets of continuous curves and of other continua.*

In this paper the author shows that if H denotes the sum of the boundaries of all the complementary domains of a continuous curve M, then (1) H is strongly connected if and only if the set of points $M-H$ does not separate the plane even in the weak sense; (2) if $M-H$ contains no continuum, then H is regular (connected im kleinen). Result (1) is true for any continuum M such that the diameters of its complementary domains may be arranged into a null-convergent sequence. If M is a continuous curve every subcontinuum of which is a continuous curve, then (a) the set H, above, is regular; (b) if R is any arcwise connected subset of M, it has property S and every limit point P of R is regularly accessible from R. If a continuum M is regular at every point of the boundary K of one of its complementary domains, then K is a continuous curve. Every connected subset of a "regular curve" (sense of K. Menger) is connected im kleinen.

12. Professor C. C. MacDuffee: *A correspondence between quadratic ideals and matrices.*

The correspondence discovered by Poincaré between the elements of a linear associative algebra and matrices of a certain type implies a p-to-1 correspondence between certain of these matrices and the principal ideals of a quadratic field where p is the number of units in the field. It is found that this correspondence is included in a p-to-1 correspondence between the matrices of a more extensive class and all ideals of the field. For every ideal class it is possible to choose two matrices B_1 and B_2, the first corresponding to a reduced ideal, such that the totality of matrices corresponding to ideals of the class is given by xB_1+yB_2 where x and y range independently over all rational integers, not both zero.

13. Professors Frank Morley and A. B. Coble: *Eliminants.*

Sylvester's dialytic method for eliminating one variable from two algebraic equations of any order was extended by Professor Morley at the Toronto Congress to cover the simultaneous elimination of two variables from three equations of the *same* order. This paper gives a method for simultaneous elimination which is much more comprehensive in scope.

14. Professor L. E. Dickson: *Simpler proofs of Waring's theorem on cubes, with various generalizations.*

Write C_n for the sum of the cubes of n undetermined integers $\geqq 0$. The following forms represent all positive integers: tx^3+C_8 for $1\leqq t\leqq 23$, $t\neq 20$; $tx^3+2y^3+C_7$ for $1\leqq t\leqq 34$, $t\neq 10$, 15, 20, 25, 30; $tx^3+3y^3+C_7$ for $1\leqq t\leqq 9$, $t\neq 5$. The paper will appear in the Transactions of this Society.

15. Professor L. E. Dickson: *Generalizations of Waring's theorem on fifth powers.*

Denote $ax^5+\cdots+kw^5$ by $(a, \cdots, k)$, and call $a+\cdots+k$ its weight. If a form of weight 37 represents all positive integers p, its order

exceeds 7. The only two forms of order 8 which represent every $p \leqq 6500$ are (1 1 2 3 4 6 8, 12) and (1 1 2 3 4 5 7, 14). From these and two others we obtain by partitioning a coefficient into two parts 58 distinct forms of order 9 which represent all integers $\leqq 3137$. Certain types of order 9 are fully determined. R. C. Shook and K. C. Yang have investigated sixth and seventh powers.

16. Professor L. E. Dickson: *Extension of Waring's theorem on sixth powers.*

Every positive integer is represented by both $(1_{10}, 2_{90}, 3_{72}, 4_{18})$ and $(1_{178}, 8_{99})$, which give the coefficients of the sixth powers in the two forms. Here c_n denotes that n coefficients are equal to c.

17. Professor L. E. Dickson: *Every positive integer is a sum of* 17 *biquadrates and the doubles of* 10 *biquadrates.*

For $s = 0, 1, \cdots, 10$, it is proved that every positive integer is a sum of s doubles of biquadrates and $37 - 2s$ biquadrates. The case $s = 0$ gives the best complete result to date on single biquadrates. Papers 16 and 17 combined will appear in the American Journal of Mathematics.

18. Professor O. E. Glenn: *On the generalization of the algebra of lower number theory. Invariants of the cyclic transformations.* Preliminary communication.

A non-homogeneous form in e variables and with integral coefficients may have a modular period of $s = e + \alpha - 1$ residues γ_i of an integral modulus n, such that

(1) $F(\gamma_1, \gamma_2, \cdots, \gamma_e) \equiv 0$, $F(\gamma_2, \gamma_3, \cdots, \gamma_e, \gamma_{e+1}) \equiv 0, \cdots, F(\gamma_\alpha, \cdots, \gamma_s) \equiv 0$, $F(\gamma_{\alpha+1}, \cdots, \gamma_s, \gamma_1) \equiv 0, \cdots, F(\gamma_s, \cdots, \gamma_{e-1}) \equiv 0 \pmod{n}$.

The necessary and sufficient condition is obtained by forming the eliminant of (1) by Poisson's method as modified for congruences by Stieltjes. The result is a congruence in one γ, as $Q(\gamma_1) \equiv 0 \pmod{n}$. The period is the least s for which this congruence can be satisfied by a γ. If F is linear and homogeneous, Q is a cyclic determinant of order s and an invariant of F in reference to a general cyclic transformation. If $e = 2$, Q is Euler's binomial $a^s - (-b)^s$ the period being the least s for which $Q \equiv 0 \pmod{n}$. We make generalizations to the periods of $e - 1$ forms in e variables, the case $e = 3$ being a modular theory of the two abscissas that can be drawn from a point in the (η, ξ) plane to two surfaces.

19. Mr. E. G. Keller: *A necessary condition for a planet to be of annular origin.*

This paper gives a necessary condition which must be satisfied for a Laplacian ring to have condensed into a satellite. The condition is derived from the moment of momentum and energy equations of the ring, primary, and resulting satellite. The meridian section of the ring is circular or elliptical. The particles of the ring describe circular orbits about the primary according to the Newtonian law. In the first section of the paper

the matter of the ring is supposed homogeneous. In the second section the condition is developed for a ring of variable density. The criterion, when applied to the solar system, indicates that Mercury and four satellites each of Jupiter and Saturn can not be of annular origin.

20. Professor G. Y. Rainich: *On the field of radiation.*

The formulas for singular Lorentz transformations given by the author in the February number of the Proceedings of the National Academy of Sciences are applied in this paper to the study of fields which remain invariant under transformations not affecting the path of a light ray. The field of electric and magnetic forces, the field of stresses and the curvature field of the "general relativity theory" are treated from this point of view.

21. Professor P. R. Rider: *The devil's curve and abelian integrals.*

The curve $y^4-x^4+ay^2+bx^2=0$ is called by French geometers *la courbe du diable.* It has been a favorite example in curve tracing. This paper discusses the Abelian integrals connected with the curve. It has appeared in the American Mathematical Monthly (vol. 34 (1927), p. 199).

22. Professor H. T. Davis: *A note on the factoring of Fredholm minors.*

L. Tocchi has proved that a necessary and sufficient condition for the factoring of a Fredholm minor of the nth order is that the minors of lower order shall vanish for some characteristic value of the parameter. The present note derives this theorem from the fact that a necessary and sufficient condition that a function $D(x,y)$ shall be factorable is that it satisfies the equation

$$D(x,y)\frac{\partial^2 D}{\partial x \partial y} - \frac{\partial D}{\partial x}\frac{\partial D}{\partial y} = 0.$$

23. Professor Solomon Lefschetz: *On intermediary functions.*

An intermediary function of p variables is a function defined by certain properties of periodicity analogous to those of theta functions. In a paper published in Crelle, vol. 95, Frobenius has made a thoroughgoing investigation of these functions but his methods are exceedingly involved. The main object of this note is to point out that the work of Frobenius can be greatly simplified. Together with a paper by Appell in Journal de Mathématiques, 1891, it may form the basis for a most elegant and direct presentation of the chief theorems on multiply periodic functions.

24. Dr. L. E. Ward: *Expansion of functions.*

The author considers the expansion of functions in series whose terms are solutions of the differential equation $u'''+\rho^3 u=0$ and three boundary conditions which are linear and homogeneous in the dependent variable and its first two derivatives with real coefficients. The boundary conditions are restricted so that two (called the first two) bear at one point only and the third at a second point and possibly also at the first point. It is found that a uniformly convergent series of the characteristic functions necessarily

converges to an analytic function satisfying two conditions closely related to the first two boundary conditions, and that the region of convergence is the interior of an equilateral triangle centered at the point at which the first two conditions bear. Subject to further minor restrictions on the function expanded, the formal series is proved to converge uniformly to the generating function. Use is made of a contour integral for the sum of the first n terms of the formal series.

25. Dr. L. E. Ward: *Expansion of functions.*

The author considers the problem corresponding to that stated in the preceding abstract but with special boundary conditions of which the first bears at one real point, the second at one complex point, and the third at both points. The characteristic functions are shown to fall into two sets, each set being identical with a set of characteristic functions arising from three boundary conditions of the type considered in the preceding paper. The type of function expansible in uniformly convergent series of these characteristic functions is obtained, and convergence is proved. The corresponding problem with boundary conditions bearing one at each point, two of which are complex, is also considered. The characteristic functions fall into three sets, each set being identical with a set of characteristic functions of the type considered in the preceding paper; and, subject to certain minor restrictions, an arbitrary analytic function can be expanded in a uniformly convergent series of these characteristic functions.

26. Professor Henry Blumberg: *Remarks concerning a theorem on arbitrary real functions.*

At the 1926 Philadelphia meeting of the Society, the author presented a theorem, of which the following is a corollary: If $f(x,y)$ is a real function, l a straight line in the xy-plane, and d_1, d_2 two directions approaching l on the same side, then lim sup $f(x,y)$ as (x,y) approaches a point P of l along d_1 is not less than lim inf $f(x,y)$ as (x,y) approaches P along d_2, with possibly a countable number of exceptional positions of P on l where this inequality may be invalid. If $f(x,y)$ is symmetric in its arguments, in particular, if it is an "interval function," it follows that lim sup $f(x, x+0) \geqq$ lim inf $f(x, x-0)$ except possibly for an enumerable set of x's. By defining various interval functions relative to a given function of a single variable, various theorems are obtained concerning unconditioned functions of one variable, among them the theorem of W. H. Young on the right and left limits of a function (Quarterly Journal of Mathematics, vol. 39), the theorem of G. C. Young on right and left derivatives (Acta Mathematica, vol. 37), the theorems of Blumberg concerning the equality of the right and left saltus (this Bulletin, vol. 24) and the theorem of Kempisty on the approximate limits of a function in the sense of Lebesgue measure (Fundamenta Mathematicae, vol. 6).

27. Professor Henry Blumberg: *On the character of the set of points of a surface where there is a vertical, conical lacuna.*

The "surface" $z=f(x,y)$, where f is any real function whatsoever, is said to have a conical, vertical lacuna at the point $P \equiv (x,y)$, if there is a

vertical (double) cone, with vertex P and axis parallel to the z-axis, having no surface points in it other than P in a sufficiently small neighborhood of P. This paper shows that the set of points where there is such a lacuna is an F_σ, i.e., the sum of a countable number of closed sets; and conversely, if S is a point set in the (x,y) plane representable as an F_σ, then a function $f(x,y)$ exists such that $z=f(x, y)$ has a vertical conical lacuna precisely at the points of S but nowhere else.

28. Dr. C. F. Roos: *A general problem in the calculus of variations.*

In order to develop a general theory of depreciation in economics it is necessary to consider a problem similar to a Lagrange problem with variable end points but different from the Lagrange problem in that the variable end values occur in the integrand. The problem somewhat resembles the brachistochrone problem of determining the curve of quickest descent from a fixed curve to a fixed point in a resisting medium. The depreciation problem, the brachistochrone problem just referred to, and a problem proposed by Bolza to include the most general Lagrange problem with general boundary conditions and the Mayer problem with general boundary conditions as special cases, are all special cases of the general problem discussed in this paper. The Euler-Lagrange multiplier rule and the transversality condition for this general problem are obtained by modern calculus of variations methods.

29. Dr. C. F. Roos: *A general problem of minimizing an integral with discontinuous integrand.*

In this paper it is shown that the problem of determining when to replace one machine by another in such a way that there results a maximum profit for a given period of time is a type of Lagrange problem with a discontinuous integrand. The problem differs from the ordinary problem with discontinuous integrand discussed by Bliss and Mason in that the integrand is a function of the end values which are variable. In addition to the Euler-Lagrange multiplier rule and the transversality conditions which follow readily by the analysis of my paper *A general problem in the calculus of variations*, corner conditions and further necessary conditions which lead to sufficient conditions are obtained for a general problem which includes this replacement problem as a special case.

30. Professor J. H. Taylor: *Parallelism and transversality in a subspace of a general (Finsler) space.*

An arbitrary integral of the form $\int F(x, \dot{x})dt$, where F satisfies the conditions for a *regular* problem in the calculus of variations, is taken as the definition of arc length in a space N of n dimensions. In terms of parallelism and angle developed by the author in an earlier paper (*A generalization of Levi-Civita's parallelism and the Frenet formulas*, Transactions of this Society, vol. 27 (1925), pp. 246–264), it is shown that parallelism of vectors in the space N implies parallelism in any subspace of N in which they lie, and that the angle between vectors is the same if measured with

respect to the metric of the space N or with respect to the metric of the subspace. Transversality is a special case of orthogonality in the sense here used; hence the transversality condition is preserved in a subspace.

31. Professor J. V. Uspensky: *On the convergence of quadrature formulas related to an infinite interval.*

Denoting by $p(x)$ any positive function, defined in an infinite interval (a, ∞) and such that all the constants $c_n = \int_a^\infty p(x)x^n dx$ exist, the author shows that the quadrature formula of the Gaussian type $\int_a^\infty p(x)f(x)dx = \sum_1^n A_k f(x_k) + R_n$ converges for indefinitely increasing n, whenever the following conditions are fulfilled: 1. There exists a function $\phi(n)$ such that $\phi(n)/n \to 0$ and $c_n(\phi(n)/n)^{2n} \to 0$ as $n \to \infty$; 2. The function $f(x)$ satisfies the condition $|f(x)| < x^m$ for sufficiently large values of x, where m is a positive number, arbitrarily large. The paper states the consequences of this theorem for the convergence of a certain class of continued fractions and for the fundamental theorem in the calculus of probabilities.

32. Professor J. A. Shohat: *A simple method for normalizing Tchebycheff polynomials and evaluating the elements of the associated continued fraction.*

This paper appears in full in the present number of this Bulletin.

33. Dr. N. B. Conkwright: *The summability of Birkhoff series.*

It is known that the sum of the first k terms of a Birkhoff series may be expressed as an integral around a contour C_k which incloses the first k poles of the Green's function associated with a certain linear differential equation and a set of regular boundary conditions at two points a and b. This paper introduces into the integrand functions $\phi(k, \lambda)$, analytic in λ, such that the limit of the integral as k becomes infinite represents an arbitrary function $f(x)$, absolutely integrable in the interval ab. The introduction of this factor is equivalent to multiplying the first k terms of the series by quantities $d_{kl}(l = 1, \cdots, k)$, thus summing the series by a method of mean values with finite reference. If $f(x)$ is continuous in a sub-interval $\alpha\beta$ of ab, the Birkhoff series is uniformly summable by the methods under consideration in the interval $\alpha + \delta \leqq x \leqq \beta - \delta$, δ being a positive constant. The summability of multiple Birkhoff series may be investigated by a repetition of the argument for a single series. The method may also be applied to series arising from boundary conditions applied at more than two points.

34. Professor W. M. Whyburn: *Existence and oscillation theorems for linear non-self-adjoint differential systems of the second order.*

This paper studies the pair of differential equations $y' = K(x, \lambda)z$, $z' = G(x, \lambda)y$ together with boundary conditions that apply to more than two points of the interval. Existence and oscillation theorems are estab-

lished for cases where one of the conditions involves only one point of the interval while the other involves two or more points. K and G are summable functions of x on $X(a \leqq x \leqq b)$, for each λ on $L(L_1 \leqq \lambda \leqq L_2)$, continuous functions of λ on L for each x on X, and bounded numerically by a function $M(x)$ that is summable on X. Certain non-self-adjoint systems of the two-point type are treated as special cases of these more general systems.

35. Professor W. M. Whyburn: *On the polynomial convergents of power series.*

In a paper published in the Annals of Mathematics, (2), vol. 8 (1906), pp. 189–192, M. B. Porter defined a particular set of convergents for a power series and discussed certain properties of this set. In the present paper it is shown that every point on the boundary of the circle of convergence is a limit point of the zeros of Porter's set of convergents. This work includes the theorem of Jentzsch (see Landau, Funktionentheorie, Berlin, 1916, p. 80) as a corollary. Other properties of these convergents are also discussed.

36. Mr. M. E. Mullings: *First-order vector differential equations of scalar functions.*

This paper demonstrates the existence of a unique solution of the vector differential system $\nabla u(\rho) = \theta(u,\rho)$, and $u(\rho_0) = u_0$ under very general conditions. Components are avoided by use of a set of definitions extending certain notions of real variables to vectors. Four theorems are obtained, the first dealing with continuity. The second shows the existence of the gradient of a scalar function under the given conditions, the third shows the existence of a line integral of the gradient that is independent of the path between two points, and the last theorem secures a unique solution for the given vector system.

37. Professor H. E. Bray: *An auxiliary theorem in the theory of harmonic functions.*

If the function $u(M)$ is harmonic at all interior points M of the unit sphere, then the quantity $F(r,s) = \int u(M)dM$ defines an additive function of segments s, the integration extending over the interior of the segment s, on the sphere of radius $r<1$, and bounded by circles of latitude θ', $\theta'' =$ const. and by meridians ϕ', $\phi'' =$ const. Thus $F(r,s) = F(r,\theta',\theta'',\phi',\phi'')$ is a function of the four variables θ', θ'', ϕ', ϕ'', with parameter r. The theorem is that if $F(r_i,s)$ is of uniformly limited variation for all r_i of a sequence $[r_i]$ converging to unity, then there exist two sets of numbers, $E'(\theta)$ and $E''(\phi)$, dense in the respective intervals $(0, \pi)$ and $(0, 2\pi)$, and a subsequence $[r_i']$ of $[r_i]$, such that $\lim F(r_i', \theta', \theta'', \phi', \phi'')$ exists, as $i = \infty$, provided that θ', θ'' belong to $E'(\theta)$ and ϕ', ϕ'' to $E''(\phi)$. The analogous result for the circle was proved by Evans, by the use of Fourier series. This paper shows that the sequence of functions $F(r_i, \theta', \theta'', \phi', \phi'')$ converges in the mean of order 1, as $i = \infty$.

38. Dr. A. H. Copeland: *Note on the Fourier development of continuous functions.*

This paper will appear in full in an early issue of this Bulletin.

39. Professor G. C. Evans: *A general Neumann problem for the sphere.* Preliminary communication.

Let v be harmonic within a sphere S of unit radius, and let w_r be a closed regular curve on the concentric sphere S_r of radius $r<1$, whose projection on S is w. The author considers the problem of determining v so that $\int(\lambda v+\partial v/\partial r)d\sigma$ extended over the region bounded by w_r takes on given values $H(w)$ as r approaches 1, $H(w)$ being a bounded additive function of regular curves on S. For the class of functions for which $\int|\partial v/\partial r|d\sigma$, extended over S_r, remains bounded as r approaches 1, there is a unique solution of the problem, provided λ is not one of a set of characteristic values $\leqq 0$. For the characteristic value $\lambda=0$, there is a unique solution provided $H(S)=0$; in this last case, if $H(w)$ is absolutely continuous the problem reduces to the Neumann problem for boundary values summable (L). The method used is to write v as the potential of a simple distribution of mass, for which the mass function is given in terms of $H(w)$ by means of a Stieltjes integral equation.

40. Mr. J. J. Gergen: *On generalized lacunae.* Preliminary communication.

According to S. Mandelbrojt the generalized lacunae of the series $f(x)=\sum a_n x^n$ are the lacunae of the series $F(x)=\sum g(a_n)x^n$, $g(z)=\sum\alpha_m z^m$ being an entire function without constant term. The author proves theorems about the singularities of a series with generalized lacunae, defining the notion of the exponential order of the singularities of a uniform function. If $f(x)$ has as its only possible singularities the p pth roots of unity and the point at infinity, no singular point of $f(x)$ being of exponential order $>q$, then the only possible singularities of $F(x)$ are those of $f(x)$, provided $|\alpha_m|^{1/m}=\sigma(e^k)$ where $k=-2^{(q+1)m}$. The lacunae of $F(x)$, determining the nature of its singularities, likewise determine the nature of those of $f(x)$. In particular, making use of Carlson's theorem, if the sequence (a_n), with superior limit $|a_n|^{1/n}=1$, is a sequence of complex integral numbers consisting of two infinite subsequences $\{a_{\lambda_n}\}$, $\{a_{\mu_n}\}$ if $g(a_{\mu_n})=0$, if the superior limit of $|g(a_{\lambda_n})|^{1/\lambda_n}=1$, and if $\lim(\lambda_{n+1}-\lambda_n)=0$, then the circle of convergence of $f(x)$ is a cut.

41. Dr. D. V. Widder: *The singularities of a function defined by a Dirichlet series.* Preliminary communication.

The purpose of the present paper is to generalize a familiar theorem of Hadamard concerning the multiplication of singularities of functions defined by Taylor's series. A Dirichlet series, $f(s)=\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, may be regarded as a generalization of Taylor's series. If $\phi(s)=\sum_{n=1}^{\infty} b_n e^{-l_n s}$, then it is found that under certain conditions the function defined by the series $\sum_{n=1}^{\infty} b_n e^{-l_n z}\sum_{\lambda_m<l_n} a_m(l_n-\lambda_m)^{\mu}$ can have no other singularities in the whole

plane but points $\alpha+\beta$ and β, α designating a singularity of $f(s)$ and β a singularity of $\phi(s)$. Here μ is a suitably chosen positive number. The proof is made by replacing the integral of Parseval, of use in the proof of Hadamard's theorem, by the integral $(1/2\pi i)\int_c f(s)\phi(s)/s^{\mu+1}$ suggested to the author by S. Mandelbrojt.

42. Dr. C. C. Camp: *A differential system in p independent parameters which leads to a multiple series development.* Preliminary communication.

For the simplest case $p=2$ one has the system $X'+(\lambda a+\mu b)X=0$, $Y'+(\lambda c+\mu d)Y=0$ and the boundary conditions $X(\pi)=X(-\pi)$, $Y(\pi)=Y(-\pi)$. By assuming $\Delta\equiv A_1D_1-B_1C_1\neq 0$, where $2\pi A_1=\int_{-\pi}^{\pi} a(x)dx$, etc. one insures the existence of a double set of principal parameter values alike for this system and its adjoint. In the biorthogonality condition one has the determinant $a(s)d(t)-b(s)c(t)$ as a multiplier. The transformation $v_1=\lambda A_1+\mu B_1$, $v_2=\lambda C_1+\mu D_1$ leads to a comparatively simple Green's function. In the extension of Birkhoff's contour method the evaluation by integrations by parts is effected by somewhat unusual algebraic rearrangements of the integrands. The system may be arranged so that $\Delta>0$ and if one restricts the coefficients a, b, c, d so that each one is either always of one sign or identically zero a satisfactory development of an arbitrary function $f(x, y)$ of the usual sort is obtained with a double series so arranged that λ, v_1, v_2 become infinite together. The author is extending the theory to include any number of parameters.

43. Mr. H. S. Wall: *On the Padé approximants associated with the continued fraction and series of Stieltjes.* Preliminary communication.

The sequence of odd convergents and that of even convergents of the continued fraction of Stieltjes converge to the same or to distinct functions according as $\sum a_i$, the series of coefficient of the continued fraction, diverges or converges. The writer finds that when $\sum a_i$ converges, every diagonal file of approximants parallel to the principal diagonal in the Padé table for the series of Stieltjes converges, no two to the same meromorphic function. When $\sum a_i$ diverges two cases arise: either every diagonal file converges to the same function, which is analytic except for the whole or a part of the negative real axis, or else in a certain portion of the table they converge to one and the same meromorphic function, and in another portion to different meromorphic functions. The remainder after n terms of a Stieltjes series being a series of the same character, the question arises, whether Stieltjes series exist of which a given Stieltjes series is the remainder after n terms. When $\sum a_i$ converges, such series exist: an infinity for each n; when $\sum a_i$ diverges, and $n=1$, if and only if $\sum a_{2i}$ converges. In this case, when the series exists for a given n, all its coefficients are uniquely determined except the first.

44. Professor Louis Brand: *Vector forms of the Euler equation in the calculus of variations and their relation to the Hilbert integral.*

The integral $\int F(r, T)ds$ over plane curves joining two fixed points is first considered, r being the position vector and T a unit tangent vector in the positive sense. Using normal variations, it is shown that $\nabla_r F - dQ/ds = 0$, where $Q(r, T) = FT + \nabla_T F$, along an extremal; tangential variations yield only an identity in partial differentiation. If ∇_r is always normal to the constant vector e, $e \cdot Q$ is a first integral of the above equation. In a field of extremals for which $T = a$, rot $Q(r, a) = 0$; hence $Q = \nabla W$ and $\int Q \cdot T\, ds$ is independent of the path. This is the Hilbert integral. The curves $W = \text{const.}$ are the transversals. If the comparison curves lie on a surface of unit normal n, $\nabla_r F - (n \times dQ/ds) \times n = 0$ along the extremals; $n \cdot \text{rot } Q = 0$ in a field and $\int Q \cdot T\, ds$ again represents the Hilbert integral. When $F = 1$ the equation states that the geodesic curvature is zero. For surface integrals $\int F(r, n)d\sigma$ limited by a fixed closed curve, $n \cdot \nabla_r F + \text{Div}(Fn + \Delta_n F) = 0$ over the extremals; here Div denotes the *surface* divergence. In a field for which $n = c$, div $P = 0$, where $P = Fc + \nabla_0 F$ and the invariant integral is $\int P \cdot n\, d\sigma$. When $F = 1$ the equation states that the mean curvature is zero.

45. Professor J. B. Shaw: *On linear algebra.*

The main developments of linear algebra have arisen from the consideration of the hypernumbers as linear operators, whence the characteristic equation and resulting properties. The present investigation is based upon the properties of idempotents and nilpotents, that is, the hypernumber is viewed from the standpoint of quadratic forms; a study of the structures of algebras (non-associative and associative) shows the existence of an ever increasing complexity.

46. Professor I. A. Barnett: *Conformal transformations in function space.*

G. Kowalewski has found the most general regular infinitesimal transformation possessing the property of preserving the angle between two curves in function space. In this paper, the class of all finite transformations which can be generated by a certain subgroup of the above transformations is obtained. Analytically, this amounts to solving a set of linear integro-differential equations. These equations are completely solved and it is shown that the finite transformations generated by the above infinitesimal conformal transformation constitute a one-parameter family of non-singular conformal transformations.

47. Professor Dunham Jackson: *A problem in minima.*

Various problems have arisen by generalization of the least-square property of Fourier series, according to which the partial sum of the series, regarded as an approximation to the given function $f(x)$, is distinguished among all trigonometric sums of like order by the fact that the integral of the square of the error is a minimum. If the integral is looked upon as a rudimentary Gramian determinant, of the first order, one form of generaliza-

tion is obtained by supposing that two functions $f(x)$ and $\phi(x)$ are given, and seeking a trigonometric sum $T_n(x)$, of specified order n, to minimize the Gramian of the functions $f(x)-T_n(x)$ and $\phi(x)-T_n(x)$. In the geometry of function space, the elementary problem calls for the minimizing of a distance, and the new one for the minimizing of the area of a triangle. This paper discusses the existence of a solution of the triangle problem for specified n, and the convergence of the sums $T_n(x)$ which it defines, as $n\to\infty$.

48. Dr. Lulu Hofmann: *On some special congruences of rays connected with analytic functions.*

Given an analytic function $w=f(z)$, we choose two arbitrary planes (w) and (z) forming an angle α in ordinary three-dimensional space for the Gaussian planes of w and z and study the congruences of rays obtained by joining two corresponding points of (w) and (z). (For the general investigation of such congruences in the case of parallel planes, see Wilczynski, Transactions of this Society, 1919.) For a general analytic function invariance with regard to α is stated for the curves cut out in (w) and (z) by any developable of the congruence and for the anharmonic ratio formed on any line of the congruence by its focal points and its intersections with (w) and (z). The congruences of the functions $w=cz^k$, where k is a positive integer, are examined in detail for the special position of the two Gaussian coordinate-systems in which the two points of origin and the two real axes coincide. Under the same assumption the congruence of $w=c/z$ is studied. The focal surfaces are investigated by means of the representative system of plane curves according to the methods of algebraic geometry.

49. Professor H. A. Simmons: *On the existence of a solution of the Diophantine equation* $\sum 1/(x_1\cdots x_s)=1$.

The papers by Kellogg (American Mathematical Monthly, 1921, p.300–303) and Curtiss (ibid., 1922, p. 330–337) on the Diophantine equation in n variables $\sum 1/x_1=1$, led the author to study the equations $\sum 1/(x_1x_2)=1$, $\sum 1/(x_1x_2x_3)=1$. In obtaining these solutions certain principles appeared which are independent of the number of variables in each unit fraction. Thus he was led to a solution of the general equation $\sum 1/(x_1x_2\cdots x_s)=1$, s any positive integer $\leqq n$. The solution obtained for this equation is a perfect analog of Kellogg's solution and indeed reduces to it in the case $s=1$. On account of Curtiss's proof that Kellogg's solution contains the maximum x that can appear in a solution of the equation $\sum 1/x_1=1$ and additional facts, the author believes that his solution contains the largest x that can appear in a solution of the equation $\sum 1/(x_1x_2\cdots x_s)=1$.

50. Professor J. S. Turner: *Positive determinants of the Seelhoff type.*

The present paper contains a number of positive determinants which can be used in a similar manner to those of Seelhoff for testing the prime of composite character of large integers.

Arnold Dresden,
Assistant Secretary.

THE MAY MEETING IN NEW YORK

The two hundred fifty-fifth regular meeting of the American Mathematical Society was held at Columbia University, on Saturday, May 7, 1927, extending through the usual morning and afternoon sessions. Attendance included the following eighty members of the Society:

Alexander, Archibald, W. L. Ayres, F. W. Beal, A. A. Bennett, Blichfeldt, B. H. Camp, G. A. Campbell, Carmichael, W. B. Carver, Curry, Douglas, Edmondson, Farnum, Fite, D. A. Flanders, Fort, Philip Franklin, Gehman, Gill, Gronwall, C. C. Grove, Haskins, Hedlund, Himwich, Hollcroft, Ingraham, Joffe, W. A. Johnson, O. D. Kellogg, Kholodovsky, Kline, Koopman, Kormes, Langford, Langman, Lefschetz, Littauer, Litzinger, Lotka, MacColl, W. A. Manning, Michal, H. H. Mitchell, L. T. Moore, Mullins, Neelley, K. E. O'Brien, F. W. Owens, H. B. Owens, Paradiso, Pell-Wheller, Pfeiffer, Phalen, Pierpont, Post, R. G. Putnam, Raynor, R. G. D. Richardson, Ritt, Ruger, Rutt, Seely, Siceloff, Simons, Smail, Virgil Snyder, M. H. Stone, Teach, J. M. Thomas, Tracey, Veblen, Wedderburn, Weida, Weisner, H. S. White, Whittemore, Wiener, J. W. Young, Margaret M. Young.

The Secretary announced the election of the following eighteen persons to membership in the Society:

Miss Margaret Charlotte Amig, St. Catherine's School, Richmond;
Dr. Walter Bartky, University of Chicago;
Mr. John Montgomery Clarkson, Duke University;
Mr. Hubert Leron Clary, radio engineer, Milwaukee;
Mr. Thomas Freeman Cope, Adelbert College, Western Reserve University;
Mr. Henry Leslie Garabedian, University of Rochester;
Mr. José Treviño García, civil engineer, Parras, Mexico;
Mr. John Jay Gergen, Rice Institute;
Dr. Lulu Hofmann, Springfield, Ohio;
Professor Robert E. Hundley, University of Cincinnati;
Mr. Vern James, Stanford University;
Professor Lida Belle May, Kidd-Key College;
Professor Harry L. Miller, University of Cincinnati;
Mr. Bernard Preston, certified public accountant, New York City;
Mr. Ralph Grafton Sanger, University of Wisconsin;
Professor Harold Ward Sibert, University of Cincinnati;
Dr. Waldemar Joseph Trjitzinsky, University of Texas;
Professor Charles Newman Wunder, University of Mississippi.

The meeting of the Board of Trustees took place in the Murray Hill Hotel on the evening of May 6, and that of the

Council on May 7 at 9:30 A.M., in Columbia University.

It was voted by the Trustees that all funds bequeathed to the Society should be added to endowment unless specifically designated for other purposes.

Professor E. W. Brown was invited to give the fifth Josiah Willard Gibbs Lecture in connection with the Annual Meeting to be held at Nashville.

The following appointments were announced: to represent the Society at the Bicentennial of the American Philosophical Society in Philadelphia, April 27–29, Professors G. D. Birkhoff, L. E. Dickson, and H. B. Fine; to represent the Society at the Centennial of Lindenwood College, St. Charles, Missouri, Professor W. H. Roever.

It was announced that Professor R. L. Moore has accepted the invitation to give a Colloquium at the Summer Meeting of 1929 in Boulder, and that Professor J. H. M. Wedderburn has been appointed an associate editor of the Bulletin.

An invitation was received from Amherst College to hold a mathematical conference of a few weeks duration about the time of the Summer Meeting which is to be held there in 1928. Brown University presented an invitation for the Summer Meeting of 1930 or 1931.

The following committee was appointed to nominate to the Council a list of Officers and Members of the Council for 1928: Professors D. R. Curtiss (chairman), W. B. Ford, D. C. Gillespie, E. R. Hedrick, and W. H. Roever.

Professor Hollcroft presided at the beginning of the morning session, relieved by Ex-President Veblen and President Snyder. Vice-President Kellogg and President Snyder presided at the afternoon session.

At the request of the Program Committee, Professor J. R. Kline delivered an address, at the beginning of the afternoon session, entitled *Separation theorems and their relation to recent developments in analysis situs*. Titles and abstracts of the other papers read at the meeting follow below. The papers of Alexander, Bray, Dodd, Douglas (third and fourth papers), Flanders, Garver, Haskell, Kasner, McVey

and Babb, Maria, Merriman, Murnaghan, Musselman, Richmond, Serghiesco, Weisner (second paper), and Whyburn were read by title. Professor Jules Drach was introduced by Professor R. G. D. Richardson, Mr. McVey by Professor Babb, Mr. Serghiesco by Professor Ritt, and Professor Struik by Professor Wiener.

1. Dr. Louis Weisner (National Research Fellow): *On m-dimensional cross ratios.*

Regarding the projective invariance of the cross ratio of four collinear points as its essential property, many writers have proposed unsatisfactory generalizations. The characteristic property of the cross ratio is not its invariance under projection, as an arbitrary function of the cross ratio also enjoys this property, but is that expressed by the following theorem: A necessary and sufficient condition that two sets of four collinear points be projective is that their respective cross ratios be equal. In generalizing to m-space, we seek functions which, in a similar way, provide conditions that two sets of $m+3$ points in m-space be projective. Such functions (m of them in m-space) have been proposed in Veblen and Young's *Projective Geometry*, vol. 2, p. 55. It is shown in the present paper that these cross ratios have properties which are generalizations of the known properties of the classical cross ratio. Applications are made to Coble's associated point sets.

2. Dr. Louis Weisner: *A new reciprocity law in invariant theory.*

The reciprocity law is a result of the observation that from an invariant I of degree d of an m-ary form of order n, expressed as the product of symbolic determinants of order m, we may form an invariant I' of degree d of a k-ary form of order kn/m, where $k=d-m$, by replacing each determinant D of I by the determinant of order k consisting of those symbols in I which are not contained in D. By the same rule I is obtained from I'. For example, from the invariant $(ab)^2(ac)^2(de)^2(bd)(be)(cd)(ce)$ of the binary quartic we form the invariant $(cde)^2(bde)^2(abc)^2(ace)(acd)(abe)(abd)$ of the ternary sextic. The law breaks down if one and only one of the two corresponding invariants vanishes. That this may actually occur is shown by an example. However, if one or two corresponding *absolute* invariants has the value 0, ∞, or 0/0, the other also has this value. The reciprocity law is therefore valid for absolute invariants without exception.

3. Dr. Louis Weisner: *The functional equation defining diophantine automorphisms.*

This paper will appear in full in an early issue of this Bulletin.

4. Dr. L. T. Moore and Dr. J. H. Neelley: *Rational tacnodal and oscnodal quartic curves considered as plane sections of certain quartic surfaces.*

This paper develops a system of invariants, which are functions of the coefficients of the cutting plane, for both the tacnodal and the oscnodal rational quartic curves. We also distinguish between the two types of tacnodes by means of an invariant.

5. Dr. L. T. Moore: *Note on the nodes of the rational plane quartic.*

The nature of the nodes of the rational quartic, given parametrically or as a section of the Steiner quartic surface, is determined from the invariants of the curve.

6. Professor T. R. Hollcroft: *The generalized Hessian.*

The definitions of the Hessian, Steinerian, and Cayleyan were extended by Salmon to include the covariant curves similarly related to the systems of first, second, third, etc., polars of an algebraic curve, but he did little more than define them. Any generalized Steinerian is also a generalized Hessian. The ordinary Steinerian now appears as a Hessian with respect to the net of polar conics. All generalized Hessians have distinct nodes and cusps except the first Hessian. The characteristics of the generalized Hessian are found for a non-singular basis curve and for a basis curve with simple or compound singularities. The order and other characteristics of certain generalized Hessians are reduced by singularities of the basis curve. For every curve of odd order, a certain generalized Hessian coincides with its corresponding Steinerian. This curve is given the name mid-Hessian. The Hessian of the cubic is the simplest example. A property of the mid-Hessian is that it can never have more than one singular point other than distinct nodes and cusps and this singularity can not be compound.

7. Mr. A. J. Lotka: *The progeny of a population element.*

In connection with population statistics as well as with the mathematical theory of evolution, it is of interest to know how the successive generations descended from an initial population element are distributed in time. Given a frequency distibution in any one generation, the nth, say, it is shown that the kth seminvariant of the frequency distribution of the $(n+1)$th generation is obtained by adding to the kth seminvariant of the nth generation the kth seminvariant of the function $f(a)=s(a)\beta(a)$, where $s(a)$ is the fraction surviving to age a, out of a batch of births, while $\beta(a)$ is the frequency of reproduction at age a. (The discussion is restricted to the case in which $s(a)$ and $\beta(a)$ are independent of the time, and are functions of the age a only. As corollaries we have the following: (1) the frequency quency distribution of births in advanced generations is essentially independent of the frequency distribution in the initial population element; (2) the frequency distribution of births ultimately approaches more and more nearly to the "normal" or Gaussian distribution. Formulas are also developed for (a) the frequency distribution expressing the relative contribution of coexisting "successive" generations, to the total birth rate at a given time, and (b) the total birth rate at any instant.

8. Professor Philip Franklin: *A qualitative definition of the sub and super harmonic functions.*

An integral characterization of functions satisfying Poisson's equation with non-negative density has recently been given by F. Riesz (Acta Mathematica, vol. 48, pp. 329 ff.) which identifies them with the super harmonic functions studied in other connections. In the present paper we give a set of postulates characterizing these functions which are of a more qualitative nature, in that they involve neither derivatives nor integrals. They are analogous to a set previously given for the harmonic functions themselves (this Bulletin, vol. 30, pp. 41 ff.).

9. Dr. Jesse Douglas (National Research Fellow): *The most general geometry of paths.*

A system of paths in an n-dimensional space S_n is defined as any analytic system of curves in S_n such that a unique curve passes through any given two points, or through any given point in any given direction. According as the paths are considered in a fixed parameterization or independent of any parameterization, we speak of an *affine* or *projective* space of paths. The principal theorems of this paper are the following: (1) Any affine space of paths is representable by $d^2x^i/dt^2 = H_2{}^i(x, p)$, $(p = dx/dt)$, where $H_2{}^i(x\ p)$ denotes a function homogeneous of the second degree in p. (2) Any projective space of paths is representable by $d^2x^i/dt^2 = H_2{}^i(x, p) + p^iH_1(x\ p)$, where $H_2{}^i(x, p)$ are fixed functions of their kind, and $H_1(x, p)$ denotes an arbitrary homogeneous function of the first degree in p, whose variation affects only the parameterization of the paths, not the paths themselves. These results and their proofs will appear in the Proceedings of the National Academy, and an elaboration developing the basis of the geometry of a general system of paths in an early number of the Annals of Mathematics.

10. Dr. Jesse Douglas: *A method of numerical solution of the problem of Plateau.*

By replacing the partial derivatives in the equation of minimal surfaces $(1+q^2)r - 2pqs + (1+p^2)t = 0$ by their finite difference approximations, a formula is derived giving the value of a solution at the center of a small square in terms of the values at the midpoints of the sides and the vertices. A numerical solution of the problem of Plateau is based on the use of this formula in connection with successively finer and finer nets of square meshes constructed over the xy-plane. Applied to a chosen contour on the catenoid $z = \cosh^{-1}(x^2+y^2)^{1/2}$, this method gives numerical results agreeing to four decimal places with a table of hyperbolic functions. The method is then applied to a contour where the solution is not known in advance.

11. Dr. Jesse Douglas: *A method of numerical solution of the problem of Dirichlet for a square.*

The value at the center of a small square, side $2h$, in the xy-plane, of any solution of Laplace's equation $\partial^2z/\partial x^2 + \partial^2z/\partial y^2 = 0$ is, up to infinitesimals of the order h^4, the average of the values at the midpoints of the sides;

also the average of the values at the vertices. It is found that if the former values are weighted four times as heavily as the latter, the resulting average gives the value at the center up to infinitesimals of the order of h^6. On the basis of this relation, a numerical solution of different particular cases of the problem of Dirichlet for a square is effected by a successive approximation procedure involving the use of a sequence of nets with square meshes. It is expected in a future paper to apply this method to numerical computation of the function $\wp(u)$ for which $g_2=1$, $g_3=0$.

12. Dr. Jesse Douglas: *Contact transformations of 3-space which convert a system of paths into a system of paths.*

The following theorem is proved: Given a proper contact transformation Γ(i.e., one not reducing to a point transformation) of a 3-space S_3 into another S_3'; if there exists a system of paths in S_3 (in the sense of our first paper, above) which is converted by Γ into a system of paths in S_3', then both systems of paths must be linear (equivalent by point transformation to the straight lines of a projective space), and Γ must have the form PDP_1, where P, P_1 are any point transformations and D is a dualistic transformation.

13. Professor James Pierpont: *On the geometry whose absolute is a ruled quadric.*

The equation of a non-degenerate quadric has one of the three following forms: $x_1^2+x_2^2+x_3^2+x_4^2=0$, $x_1^2+x_2^2+x_3^2-x_4^2=0$, $x_1^2+x_2^2-x_3^2-x_4^2=0$. Each of these taken as the absolute defines a geometry; the first gives elliptic, the second gives hyperbolic geometry. The last is a ruled quadric, and the question arises as to what kind of a geometry this defines. Poincaré (Bulletin de la Société de France, vol. 15(1886), p. 206) has stated a few properties for the plane; the author has seen no other reference. The present paper undertakes a study of this space.

14. Dr. C. H. Langford: *On inductive relations.*

Difficulties, connected with the occurrence of reflexive fallacies, appear when we attempt to give straighforward definitions of inductive series. These difficulties arise inevitably from the fact that, if a series is to be inductive, in the ordinary sense, properties of all orders must be transmitted in the series, and it would seem that no proposition, nor any finite set of propositions, can assert that properties of all orders are transmitted. In this paper, it is held that, although propositions in extension are strictly subject to differences of type, propositions in intension are not; and that propositions in intension, modal propositions, fall outside the heierarchy, and may, in an important sense, be about any proposition. This distinction of modal and material propositions leads to an analogous distinction of modal and material properties; and modal properties are important in definitions of inductive series.

15. Dr. B. O. Koopman: *On multiple-valued functions of several complex variables.*

We consider the infinitely multiple-valued monogenic function $f(z_1, \cdots, z_n)$, every branch of which is meromorphic in a finitely multiply-connected region R of the complex $z_1 \cdots z_n$-space. Evidently $f(z_1, \cdots, z_n)$ admits an infinite monodromy group. By the rank r of $f(z_1, \cdots, z_n)$ shall be meant the maximum number of functionally independent branches at any point. When $f(z_1, \cdots, z_n)$ is given, r is determined, and $1 \leqq r \leqq n$. We show that there is a $2r$-dimensional region Ω on a complex $u_1 \cdots u_r$-space which admits a group of (1-1) analytic automorphisms, isomorphic with the monodromy group, and having the same effect on the branches. In general, $r = n$. The case $r < n$ arises if and only if $f(z_1, \cdots, z_n)$ is an integral of $n-r$ independent equations of the form $Z_1(\partial f/\partial x_1) + \cdots + Z_n(\partial f/\partial z_n) = 0$, the Z's being single-valued meromorphic functions of $(z_1, \cdots, z_n)$ in R. When $r = n-1$, and when $f(z_1, \cdots, z_n)$ is real, Ω and its group of automorphisms forms an immediate generalization of the ring-transformation (surface of section) for the dynamical system $dz_1/Z_1 = \cdots = dz_n/Z_n$. Other special cases are furnished by the inverses of certain automorphic and Cremona functions.

16. Dr. B. O. Koopman: *The Riemann multiple-space and algebroid functions.*

The object of this paper is two-fold: first, we define the notion of the Riemann surface in its extension to the case of several variables from the point of view of abstract sets; the desideratum is that the result shall be sufficiently general for the purposes of the theory of functions, without having unnecessary generality. The second and more extensive part of the paper studies the result, from the point of view of analysis situs, in the case of functions defined by algebroid and algebraic equations. The methods used are those of the theory of analytic factorization (see Osgood, *Funktionentheorie*, vol. 2, Chapter 2), and of combinatorial analysis (see Veblen, The Cambridge Colloquium, Part II). After dividing the Riemann multiple-space into a complex of analytic cells, it is shown that, in the case of algebraic functions, the result is a generalized manifold (Veblen, loc. cit., p. 92), while the branch points and non-spherical points form circuits of cells of the complex, of certain specified sorts. Corresponding theorems may be stated in the case of algebroid functions.

17. Mr. S. Serghiesco: *On an integral giving the number of the common roots of a system of simultaneous equations.*

First suppose that two equations $f = 0$, $\phi = 0$ are given. We assume that the conditions that Picard's well known formula giving the common roots of this system shall reduce to a line integral are satisfied. In the present paper a new and general integral is obtained, giving the same number of roots by making use of the differential invariants of $f + \lambda\phi = 0$. This integral is then generalized, for the case of n equations $f_1 = 0, f_2 = 0, \cdots, f_n = 0$, to an integral of order $n-1$, by means of the differential invariants of $f_1 + \lambda f_2 + \mu f_3 + \cdots = 0$. A verification of Picard's results for the case of polynomials is also given through this new formula.

18. Mr. R. N. Haskell: *A note on Stieltjes integrals.*

Given $h(P)$, a continuous function of the point P, and $F(w)$, a bounded additive function of regular curves w, and a sequence $\{\delta_n\}$ of positive numbers converging to zero; if the fundamental region Δ be divided into finite number K_n of simply connected regions bounded by regular curves $\{W_n^i\}$, with diam.$(W_n^i) < \delta_n$, $(i=1, 2, \cdots, K_n)$, then the Riemann sum $\sum_i h(P_n^i)F(W_n^i)$ approaches a unique limit $\int_\Delta h(P)dF(w)$ as $n \to \infty$, independently of the system of curves W_n^i by which it is calculated. This limit is therefore equal to the Stieltjes integral $\int h(P)dF(s)$ as usually defined by means of segments, s.

19. Dr. G. M. Merriman (National Research Fellow): *Concerning the summability of double series of a certain type.*

The discussion in the present paper is pointed to a general theorem containing, for example, the result that if $a_{m,n}$ is the general term of a double series summable (C, r, s), then $a_{m,n}m^{-h}n^{-k}$ is the general term of a double series which is summable $(C, r-h, s-k)$. The discussion is carried out through generalizations to double series of the Rieszian summation means, here defined for the first time; and it includes, necessarily, a proof of the equivalence, as a means of summation, of certain of these Rieszian means to the double Cesàro means. A by-product is the important theorem that a series summable (C, r, s) is also summable (C, r', s'), $r' > r$, $s' > s$; this result has been proved before only for integral values of r, r', s, s'.

20. Professor F. D. Murnaghan: *On the degree of arbitrariness of the wave equation in the new quantum mechanics.*

For a dynamical system with fixed constraints, moving in a stationary conservative force field, the action, $W = -Et + S(q)$, satisfies the Hamilton equation $\partial W/\partial t + H(q, \partial W/\partial q) = 0$. Here q denotes the positional coordinates of the system, and E the constant total energy, both kinetic and potential, of the system. The partial differential equation of the family of moving surfaces $F(q, t) = 0$, on each of which W takes a constant value, is found by putting $F = f(W)$, whence $\partial W/\partial q = (\partial W/\partial t)((\partial F/\partial q)/(\partial F/\partial t)) = -E((\partial F/\partial q)/(\partial F/\partial t))$, and $H(q, -E(\partial F/\partial q)/(\partial F/\partial t))$. For a moving particle we get the familiar equation $(\partial F/\partial x)^2 + (\partial F/\partial y)^2 + (\partial F/\partial z)^2 = (2M(E-V)/E^2)(\partial F/\partial t)^2$. The wave equation is to have the surfaces $F = 0$ for its wave fronts, i.e. characteristics. It may be written $g_{11}\partial^2\psi/\partial x^2 + g_{22}\partial^2\psi/\partial y^2 + g_{33}\partial^2\psi/\partial z^2 = g_{44}\partial^2\psi/\partial t^2 + \phi$ where g_{11}, for example, is any function of $(x, y, z, t, \psi, \partial\psi/\partial x, \partial\psi/\partial y, \partial\psi/\partial z, \partial\psi/\partial t)$ which reduces to 1 when ψ and the first-order derivatives are zero. ϕ is any function of the same arguments reducing to zero when ψ and its first-order derivatives vanish. The introduction through ϕ of terms in the first order derivative of ψ is interesting in connection with the theory of a magnetic field.

21. Professor E. L. Dodd: *The probability law for the intensity of a trial period, with data subject to the Gaussian law.*

This paper will appear in full in an early issue of this Bulletin.

22. Dr. Raymond Garver: *A series solution for certain rational cubics.*

Any rational cubic, with minor exceptions, can be transformed rationally into an equation of the form $x^3+A(x+1)=0$. By means of certain substitutions and an inversion we obtain an expression for a real root of this equation as a power series in $1/(A+m)$. The series for $m=4$ is studied.

23. Dr. Raymond Garver: *Arccotangent relations and expressions for* π.

A number of arccotangent relations are developed (some of which the author has not seen before); these lead to several new arctangent expressions for π

24. Dr. Raymond Garver: *A type of function with k discontinuities.*

An analytic expression is given for a type of function which is, in general, discontinuous at k points, where k is any positive integer.

25. Professor J. R. Musselman: *On an imprimitive group of order* 5184.

In an earlier paper, read before the Society December 29, 1924, the author discussed a certain configuration in three-dimensional space. In the present paper he considers the imprimitive group of order 5184 under which the configuration is invariant. Associated with the configuration are eight Eckhardt cubic surfaces and twenty-seven sets of desmic tetrahedra. A similar problem in the plane leads to an imprimitive group of order 216 associated with four Hesse configurations. Both papers will appear together in the American Journal of Mathematics.

26. Dr. G. T. Whyburn: *Concerning irreducible cuttings of a continuum.*

A subset K of a continuum M is a *cutting* of M provided $M-K$ is not connected; K is an *irreducible cutting* of M if no proper subset of K cuts M; K is a *cutting of M between the points A and B of M* provided $M-K$ is the sum of two mutually separated sets M_a and M_b containing A and B respectively. Let M denote any continuum, A and B any two points of M, and K an *irreducible cutting of M between A and B*, (if one exists). It is shown in this paper that (1) if $M-K=M_a+M_b$, as above, then M_a+K and M_b+K are continua; (2) if M is indecomposable, no set K can exist; (3) if M is a continuous curve, then every cutting of M between A and B contains a set K; (4) if every subcontinuum of M is a continuous curve, then every cutting of M contains an irreducible cutting of M; and (5) if M is a bounded plane continuum, and H is an irreducible cutting of M which has more than two maximal connected subsets, then $M-H$ is the sum of two mutually separated *connected* point sets.

27. Dr. G. T. Whyburn: *On the separation of indecomposable continua and other continua.*

In this paper it is shown that if M is any indecomposable continuum and $[A_i]$ is any countable collection of mutually exclusive subcontinua of M, then no subset of $\sum A_i$ disconnects M. It is also shown that if M is any continuum whatever, X and Y are any two points of M, and K denotes the set of all those points of M which separate X from Y in M, and H is any connected subset of K, then (1) every point of H, save possibly two, is a cut point of H; (2) if H is closed, it is a simple continuous arc; (3) if A is a cut point of H, then $H-A$ is the sum of two mutually separated connected point sets; (4) if A and B are any two points of H, then H contains one and only one connected subset which is irreducibly connected between A and B; (5) if H has two non-cut points A and B, then H is irreducibly connected between A and B; and conversely, if H is irreducibly connected between some two of its points, these points are the non-cut points of H.

28. Professor H. F. Blichfeldt: *The reduction to six terms of the general quadratic differential form in four variables.*

This paper shows that the general quadratic differential form in four variables, $\sum\sum\alpha_{ij}dx_idx_j$, $(i,j=1,2,3,4)$, of non-vanishing determinant, can, by a suitable choice of variables, $y_1, \cdots, y_4$, be reduced to the form $\beta_{11}dy_1^2+2\beta_{12}dy_1dy_2+\beta_{22}dy_2^2+2'\beta_{14}dy_1+\beta_{24}dy_2+\beta_{34}dy_3)dy_4$. Corresponding reductions are given for the general quadratic form in n variables.

29. Professor W. A. Manning: *The primitive groups of class fourteen in which the positive subgroup is of class greater than fifteen.*

All the primitive groups of class 14 contain negative permutations. Those in which the positive subgroup is of class 15 are best studied in connection with the primitive groups of class 15. The author announces that there are but four primitive groups of class 14 in which the positive subgroup is of class greater than 15. There are (1) the simply transitive primitive group according to which the symmetric group of degree 9 permutes its 36 transpositions; (2) the simply transitive primitive group of degree 49 and order $3(7!)^2$ isomorphic to the group generated by (ab), $(bcdefg)$, $(\alpha\beta)$, $(\beta\gamma\delta\epsilon\zeta\eta)$, and $(a\alpha)(b\beta)(c\gamma)(d\delta)(e\epsilon)(f\zeta)(g\eta)$; (3) the triply transitive group of degree 22 and order $22\cdot21\cdot20\cdot96$ which occurs in Mathieu's quintiply transitive group of degree 24 as a transitive constituent of the largest subgroup in which the subgroup that leaves two letters fixed is invariant; (4) the doubly transitive subgroup of (3) of degree 21 and order $21\cdot20\cdot96$.

30. Professor Edward Kasner: *Irregular differential invariants.* II: *The group of all analytic transformations.*

This paper is a continuation of an earlier paper by the author, read at the February, 1927, meeting of the Society; it gives a qualitative classification of irregular analytic elements (represented by fractional power series). Some species, like $(p=2, q=3)$, have no invariants; others, like $(p=5, q=6)$, have invariants.

31. Professors D. J. Struik and Norbert Wiener: *The quantum theory as a corollary of relativity.*

In a general manifold of four dimensions, we assume a linear partial differential equation of the second order and the hyperbolic type. By means of the appropriate invariant theory, we cast it into a unique normalized form, involving a tensor of the second order, a vector, and an invariant scalar, representing respectively the gravitational field, the electromagnetic field, and a term connected with the quantum theory. On the assumption of the Einstein field equations for the gravitational field, our equation assumes a form which has already been given for the Schrödinger wave equation of quantum mechanics.

32. Professor G. A. Pfeiffer: *Certain sequences of curves which approach a rectifiable boundary from within.*

In this paper there is established the fact that if the boundary of a region G(a set of inner points) is a simple rectifiable closed curve C, then there exists a sequence of simple rectifiable closed curves, C_i, such that (1) every curve C_i is in G, (2) every point of G is in the interiors of all but a finite number of the curves C_i, and (3) if l_i is the length of C_i, then the length of C is the limit of the sequence $\{l_i\}$. The above theorem is obtained by means as elementary as the matter seems to permit. It is then shown that the curves of the sequence $\{C_i\}$ may be taken as equipotential lines of the region G.

33. Mr. W. L. Ayres: *Concerning the arc curves and basic sets of a continuous curve.*

If K is a subset of a plane continuous curve M, the arc curve of K with respect to M (denoted by $M(K)$) is the set of all arcs of M whose end points belong to K. Any subset of M whose arc curve is M itself is called a basic set of M. A large number of the properties of arc curves are obtained in this paper. Probably the most important are the following: (1) every point which is a limit point of $M(K)$ but not a point of $M(k)$ is a limit point of K; (2) $M(K)$ is arc-wise connected im kleinen; (3) whenever K is closed, $M(K)$ is a continuous curve; (4) every interior point of an arc of M whose end points are points or limit points of $M(K)$ belongs to $M(K)$. Necessary and sufficient conditions are given in order that a subset of M be a basic set of M and in order that a basic set of M be irreducible.

34. Mr. W. L. Ayres: *On the structure of a plane continuous curve.*

In this paper the limiting arc curve of a continuous curve M at a point is defined, and the following statements are proved: (1) if P is a point of M lying on no simple closed curve of M and α is any arc of M containing P, then P is a limit point of the cut points of M which lie on α; (2) the set of all simple closed curves of M containing a given point P is a continuous curve; (3) if P is a point of M, the limiting arc curve of M at P(a) exists and consists of the single point P, (b) exists and is the set of all simple closed curves of M containing P, (c) does not exist, if and only if (a) P lies on no simple closed

curve of M, (b) P is a non-cut point of M and lies on some simple closed curve of M, (c) P is a cut point of M and lies on a simple closed curve of M; (4) a continuous curve M is acyclic if and only if the limiting arc curve of M at P is P for every point P of M; (5) a continuous curve M is cyclically connected if and only if it contains a point P such that the limiting arc curve of M at P is the set M itself.

35. Mr. N. E. Rutt: *Concerning the cut points of a continuous curve when the arc curve, ab, contains exactly n independent arcs.*

In this paper the author proves that if a and b are two distinct points of a continuous curve E, and if there exists a number n such that in E from a to b there are n independent arcs and only n, then there are in E n points $P_1, P_2, \cdots, P_n$ such that $E - \sum_{i=1}^{n} P_i$ is disconnected and a and b are not in the same connected subset.

36. Mr. D. A. Flanders: *The double elliptic geometry in terms of point, order, and congruence.*

The set of axioms presented in this paper possessess two properties of particular interest. The first may be termed "dimensional categoricity," by which is meant that every proposition in space of a given dimensionality, n, is demonstrable in terms of axioms whose hypotheses predicate the existence of not more than n dimensions. The second may be described by saying that the space is characterized by its one-dimensional axioms; i.e., in order to extend the space beyond the first dimension, the only new axioms required are existence and closure axioms.

37. Dr. D. E. Richmond (National Research Fellow): *Number relations between types of extremals joining a pair of points.*

Let A and B be a pair of points in an extremal-convex region for a calculus of variations problem. An extremal arc joining A to B may be assigned a type number equal to the number of points on the arc conjugate to A. Existence theorems are obtained for certain relations between the numbers of extremals of different types which join A to B, by an appropriate choice of integrand and by application of envelope theory.

38. Professor H. E. Bray: *Is a function of écart fini necessarily of limited variation?*

In Liouville's Journal, (4), vol. 8 (1892), p. 166, Hadamard raised the question as to whether a continuous function of écart fini is necessarily of limited variation. In this note it is proved that the function $x \sin (1/x)$ is of écart fini. The answer is therefore in the negative, for the function in question is also continuous, but is not of limited variation in the interval $0 \leqq x \leqq 2\pi$.

39. Dr. A. J. Maria (National Research Fellow): *Derivatives of functions of plurisegments.*

The author obtains theorems concerning the derivative of a function of plurisegments with respect to an absolutely additive function of point sets. Use is made of the results contained in an article by the author which is to appear in the Annals of Mathematics.

40. Dr. A. J. Maria: *Derivatives of functions of several variables.*

In this paper the author shows that a function of bounded variation $f(x, y)$, defined in the rectangle R $(0 \leqq x \leqq a,\ 0 \leqq y \leqq b)$ can be represented as the sum of two functions $s(x, y)$ and $a(x, y)$. The function $s(x, y)$ is such that $\partial^2 s/\partial x \partial y$ is zero nearly everywhere in R, and $a(x, y)$ is equal to $\int_0^x \int_0^y (\partial^2 f/\partial x \partial y) dx dy$. The theorem rests on the theory of functions of plurisegments developed by the author in a previous article (Transactions of this Society, vol. 28 (1926), pp. 448–471).

41. Mr. P. McVey and Professor M. J. Babb: *On $(n\text{-}k)$ numbers.*

A number n the sum of whose factors is nk is called an $(n\text{-}k)$ number. A technique is developed for even $(n\text{-}k)$ numbers, and it is shown that no odd $(n\text{-}k)$ numbers exist.

42. Professor J. W. Alexander: *Topological invariants of knotted curves in 3-space.*

In this paper, certain topological invariants of knotted curves in 3-space are derived. The principal invariants are in the form of polynomials with integer coefficients.

43. Professor Jules Drach: *Integration by linear systems of equations of the type $r+f(s, t)=0$.*

This paper gives some unexpected results on the general integration by linear systems of equations (1): $r+f(s, t)=0$, where $dz=pdx+qdy$, $dp=rdx+sdy$, $dq=sdx+tdy$. If m_1, m_2 are the roots of $m^2-m(\partial f/\partial s)+\partial f/\partial t=0$, we have for the *characteristics* of Cauchy two integrable combinations: $ds+m_2dt=Ad\alpha$, $ds+m_1dt=Bd\beta$. With the new variables α, β, equation (1) can be replaced by (2): $F(Z)=(\partial^2 Z/\partial\alpha\partial\beta)\div(4(\partial Z/\partial\alpha)\cdot(\partial Z/\partial\beta))=\lambda^2$, and $\lambda(\alpha, \beta)$ is such that r and t are two solutions of (2). Equation (2) may be transformed by $dZ=u^2d\alpha+v^2d\beta$ into a linear system (3): $\partial u/\partial\beta=\lambda v$, $\partial v/\partial\alpha=\lambda u$. For each form of $\lambda(\alpha, \beta)$, we have, for *two arbitrary solutions* of (3), $dr=u_2^2d\alpha+v_2^2d\beta$, $ds=u_1u_2d\alpha+v_1v_2d\beta$, $dt=u_1^2d\alpha+v_1^2d\beta$, and we may then obtain equation (1). With the *general solution* u, v of (3) we have $u_1y+u_2x=u$, $v_1y+v_2x=v$, and $z=\frac{1}{2}Z+\frac{1}{2}(rx^2+2sxy+ty^2)-(xP+yQ)$, where $dP=uu_2d\alpha+vv_2d\beta$, $dQ=uu_1d\alpha+vv_1d\beta$, $dZ=u^2d\alpha+v^2d\beta$; these equations give x, y, z as functions of α, β. By means of (2) and a transformation due to Goursat, we may obtain, in a regular way, all the equations (1) that may be integrated by quadratures only.

R. G. D. Richardson,
Secretary.

A CLASS OF TRANSCENDENTAL NUMBERS

BY S. MANDELBROJT

The purpose of this short note is to determine a class of transcendental numbers by considerations relating to series of powers. I shall give here a theorem quite different from those that I have given previously.* I shall prove the following theorem.

THEOREM I. *Let* $\sum a_n x^n = F(x)$ *be a power series having singularities only within the circle with center at the point* 1 *and radius* $1/(K-1)$, K *an integer, there being at least one essential singularity within the circle, not at the center.*

If there exists a number ζ *and an integer* $K' < \zeta$ *such that the quantities*

$$K'^n\left(a_n + \frac{1}{\zeta^{n+1}}\right) = N_n \qquad (n = 1, 2, \cdots)$$

are integers, then ζ *is a transcendental number.*

I have given elsewhere the following theorem.†

THEOREM II. *If the series* $\sum b_n x^n = f(x)$ *represents a function having singularities only within the circle with center at* 1 *and radius* $1/(K_1-1)$, *where* K_1 *is an integer, and if the quantities*

$$b_n K_1^n = N_n' \tag{1}$$

are integers, then we have

$$f(x) = \frac{P(x)}{(1-x)^h},$$

in which h *is an integer and* $P(x)$ *a polynomial.*

From this theorem, I then proved the following theorem.

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* Comptes Rendus, Feb. 1, 1926; Journal de Mathématiques, 1926.

† Journal de Mathématiques, 1926.

THEOREM III. *If a series* $\sum c_n x^n = \phi(x)$ *has at least one singular point distinct from the point* 1, *and if an integer* K_2 *exists so that the quantities*

$$K_2^n c_n = N_n^{(2)}, \qquad (n = 1, 2, \cdots)$$

are also integers, then, if we denote by x_0 *the argument of the singularity of* $\phi(x)$ *which is farthest from the point* 1, *we shall have*

$$|x_0 - 1| > \frac{1}{K - 1}. \tag{2}$$

In these theorems, the point at infinity was counted with the other points; that is, in Theorem II, we assumed the point at infinity to be a regular point for $\sum b_n x^n$. Likewise in Theorem III, if the point at infinity happens to be a singular point, the inequality (2) is devoid of meaning. But we can now generalize Theorem II by adding that if the point at infinity is a pole for $f(x)$, and if the other singularities be within the circle above mentioned, we shall still have

$$f(x) = \frac{P(x)}{(1 - x)^h},$$

if the property (1) is satisfied.

The proof in this case will be almost the same as for II*. We will only remark that Hurwitz's theorem can be generalized to series of the form

$$f(x) = \sum_0^\infty \frac{\alpha_n}{x^{n-m}}, \quad \phi(x) = \sum_0^\infty \frac{\beta_n}{x^{n-m}},$$

viz., that the series

$$\frac{A_m}{x^m} + \cdots + \frac{A}{x} + A_0 + \sum \frac{\alpha_{n+m}\beta_{m+1} - C_n' \alpha_{n+m-1}\beta_{m+1} + \cdots + (-1)^n \beta_{m+n}\alpha_{n+1}}{x^n}$$

* See Journal de Mathématiques, 1926.

has as singularities, besides the point at infinity, only points of the form $\alpha+\beta$, where α is a singular point of $f(x)$ and β a singular point of $\phi(x)$.

A. We can therefore see that if the series $\sum a_n x^n$ is such that there exists an integer K so that the quantities $a_n K^n$ are integers, then if $P(x)$ is a polynomial with integral coefficients, the function

$$P(x)\sum a_n x^n = \sum b_n x^n$$

is such that there exists an integer $K_1 \leqq K$, so that the quantities $K_1^n b_n$ are integers. This results immediately from the form of the coefficients b_n:

$$b_n = a_n A_0 + a_{n-1}A_1 + \cdots + a_{n-p}A_p,$$

where

$$P(x) = A_0 + A_1 x + \cdots + A_p x^p.$$

We write then, returning to the theorem to be proved,

$$\sum c_n x^n = \sum\left(a_n + \frac{1}{\zeta^n}\right)x^n = F(x) + \frac{1}{\zeta - x}.$$

If we suppose that ζ is an algebraic number, there will be a polynomial with integral coefficients such that $P(x)/(\zeta - x)$ will be regular in the entire plane except for the point at infinity. Hence the function $\theta(x) = P(x) \cdot F(x)$ has singularities only within the circle with center at the point 1 and radius $1/(K'-1)$, except for a pole at infinity.

This results from the remark A, that there is an integer $K_1 \leqq K'$, such that the quantities $c_n K_1^n$ are integers, and also from the fact that the singularities of $\theta(x)$, except for the point at infinity, lie within the circle with center at 1, and radius $1/(K_1-1)$. Accordingly, by Theorem II, as generalized above, it follows that $\theta(x)$ must be of the form $P(x)/(1-x)^h$. But this is impossible, since we assumed $\theta(x)$ to have a singularity at some point other than the center of the circle. Hence ζ must be a transcendental number.

THE RICE INSTITUTE

CAUCHY'S CYCLOTOMIC FUNCTION AND FUNCTIONAL POWERS*

BY E. T. BELL

1. *Introduction.* Let $\phi(n)$ as usual denote the totient of n. Cauchy's function is the polynomial $F_n(x)$ of degree $\phi(n)$, whose zeros are the primitive nth roots of unity. If $a, b, c, e, \cdots$ are the distinct prime divisors (>1) of n,

$$F_n(x) = N_n(x)/D_n(x), \tag{1}$$

where

$$N_n(x) \equiv (x^n - 1)\prod(x^{n/ab} - 1)\prod(x^{n/abce} - 1)\cdots,$$
$$D_n(x) \equiv \prod(x^{n/a} - 1)\prod(x^{n/abc} - 1)\cdots.$$

An anonymous writer† in L'Intermédiaire des Mathématiciens (vol. 24 (1917), pp. 5–6), stated that $\prod F_d(1) = n$, where d ranges over all divisors of n, also that the value of $F_n(-1)$ is 2 if n is a power of 2, $F_p(1)$ if n is double an odd prime p, but is 1 in all remaining cases. The second part seems to be incorrect. For example, $F_2(x) = x+1$, $F_{18}(x) = x^6 - x^3 + 1$, $F_{50}(x) = x^{20} - x^{15} + x^{10} - x^5 + 1$, and hence for $n = 2, 18, 50$ the respective values of $F_n(-1)$ are 0, 3, 5, not 2, 1, 1 as demanded by the assertion. To obtain in §3 the correct values of $F_n(-1)$ it is convenient to prove in §2 the first result stated, which in turn depends upon a theorem of Trudi, which we shall also prove in §2 as several details of the proof are useful. Finally, in §§4, 5 we connect $F_n(x)$ in a new way with other numerical functions, to illustrate what are here called *functional powers*.

We shall need $G_n(x)$ defined by

$$G_n(x) = M_n(x)/R_n(x), \tag{2}$$

* Presented to the Society, San Francisco Section, June 18, 1927.

† Quoted from the *Report on Algebraic Numbers*, Bulletin of the National Research Council, vol. 5, Part 3, No. 28 (1923), p. 32, where it is also stated that no reply is given in L'Intermédiaire for 1917–19. References to other writers cited here will be found on pp. 31, 32 of the *Report*.

where

$$M_n(x) \equiv (x^n + 1)\prod(x^{n/ab} + 1)\cdots,$$
$$R_n(x) \equiv \prod(x^{n/a} + 1)\prod(x^{n/abc} + 1)\cdots.$$

Hence if $n = 2^\alpha m$, m odd, $\alpha > 0$, we have

$$(3) \qquad F_m(x^{2^\alpha}) = G_m(x^{2^{\alpha-1}})G_m(x^{2^{\alpha-2}})\cdots G_m(x)F_m(x).$$

For the same n, α, m, it follows from (1) on resolving each difference of 2 squares into a product of a sum and difference, and comparing with (1), that

$$(4) \qquad F_n(x) = F_m(x^{2^\alpha})/F_m(x^{2^{\alpha-1}}),$$

which is the special case of Dickson's recursion formula (his prime p here $=2$, *Report*, p. 32) which will be required presently.

2. *Proof that* $\prod F_d(n) = 1$. Let $\mu(n)$ be Möbius' function; $\mu(1) = 1$, $\mu(n) = 0$ if n is divisible by a square > 1, and in the contrary case $\mu(n) = 1$ or -1 according as the number of prime divisors of n is even or odd. Then we may write from (1),

$$(5) \qquad F_n(x) = \prod(x^d - 1)^{\mu(\delta)},$$

where $\prod$ extends to all pairs (d, δ) of conjugate divisors > 0 of n, so that $d\delta = n$. If n is divisible by precisely t distinct primes, $\mu(\delta)$ takes the value 1 for exactly

$$\binom{t}{0} + \binom{t}{2} + \binom{t}{4} + \binom{t}{6} + \cdots = 2^{t-1}$$

values of δ, and it takes the value -1 for precisely

$$\binom{t}{1} + \binom{t}{3} + \binom{t}{5} + \cdots = 2^{t-1}$$

values of δ. Hence 2^{t-1} factors $x - 1$ may be cancelled in $F_n(x)$ as written in (1) or (5), after which no factor in the numerator or denominator vanishes when $x = 1$. Since

$$(x^k - 1)/(x - 1) = x^{k-1} + x^{k-2} + \cdots + x + 1$$

has the value k when $x=1$, we find, after cancellation as indicated,

$$F_n(1) = \frac{n\prod(n/ab)\cdots}{\prod(n/a)\prod(n/abc)\cdots},$$

and hence, by the same argument as before, we may cancel all n's and get

$$F_n(1) = \frac{\prod(a)\prod(abc)\cdots}{\prod(ab)\prod(abce)\cdots}.$$

Consider all the products j at a time of the t distinct primes $a, b, c, e, \cdots$ and let their product be P. Among these products there are precisely $\binom{t-1}{j-1}$ that contain a given prime, say a; for with a may be taken all products $j-1$ at a time of the remaining $t-1$ primes. By convention $\binom{s}{0}=1$, $s\geqq 0$. Thus each prime factor in all the $\binom{t}{j}$ products occurs precisely $\binom{t-1}{j-1}$ times, and therefore the product of all the $\binom{t}{j}$ products is P^k, $k\equiv\binom{t-1}{j-1}$. In the numerator of $F_n(1)$ the possible values of j are $1, 3, 5, \cdots$; in the denominator, provided $t>1$, they are $2, 4, 6, \cdots$. Hence, if $t>1$, $F_n(1)=P^u/P^v$, where $u=v=2^{t-2}$. If $t=1$, we have $n=p^\alpha$, p prime, and from (1),

$$F_n(x) = (x^{p^\alpha}-1)/(x^{p^{\alpha-1}}-1),$$

which after cancellation of $x-1$ as before, gives $F_n(1)=p^\alpha/p^{\alpha-1}=p$. Finally then we have Trudi's result that

$$F_n(1) = 1, \quad \text{or} \quad p, \tag{6}$$

according as n is not or is a power p^α of a prime p.

Let $n=a^\alpha b^\beta\cdots c^\gamma$ be the resolution of n into its prime factors. By (6), among all the divisors d of n only

$$a^A, b^B, \cdots, c^C \quad (0<A\leqq\alpha, 0<B\leqq\beta, \cdots, 0<C\leqq\gamma) \tag{7}$$

give values of d for which $F_d(1)=1$, while for the typical case $d=a^A$, $F_d(1)=a$. Hence from (6), (7),

$$\prod F_d(1) = a^\alpha b^\beta\cdots c^\gamma = n,$$

where $\prod$ refers to all divisors d of n.

3. *Value of* $F_n(-1)$. Referring to (1), and noting that there are the same number of binomial factors in $N_n(x)$, $D_n(x)$, also that $x^m-1=-2$ when m is odd and $x=-1$, we see that

$$F_m(-1) = 1, \qquad (m \text{ odd}). \tag{8}$$

Henceforth, let $n=2^\alpha m$, m odd, >1. If in (2) we cancel all factors $x+1$ and set $x=-1$ in the result, we get (by §2),

$$G_m(-1) = F_m(1). \tag{9}$$

Again, if $\alpha=1$, it follows from (4), (3) that

$$F_{2m}(x) = G_m(x), \tag{10}$$

and hence by (9)

$$F_{2m}(-1) = F_m(1), \tag{11}$$

the value of which is known from (6). If $\alpha>1$, we see from (4), (3) that

$$F_n(x) = G_m(x^{2^{\alpha-1}}), \tag{12}$$

which by (10) holds also when $\alpha=1$. Now if $\beta>0$, the value of each binomial factor in $G_n(x^2)^\beta$ for $x=-1$ is 2, and since (see(2)) $M_n(x)$ and $R_n(x)$, by §2, have equal numbers of such factors, it follows from (12) that $F_n(-1)=1$ when $\alpha>1$. There remains only the case $F_s(-1)$, $s=2^\alpha$, $\alpha>0$. We have $F_2(x)=x+1$, $F_2(-1)=0$; while if $\alpha>1$,

$$F_s(x) = (x^{2^\alpha}-1)/(x^{2^{\alpha-1}}-1) = x^{2^{\alpha-1}}+1,$$

and hence $F_s(-1)=2$ for this s.

Hence if $n=2^\alpha\mu$, where $\mu\geqq 1$ is odd, the value of $F_n(-1)$ is 0 if $\alpha=\mu=1$; it is 2 if $\alpha>1$, $\mu=1$; it is p if $\alpha=1$ and $\mu=p^\beta$, $\beta>0$, $p>1$ prime (odd or even); $F_1(-1)=-2$; in all other cases $F_n(-1)=1$.

4. *Functional Powers.* It will be necessary to recall a few definitions and theorems from previous papers. A second theorem of Trudi's ((14) below) is a simple example of

what may be called the extraction of functional roots, for reasons which we now explain.

We call $f(x)$ a *numerical function* of x if $f(1)\neq 0$ and if $f(x)$, for each integral value >0 of x, takes a single finite value. Let $f_j(x)$ $(j=1, \cdots, k)$ be numerical functions, and let $\sum$ refer to all sets $(d_1, d_2, \cdots, d_k)$ of positive integral solutions of $d_1, d_2, \cdots, d_k=n$ for n fixed. Then, as explained in a previous note,* $\sum f_1(d_1)f_2(d_2)\cdots f_k(d_k)$ is written as a *symbolic product* $f_1 f_2 \cdots f_k$, called simply the *product* of the numerical functions $f_1, f_2, \cdots, f_k$ with the *parameter* n. Two such products are *equal*, $f=g$, only when the sums which they represent are equal for all values of the parameter, which may therefore be suppressed in the notation. The multiplication thus defined is abstractly identical with that in common algebra, being associative, commutative, and distributive, and having the unique unity ϵ, where $\epsilon(1)=1$, $\epsilon(n)=0$, $n>0$. If f is any given numerical function, there exists a unique numerical function f', called the *reciprocal* of f, such that $ff'=\epsilon$, and we may write $f'\equiv\epsilon/f$. The set of all numerical functions is an abelian group under multiplication as just defined. If only functions having the same parameter are added the set is a field, but the consequences of addition are uninteresting.

In a former paper,† we assigned to f^g, where f, g are numerical functions, the meaning

$$f^g \equiv \prod [f(d)]^{g(\delta)},$$

where the product extends to all pairs (d, δ) of conjugate divisors of the arbitrary constant integer $n>0$. Equality of such functional powers, $f^g=h^k$, where f, g, h, k are numerical functions, is defined as equality for all integers $n>0$ of the corresponding products, and as before we may

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* This Bulletin, vol. 27 (1921), pp. 273–275. This method was first developed in University of Washington Publications in Science, No. 1, 1915.

† L'Enseignement Mathématique, 1923, pp. 305–308.

omit reference to the parameter n. It was shown that each of the following equalities implies the other,

$$f^g = h^k, \quad f^{gl} = h^{kl},$$

all the letters denoting numerical functions, and gl, kl being products as defined above. Hence the algebra of functional powers is abstractly identical with that of powers in common algebra. In particular then the following are equivalent relations,

$$f^g = h^k, \quad f^{\epsilon/k} = h^{\epsilon/g},$$

and the second is said to be derived from the first by *extracting the* gkth (functional) *root of both members* (or by raising both members to the $\epsilon/(gk)$th power).

In the last paper cited, and from a more comprehensive point of view in another,* I established a triple isomorphism between the several algebraic varieties (fields, rings, rays, etc.) usually considered in arithmetic and the above products $fg \cdots$ and powers f^g on the one hand, and, on the other, with the algebra of polynomials in two indeterminates in an arbitrary field. This isomorphism refers the derivation of relations between given numerical functions $f, g, h, k, \cdots$, such as $fg = hk$, $f^g = h^k$, etc., to identities between the polynomials described. It is mentioned here because the few relations required presently for purposes of illustration were obtained, without computations, by its means; we may assume them known.

5. *Cauchy's Function as a Functional Power.* Understanding the variable x in (1) we consider $F_n(x)$ as a numerical function of n and write $F_n(x) \equiv F_n \equiv F(n)$, and similarly for $X_n(x) \equiv x^n - 1 \equiv X(n)$. Then (5) is equivalent to

$$F = X^\mu. \tag{13}$$

* Transactions of this Society, vol. 25 (1923), pp. 135–154.

Let $u(n)=1$ for all integers $n>0$. Then, ϵ being the unit defined in §4, $\mu u=\epsilon$. Hence (13) implies

$$F^u = X^\epsilon \equiv X. \tag{14}$$

In full (14) is $\prod F_d(x)=x^n-1$, where $\prod$ refers to all divisors d of n, a theorem due to Trudi.

It will be of interest to write down a few further consequences of raising (13) to functional powers (including the extraction of roots). Let $w(n)=n$ for all integers $n>0$. Then $\phi=\mu w$ (ϕ as in §1), and hence

$$F^w = X^\phi, \tag{15}$$

or in full,

$$\prod [F(d)]^\delta = \prod [X(d)]^{\phi(\delta)},$$

the products extending to all pairs (d, δ) of conjugate divisors of n. Again, if $k(n)=1$ or 0 according as n is or is not the square of an integer >0, and $\lambda(n)=1$ or -1 according as the total (not the number of distinct) prime divisors of n is even or odd, $\mu k=\lambda$. Hence, raising both members of (13) to the kth power we have

$$F^k = X^\lambda. \tag{16}$$

As a last example (the number may be multiplied indefinitely by the methods indicated), let $\nu(n)=$ the number of divisors of n. Then $\mu\nu=u$, and

$$F^\nu = X^u, \tag{17}$$

which in full is

$$\prod [F(d)]^{\nu(\delta)} = \prod (x^d - 1).$$

Similarly, if we write $G_n(x)\equiv G_n\equiv G(n)$, $Y_n(x)\equiv x^n+1 \equiv Y(n)$, it follows from (2) that (F, X) may be replaced by (G, Y) in (13)–(17) and in any relation of a similar kind.

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A POINT SET WHICH HAS NO TRUE QUASI-COMPONENTS, AND WHICH BECOMES CONNECTED UPON THE ADDITION OF A SINGLE POINT*

BY R. L. WILDER

If M is a set of points, and P is a point of M, then the *quasi-component* of M determined by P is the set of points common to all possible sets M_1, where $M = M_1 + M_2$, M_1 contains P, and M_1 and M_2 are mutually separated sets.† If M is a connected set, then the quasi-component of M determined by P is M itself. If M is not connected, then it can be considered as the sum of its quasi-components. If a quasi-component consists of more than one point, it is called a *true quasi-component*. If all the quasi-components of M reduce to single points, that is, if M contains no true quasi-components, then M is totally disconnected (i. e., has no connected subset consisting of more than one point). However, the quasi-components of a totally disconnected set do not necessarily reduce to single points. Sierpinski has given‡ an example of a set of points N and a point P not in N, such that N has no true quasi-components, and such that the set $N+P$, although totally disconnected, contains a true quasi-component consisting of P and a certain point of N.

Knaster and Kuratowski have given§ an example of a totally disconnected set of points S, which becomes connected upon the addition of a certain point a; i. e., $S+a$

* Presented to the Society, February 26, 1927.

† See F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914, p. 248. Two sets are called *mutually separated* if they have no point in common, and if neither contains a limit point of the other.

‡ *Sur les ensembles connexes et non connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 81–95.

§ *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 206–255. See especially pp. 240 ff.

is a connected set. In this example, one notices that the point set S has uncountably many true quasi-components, each of which has a as a limit point.

The properties of the point sets N and S just mentioned suggest the following question: *Does there exist a point set M and a point P not in M, such that M has no true quasi-components but such that $M+P$ is connected?* It is the purpose of the present note to answer this question in the affirmative.

Consider a point set M constructed in the following manner: In the ordinary number plane, denote the points (0, 0), (2,0) and (1, 1) by A, B, and P, respectively. The interval AB of the x-axis is the sum of c (where c is the cardinal number of the continuum) mutually exclusive sets each of which is dense in AB.* Call the class of these sets X. Let $C(X)$ be a one-to-one correspondence between the elements of the class X and the real numbers $\{t\}$, $0 \leqq t \leqq 1$. Consider now the class, K, of all continua contained in triangle PAB and its interior, which have points in common with both AP† and PB, but do not contain P. Since‡ this class has the cardinal number c, there exists a one-to-one correspondence $C(K)$ between its elements and the set of real numbers $\{t\}$, $0 \leqq t \leqq 1$.

If t is any real number such that $0 \leqq t \leqq 1$, let that element of class X which corresponds to t under the correspondence $C(X)$ be denoted by X_t. Similarly, denote that element of K which corresponds to t under the correspondence $C(K)$ by K_t. Then if K_{t_1} is an element of K and X_{t_2} is an

* See Knaster and Kuratowski, loc. cit., p. 252. I wish to acknowledge here my indebtedness to this noteworthy work. It will be noticed that the suggestion for the method of attack which I employ in the present problem is contained in the construction of Knaster and Kuratowski's example (β) (pp. 245 ff.), of a connected punctiform set which is irreducible between two points.

† Hereafter in this paper, if U and V are two points, UV will denote that interval from U to V on the straight line through these two points.

‡ Cf. Knaster and Kuratowski, loc. cit., p. 253.

element of X, these two sets will be said to correspond to one another if and only if $t_1 = t_2$.

If T is any point of the interval AB, whose abscissa is x, denote the interval PT by l_x. For every real number t, $0 \leqq t \leqq 1$, there corresponds a set of points M_t defined as follows: For every interval l_x, where x is the abscissa of a point of X_t, let m_x be the point of minimum ordinate of the set of points common to K_t (the set which corresponds to X_t) and l_x; the set of all points $\{m_x\}$ constitutes the set M_t. For every real number t, $0 \leqq t \leqq 1$, there corresponds a definite set M_t, and no two of these sets have a point in common.

Denote by M' the set of all points p, such that p is contained in some set M_t. Denote by M the set $M' - M_0$.

1. The set of points M is totally disconnected, and, moreover, has no true quasi-components. For let a and b be two points of M. Let l_{x_1} and l_{x_2} be those intervals of the set $\{l_x\}$ that contain a and b, respectively. Since no l_x contains more than one point of M, x_1 and x_2 are unequal, say $x_1 < x_2$. As X_0 is dense in AB, there exists a point of X_0 whose abscissa, x_3, satisfies the relation $x_1 < x_3 < x_2$. The interval l_{x_3} contains no point of M, by definition. Let the set of all points of M which lie on intervals l_x for which $x < x_3$ be denoted by M_1, and the set of all points of M which lie on lines l_x for which $x > x_3$ be denoted by M_2. Clearly M_1 and M_2 are mutually separated sets containing a and b, respectively. Hence M is totally disconnected and has no true quasi-components.

2. The set of points $M+P$ is connected. For suppose it is not connected. Then

$$M+P = M_1+M_2,$$

where M_1 contains P and M_1 and M_2 are mutually separated non-vacuous sets. Let Q be a point of M_2. By a theorem due to Knaster and Kuratowski,* there exists a continuum

* Loc. cit., Theorem 37.

H^* which separates the plane between P and Q, and contains no point of $M+P$.

Let that interval of the set $\{l_x\}$ which contains Q be denoted by PF, where F is the intersection of this interval with AB. Then the arc $PAFQ=PA+AF+FQ$, together with PQ, forms a simple closed curve J. Denote the interior of J by R. By Lemma 2 of my paper *A connected and regular point set which has no subcontinuum*,† there exists a subcontinuum H' of H which is a subset of $J+R$ and has at least one point in common with each of the arcs $PAFQ$ and PQ. Since neither P nor Q is a point of H', it is clear that H' cannot lie wholly on PF. Let g be the smallest number such that l_g contains a point of H'. Clearly $0 \leqq g < f$, where f is the abscissa of F. Denote the point of AB whose abscissa is g by G. Let D and E be points of H' on PG and PF, respectively. Then $AD+H'+EB$ forms a continuum of class K, say K_{t_1}. If $t_1=0$, let B' be an interior point of the interval PB, and let K_{t_1} be the continuum $AD+H'+EB'$. Then $t_1 \neq 0$, and the set M_{t_1} is a subset of $M+P$. As $g<f$, and X_{t_1} is dense on AB, there exists a number e such that $g<e<f$ and that point of AB whose abscissa is e belongs to X_{t_1}. Then that point of M_{t_1} determined by the intersection of l_e and K_{t_1} is necessarily a point of H'. That is, H' and M have points in common. This is impossible, since H' is a subset of H and the latter set has no point in common with M. Hence the supposition that $M+P$ is not connected leads to a contradiction.

Thus by 1. and 2. the set of points $M+P$ is an example furnishing an affirmative answer to the question raised at the beginning of this note.

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* A continuum is a closed and connected set which contains more than one point. A continuum H is said to separate the plane between two points P and Q if P and Q do not lie in the same domain complementary to H, and neither lies in H.

† See Transactions of this Society, vol. 29(1927), pp. 332–340.

I shall close with one further observation as to the properties of the set M. Mazurkiewicz calls* a set M quasi-connected if for every point m of M there corresponds a positive number λ such that there does not exist any division of M into two mutually separated sets M_1 and M_2 such that M_1 contains m and the diameter of M_1 is less than λ. He gives† an example of a quasi-connected set which contains no true quasi-components. That the set M constructed above is another example of such a set is easily shown; indeed, the value of λ may be taken *uniformly* equal to unity for all points of M.

THE UNIVERSITY OF MICHIGAN

A SIMPLE METHOD FOR NORMALIZING TCHEBYCHEFF POLYNOMIALS AND EVALUATING THE ELEMENTS OF THE ALLIED CONTINUED FRACTIONS‡

BY J. A. SHOHAT [J. CHOKHATE]

1. *Introduction.* Consider a system

$$P_n(x), \qquad (n = 0,1,2,\cdots), \tag{1}$$

of orthogonal, but not normal, Tchebycheff polynomials corresponding to a given (finite or infinite) interval (a, b) with the characteristic function $p(x)$. The corresponding *normalized* system of polynomials will be denoted by

$$\phi_n(p\,;\,x) \equiv \phi_n(x) = a_n(p)\left[x^n - S_n(p)x^{n-1} + \cdots\right], \qquad (n = 0,1,\cdots,a_n > 0). \tag{2}$$

We have, then,

$$\int_a^b p(x)\phi_m(x)\phi_n(x)dx = \begin{cases} 0, m \neq n, \\ 1, m = n. \end{cases} \tag{3}$$

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* *Sur les ensembles quasi-connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 201–205.

† Loc. cit.

‡ Presented to the Society, April 16, 1927.

The normalization of the system (1), i. e., the evaluation of

$$\int_a^b p(x)P_n^2(x)dx,$$

in general requires considerable computation, even in the simplest cases (e. g. Legendre, Laguerre, Hermite polynomials), *where we ordinarily use the known explicit expression of the polynomials involved.* Very often we have to use other characteristic properties (differential equations, difference equations, generating functions, etc.) if we wish to know more of the polynomials concerned, for example, if we wish to evaluate S_n, the sum of the roots of $\phi_n(x)$, or $\phi_n(a)$, or the elements of the "corresponding" and "associated" continued fractions

$$(4)\qquad \int_a^b \frac{p(y)}{x-y}dy \sim \cfrac{b_1(p)}{x - \cfrac{b_2(p)}{1 - \cfrac{b_3(p)}{x - \cdots}}} = \cfrac{\lambda_1(p)}{x - c_1(p) - \cfrac{\lambda_2(p)}{x - c_2(p) - \cdots}},$$

of which the first exists in case $a \geqq 0$.

The object of this article is to develop an extremely simple and elementary method,* which in cases of polynomials of Legendre etc., gives directly, with practically no computation, the *exact value* of a_n, S_n, $\phi_n\binom{a}{b}$, λ_n, $\cdots$, and in more general cases the *asymptotic expressions* (for $n\to\infty$) of these quantities. *The method does not use any property of our polynomials, except the relations* (3), *and these only in the simplest manner.*

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* A method similar with regard to the underlying idea has been developed for a different purpose by W. Stekloff in his paper *Sur une application de la théorie de fermeture. . . .*, Mémoires de l'Académie des Sciences, St. Petersburg, vol. 33 (1914), pp. 1–59.

2. *Fundamental Formulas.* The following formula follows at once from (3):

$$(5) \qquad \int_a^b p(x)\phi_n(x) \sum_{i=0}^{n+1} g_i x^i dx = \frac{g_{n+1}S_{n+1}(p) + g_n}{a_n(p)},$$

where the g_i are arbitrary.

In connection with the continued fractions (4), we shall use also the following formulas:

$$(6) \qquad \lambda_{n+1} = \frac{a_n^2{}_{-1}(p)}{a_n^2(p)},$$

$$(7) \qquad c_{n+1}(p) = S_{n+1}(p) - S_n(p) = \int_a^b p(x)x\phi_n^2(x)dx,^*$$

$$(8) \qquad b_{2n}(p) = \frac{a_n^2{}_{-1}(p)}{a_n^2(px)}, \quad b_{2n+1}(p) = \frac{a_n^2{}_{-1}(px)}{a_n^2(p)}.$$

Assume now the existence of the derivative $p'(x)$ *in* (a, b).† In the fundamental formula

$$F(b) - F(a) \equiv F(x)\Big]_a^b = \int_a^b F'(x)dx,$$

replace $F(x)$ by

$$p(x)\phi_n^2(x)x^\epsilon, \qquad p(x)\phi_n(x)\phi_{n+1}(x)x^\epsilon,$$

with $\epsilon = 0, 1$; and we get immediately, *using* (5) *only*,

$$(9) \qquad p(x)\phi_n^2(x)\Big]_a^b = \int_a^b p'(x)\phi_n^2(x)dx,$$

$$(10) \qquad p(x)x\phi_n^2(x)\Big]_a^b = \int_a^b p'(x)x\phi_n^2(x)dx + 2n + 1,$$

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* J. Chokhate (J. A. Shohat), *Sur le développement de l'intégrale* $\int_a^b [p(y)/(x-y)]\,dy\ldots$, Rendiconti del Circolo Matematico di Palermo, vol. 47 (1923), pp. 25–46; pp. 29–30.

† This assumption is made here for the sake of simplicity and can be greatly modified.

$$(11)\qquad p(x)\phi_n(x)\phi_{n+1}(x)\Big]_a^b = \int_a^b p'(x)\phi_n(x)\phi_{n+1}(x)dx + (n+1)\frac{a_{n+1}(p)}{a_n(p)},$$

$$(12)\qquad p(x)x\phi_n(x)\phi_{n+1}(x)\Big]_a^b = \int_a^b p'(x)x\phi_n(x)\phi_{n+1}(x)dx + \frac{S_{n+1}(p)a_{n+1}(p)}{a_n(p)}.$$

These formulas give directly the values of $a_n(p)$, $S_n(p)$, $\phi_n(\begin{smallmatrix}a\\b\end{smallmatrix})$, $\lambda_n(p)$, $\cdots$, if the nature of $p'(x)$ is known.

3. *Illustrations.* In order to illustrate the application of formulas (9)–(12) we consider some special important cases.

A. *Legendre polynomials*: $(a, b) = (-1, 1)$, $p(x) \equiv 1$.

$$(13)\qquad \phi_n^2(1) = \phi_n^2(-1) = \frac{2n+1}{2}, \quad [\text{from (9),(10)}];$$

$$(14)\qquad \frac{a_{n+1}}{a_n} = \left(\frac{(2n+1)(2n+3)}{(n+1)^2}\right)^{1/2}, \quad [\text{from (11)}];$$

$$(15)\qquad S_{n+1} = 0, \quad [\text{from (12)}].$$

Replacing in (14) n by $0, 1, 2, \cdots, n-1$, and multiplying, we find

$$(16)\qquad a_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\left(\frac{2n+1}{2}\right)^{1/2},$$

since we have, in general,

$$\frac{1}{a_0^2(p)} = \int_a^b p(x)dx;$$

and therefore, for the ordinary Legendre polynomial $P_n(x)$, by (13) or (16),

$$\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}.$$

B. *Generalized polynomials of Laguerre*: $(a, b) = (0, \infty)$; $p(x) = e^{-x}x^{\alpha-1}(\alpha > 0)$; $p'(x) = -p(x) + [(\alpha-1)p(x)/x]$. We get similarly

$$c_{n+1} = 2n + \alpha, \quad S_{n+1} = (n+1)(n+\alpha), \tag{17}$$

[from (10), (7)];

$$\frac{a_{n+1}}{a_n} = \frac{1}{[(n+1)(n+\alpha)]^{1/2}}; \tag{18}$$

$$a_n = \frac{1}{[\Gamma(n+1)\Gamma(n+\alpha)]^{1/2}}, \quad [\text{from (12)}].$$

The formula (9) gives for Laguerre polynomials ($\alpha = 1$)

$$\phi_n^2(e^{-x}; 0) = 1. \tag{19}$$

C. *Polynomials of Hermite*: $(a, b) = (-\infty, \infty)$; $p(x) = e^{-x^2}$.

$$c_{n+1} = 0, \qquad S_{n+1} = 0 \ [\text{from (9) or (12)}], \tag{20}$$

$$\frac{a_{n+1}}{a_n} = \left(\frac{2}{n+1}\right)^{1/2},$$

$$a_n = \left(\frac{2^n}{\Gamma(n+1)}\right)^{1/2} = \left(\frac{2^n}{\pi^{1/2}\Gamma(n+1)}\right)^{1/2}, \quad (\text{from (11)}]. \tag{21}$$

The above formulas, combined with (6)–(8), give directly

$$\int_{-1}^{1} \frac{p(y)}{x-y}dy = \cfrac{\lambda_1}{x - \cfrac{\lambda_2}{x - \ddots}}, \tag{22}$$

with

$$\lambda_{n+1} = \frac{n^2}{(2n-1)(2n+1)},$$

$$\int_0^\infty \frac{e^{-y}y^{\alpha-1}dy}{x-y} \ (\alpha > 0) = \cfrac{b_1}{x - \cfrac{b_2}{1 - \cfrac{b_3}{x - \ddots}}} \tag{23}$$

$$= \cfrac{\lambda_1}{x - c_1 - \cfrac{\lambda_2}{x - c_2 - \ddots}},$$

with $b_{2n}=n+\alpha-1$, $b_{2n+1}=n$, $\lambda_{n+1}=n(n+\alpha-1)$, $c_{n+1}=2n+\alpha$;

$$(24)\qquad \int_{-\infty}^{\infty}\frac{e^{-y^2}dy}{x-y}=\cfrac{\lambda_1}{x-c_1-\cfrac{\lambda_2}{x-c_2-\cdots}},$$

with $\lambda_{n+1}=n/2$.

4. *Generalizations.* In more general cases we replace in the formulas (9)–(12) above, the expression $p(x)\phi_n^2(x), \cdots$ by $p(x)f(x)\phi_n^2(x), \cdots$, with a properly chosen $f(x)$. Thus we get, in a very elementary way (even with $f(x)\equiv 1$), asymptotic expressions (for $n\to\infty$) of $\phi_n\binom{a}{b}$, a_n, $\lambda_n, \cdots$, which lead to new important results. This will be developed elsewhere. We shall give one illustration only.

Let us assume that (a, b) is finite, and that $p(x)=(x-q)^{\alpha-1}q(x)$ $(\alpha>0)$ with $q'(x)/q(x)$ bounded on (a, b), and $q(b)\neq 0$.* Then a very simple combination of (9), (10)† gives

$$\phi_n(b)=\left(\frac{2n+1}{(b-a)^{\alpha}q(b)}\right)^{1/2}\left[1+o\left(\frac{1}{n}\right)\right].$$

Furthermore,

$$\phi_n^{(k)}(b)=H_k n^{2k+1/2}(1+o(1)),\qquad (k \text{ finite}, =1,2,\cdots),$$

where $H_k(>0)$ does not depend on n.

Legendre polynomials are included here as a very special case.

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* See the second footnote on page 429.

† Take (10) $-a\times$(9).

ON HILBERT'S THIRTEENTH PARIS PROBLEM*

BY H. W. RAUDENBUSH, JR.

At the Paris Congress in 1900, Hilbert† presented for proof the proposition that the function f of the three variables x, y, and z satisfying the equation

$$f^7 + xf^3 + yf^2 + zf + 1 = 0 \tag{1}$$

cannot be represented by the use of a finite number of continuous functions of not more than two arguments. In this note a small part of this problem is considered. We shall prove that the function f cannot have the form $F[\alpha(x, y), \beta(y, z)]$, where $F(\alpha, \beta)$, $\alpha(x, y)$ and $\beta(y, z)$ are analytic functions.

Before proceeding to the proof it is necessary to notice certain properties of the partial derivatives f_x, f_y, and f_z. They satisfy the identities

$$Uf_x \equiv f^3, \qquad Uf_y \equiv f^2, \qquad Uf_z \equiv f, \tag{2}$$

where $U \equiv -(7f^6+3xf^2+2yf+z) \not\equiv 0$. For finite values of x, y, and z, f is finite and does not vanish. Hence U is finite and therefore the first partial derivatives cannot vanish.

In the proof we assume that

$$f(x,y,z) \equiv F[\alpha(x,y),\beta(y,z)], \tag{3}$$

where $F(\alpha,\beta)$, $\alpha(x,y)$, and $\beta(y,z)$ are analytic functions. Since $f_x \neq 0$ and $f_z \neq 0$, $\alpha_x \neq 0$ and $\beta_z \neq 0$ for finite values of x, y, and z. The Jacobian condition for functional dependence

$$\begin{vmatrix} f_x & f_y & f_z \\ \alpha_x & \alpha_y & 0 \\ 0 & \beta_y & \beta_z \end{vmatrix} \equiv 0$$

* Presented to the Society, February 26, 1927.

† This Bulletin, vol. 8 (1901–2), p. 461.

can then be written in the form

$$A(x,y)f_x + f_y + C(y,z)f_z \equiv 0,$$

where $A \equiv -\alpha_y/\alpha_x$ and $C \equiv -\beta_y/\beta_z$. Multiplying by U, making use of (2), and subsequently dividing by f, we get

$$A(x,y)f^2 + f + C(y,z) \equiv 0. \tag{4}$$

It is now easy to show that A is linear in x. Differentiating (4) with respect to x and separately with respect to z we find by the use of (2) that $A_x \equiv C_z$. Since C does not contain x, C_z does not contain x and hence A_x is a function of y alone.

We shall now show that A does not contain x at all. The eliminant with respect to f between (1) and (4) is an identity in x that has for its term in the highest power of x the term contributed by the expansion of A^7. This term must vanish identically and hence the coefficient of x in A must vanish identically. But if A does not contain x we have by (4) $f_x \equiv 0$. This is impossible and the assumption that f satisfies (3) leads to a contradiction.

Similarly it can be shown that f cannot have either the form $F[\alpha(x,y), \beta(x,z)]$ or the form $F[\alpha(x,z), \beta(y,z)]$; all functions being assumed analytic.

COLUMBIA UNIVERSITY

GENERALIZATION OF THE BELTRAMI EQUATIONS TO CURVED n-SPACE*

BY G. E. RAYNOR

Let S be a curved n-space in which the linear element is given by the equation

$$ds^2 = \sum E_{ij}dx_i dx_j, \qquad (i,j = 1,2, \cdots ,n). \tag{1}$$

Without loss of generality, we may suppose

$$E_{ij} = E_{ji}. \tag{2}$$

Also let $U^{(i)}$, $(i=1, 2, \cdots , n)$, be a set of n independent functions of $x_1, x_2, \cdots , x_n$.

We shall say that the $U^{(i)}$ are isothermal in S provided they satisfy the relation

$$\sum (dU^{(i)})^2 = \lambda \sum E_{ij}dx_i dx_j, \tag{3}$$

where λ is a function of the x_i only.

If in (3) we express the $dU^{(i)}$ in terms of the differentials of $x_1, x_2, \cdots , x_n$ it follows from the independence of these differentials that the coefficients of corresponding terms on the two sides of the equation are equal and we obtain the $n(n+1)/2$ equations

$$\sum_{k=1}^{n} U_{x_i}^{(k)} U_{x_j}^{(k)} = \lambda E_{ij}. \tag{4}$$

Let D be the discriminant of the quadratic differential form in (1) and suppose it to be written as a determinant

$$|E_{ij}|, \tag{5}$$

in which E_{ij} is the element in the ith row and jth column. If each element of (4) be multiplied by λ and if for λE_{ij} be substituted its equal given by the left side of (4), we

* Presented to the Society, September 9, 1926.

readily see that the resulting determinant is the square of the Jacobian

$$J = \frac{\partial(U^{(1)}, U^{(2)}, \cdots, U^{(n)})}{\partial(x_1, x_2, \cdots, x_n)}.$$

Hence we have

$$J = \lambda^{n/2} D^{1/2}. \tag{6}$$

In all that follows we shall suppose J to be written as a determinant in which $U^{(i)}_{x_j}$, the derivative of $U^{(i)}$ with respect to x_j, is the element of J in the ith row and jth column.

Multiply both sides of (6) by $U^{(i)}_{x_j}$, on the left letting the factor go into the ith row of J. Now if we multiply each row of J, other than the ith, by its jth element and add the corresponding products to the elements in the ith row, (6) becomes by means of (2) and (4)

$$\lambda J_{ij} = \lambda^{n/2} D^{1/2} U^{(i)}_{x_j}, \tag{7}$$

where J_{ij} is the determinant obtained from J by replacing its ith row by the jth row of D. From (6) and (7) we obtain

$$U^{(i)}_{x_j} = \frac{J_{ij}}{J^{(n-2)/n} D^{1/n}}, \qquad (i, j = 1, 2, \cdots, n). \tag{8}$$

This last set of n^2 equations are of the form obtained, by a different method, by Hedrick and Ingold* for curved 3-space. Their equation in our notation may be written

$$U^{(i)}_{x_j} = P J_{ij} \tag{9}$$

where P is an unspecified factor of proportionality. However, it may be seen as follows that equations (9) are equivalent to (8). Replace each element of J by its expression given by (9) and we obtain

$$J = P^n |J_{ij}|. \tag{10}$$

* Transactions of this Society, vol. 27 (1925), p. 561.

Now each element J_{ij} of the determinant $|J_{ij}|$ is a determinant which has one row of D in it. If we expand this determinant by cofactors with respect to the elements of this row we find that $|J_{ij}|$ breaks up into the product

$$DA_J$$

where A_J is the adjoint of J. Hence (10) becomes

$$J = P^n D J^{n-1},$$

and

$$P = \frac{1}{J^{(n-2)/n} D^{1/n}}.$$

If $n=2$, equations (8) become

$$U^{(i)}_{x_j} = \frac{J_{ij}}{D^{1/2}}, \qquad (i,j = 1,2), \tag{11}$$

which are precisely the well known Beltrami equations of differential geometry. These equations have the property that the derivatives of either one of the $U^{(i)}$ are expressed in terms of the E_{ij} and the derivatives of the other U only. This is not the case for $n>3$ in (8), since J on the right contains $U^{(i)}_{x_j}$ which appears on the left. To obtain a more desirable form we proceed as follows.

From the sub-set of the equations in (8) obtained by keeping i fixed, we get by taking ratios,

$$U^{(i)}_{x_k} = \frac{J_{ik}}{J_{ij}} U^{(i)}_{x_j}, \qquad (k \neq j). \tag{12}$$

After substituting these expressions for $U^{(i)}_{x_k}$ in the ith row of J on the right side of (8), $U^{(i)}_{x_j}$ can be removed as a factor from this row and solving the resulting equation for $U^{(i)}_{x_j}$ we obtain

$$U^{(i)}_{x_j} = \frac{J_{ij}}{M_i^{(n-2)/(2n-2)} D^{1/(2n-2)}}, \tag{13}$$

where by M_i we mean the determinant obtained from J by replacing its ith row by the ith row of the determinant $|J_{ij}|$.

Equations (13) contain none of the derivatives of $U^{(i)}$ on the right and hence we have all the partial derivatives of $U^{(i)}$ expressed in terms of the E_{ij} and $U^{(k)}_{x_j}$, $(k \neq i)$. Hence (13) is the desired generalization of the Beltrami equations to curved n-space.

It can readily be shown, conversely, that if a set of functions satisfy (8) or (13) they also satisfy (3).*

We now proceed to find the differential equation satisfied by each of the $U^{(i)}$ singly. Let C_{ij} denote the cofactor of the element in the ith row and jth column of J. Then in (8), if we expand the J_{ij} by cofactors with respect to the row of E's contained in them, (8) can be written

$$\sum_{j=1}^{n} E_{kj}C_{ij} = J^{(n-2)/n}D^{1/n}U^{(i)}_{x_k}, \qquad (i,k = 1,2,\cdots,n). \tag{14}$$

If out of the above set of n^2 equations we solve the set of n, obtained by holding i fixed, for the C_{ij} we get

$$C_{ij} = \frac{N_{ij}J^{(n-2)/n}}{D^{(n-1)/n}}, \tag{15}$$

where N_{ij} is the determinant obtained from D by substituting the ith row of J for the jth row of D. If now J in the last equation be expanded with respect to cofactors of the ith row, (15) becomes

$$C_{ij} = \frac{N_{ij}\left\{\sum_{k=1}^{n} U^{(i)}_{x_k}C_{ik}\right\}^{(n-2)/n}}{D^{(n-1)/n}}, \quad (j = 1,2,\cdots,n). \tag{16}$$

From (16) we get, by taking ratios,

$$C_{ik} = \frac{N_{ik}}{N_{ij}}C_{ij}, \qquad (k \neq j). \tag{17}$$

Substituting these values for C_{ik}, in the right of (16), and, solving the resulting equation for C_{ij}, we have

* See Hedrick and Ingold, loc. cit., p. 562.

$$C_{ij} = \frac{N_{ij}\left\{ \sum_{k=1}^{n} U_{xk}^{(i)} N_{ik} \right\}^{(n-2)/2}}{D^{(n-1)/2}}. \tag{18}$$

Now by a well known property of Jacobians,*

$$\sum_{j=1}^{n} \frac{\partial C_{ij}}{\partial x_j} = 0. \tag{19}$$

Hence, if in (19) the expressions on the right of (18) be substituted for C_{ij}, we will have the differential equation satisfied by $U^{(i)}$ alone. It is readily seen that the form of this equation is independent of the index (i) and hence the n functions

$$U^{(1)}, U^{(2)}, \cdots, U^{(n)}$$

satisfy the same differential equation, which may be looked upon as a generalization of Laplace's equation to curved n-space.

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THE NON-EXISTENCE OF A CERTAIN TYPE OF REGULAR POINT SET†

BY R. L. WILDER

In a paper not yet published,‡ I have shown that a regular§ connected point set which consists of more than one point and remains connected upon the omission of any connected subset, is a simple closed (Jordan) curve. As a simple closed curve is a bounded point set, it is clear that there does not exist any unbounded regular connected point set which remains connected upon the omission of any connected subset.

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* Muir, *Theory of Determinants*, vol. 2, p. 230.

† Presented to the Society, December 29, 1926.

‡ See, however, this Bulletin, vol. 32 (1926), p. 591, paper No. 35.

§ That is, connected im kleinen.

In the present paper I propose to consider the following question: *Does there exist any unbounded regular connected point set which remains connected upon the omission of any bounded connected subset?* If the additional restriction that it be closed is imposed upon the point set, J. R. Kline has shown* that the answer to this question is negative. In his proof Kline is able to make use of known properties of continuous curves. For the general case of non-closed sets these properties are not available, but by establishing certain properties of connected and regular point sets it is possible, as shown below, to give a negative answer to the above question.

DEFINITION. If M is a regular point set, a *region of* M is defined as follows: P being any point of M, and C a circle with center at P, then the set of all points of M which lie, with P, in a connected subset of M which lies within C is a *region of* M. It is clear that the set of all points of M which lie in a certain neighborhood of P are in a *region of* M.

DEFINITION. If A and B are two distinct points of a regular set M, *a simple chain of regions of* M *from* A *to* B is a finite sequence of regions of M, $R_1, R_2, \cdots, R_n$, such that (1) R_1 and R_2 contain A and B, respectively, (2) R_i $(i \neq 1, n)$ has points in common with R_{i-1} and R_{i+1}, but not with any other region of the sequence, (3) R_1 and R_n have points in common with R_2 and R_{n-1}, respectively, but not with any other region of the sequence.

THEOREM 1. *If* A *and* B *are any two distinct points of a regular connected point set* M, *then there exists a simple chain of regions of* M *from* A *to* B.

The proof of Theorem 1 is similar to the proof of Theorem 10 of R. L. Moore's *On the foundations of plane analysis situs*.†

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* *Closed connected sets which remain connected upon the removal of certain connected subsets*, Fundamenta Mathematicae, vol. 5 (1924), pp. 3–10.

† Transactions of this Society, vol. 17 (1916), pp. 131–164.

THEOREM 2. *If M is a regular point set, then any region of M is a regular point set.*

THEOREM 3. *If M is a regular connected point set, bounded or unbounded, and A and B are any two distinct points of M, then both A and B lie in a bounded, regular, connected subset of M.*

Theorem 3 is a consequence of Theorems 1 and 2.

THEOREM 4. *Let C_1 and C_2 be two mutually exclusive point sets, and M a regular connected point set which has at least one point in common with each of the sets C_1 and C_2, and such that the set of points common to M and C_i $(i=1, 2)$ is closed in M. Then there exists a point set K, subset of M, such that K is connected and bounded and contains no point of either C_1 or C_2, but such that C_1 and C_2 each contain at least one point of M which is a limit point of K.*

Theorem 4 is a generalization of the result contained in my paper *A theorem on connected point sets which are connected im kleinen.** Its proof, after an application of Theorem 3, is very similar to the proof of the result of the latter paper.

THEOREM 5. *If P is a point of a connected and regular point set M such that $M-P$ is the sum of two mutually separated† sets M_1 and M_2, then M_1+P and M_2+P are connected and regular sets.*

PROOF. Let K be any circle with center at P. Since M is regular, there exists a circle T concentric with K, such that all points of M interior to T lie with P in a connected subset of M which lies wholly interior to K. Denote by k the set of all points of M that lie with P in a connected subset of M which lies wholly interior to K, and by t the set of all points of M that lie interior to T. Clearly t is a subset of k.

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* This Bulletin, vol. 32 (1926), pp. 338–340.

† Two sets are said to be mutually separated if they are mutually exclusive and neither contains a limit point of the other.

By a theorem due to Knaster and Kuratowski,* M_1+P and M_2+P are connected sets. Denote the set of points common to k and M_i $(i=1, 2)$ by k_i. Neither of these sets is vacuous, since both M_1 and M_2 have points in common with t, and hence with k. Clearly $k-P$ is the sum of the two mutually separated sets k_1 and k_2. That is, P is a cut-point of k.

It follows by the theorem of Knaster and Kuratowski referred to above that k_1+P and k_2+P are connected sets. If x is any point of M_1 interior to T, then x is a point of k and a fortiori of k_1. Then there exists a connected subset of M_1+P, namely k_1+P, which contains both x and P and lies wholly within K. That is, the set of all points of M_1+P which lie interior to T lie, with P, in a connected subset of M_1+P which lies wholly interior to K. Hence M_1+P is regular at P. That it is regular at all other points is easily seen. Similarly, M_2+P is regular.

THEOREM 6. *If M is a regular point set, R a region of M, and P a point of R, then if k is a maximal connected subset of $R-P$, $k+P$ is a regular connected point set.*

PROOF. If $R-(k+P)$ is vacuous, $k+P$ is a regular connected set by Theorem 2. If $R-(k+P)$ is not vacuous, denote it by q. Then $R-P$ is the sum of the two mutually separated sets k and q, and hence, by Theorem 5, $k+P$ is connected and regular.

DEFINITION. If M is a point set, and C_1 and C_2 are mutually exclusive point sets, and H is a connected subset of M which has no point in common with either C_1 or C_2, but has limit points which are points of M in both C_1 and C_2, then H, together with those points of M in C_1+C_2 which are limit points of H, will be called a set $K(C_1, C_2)M$.† If the set

* B. Knaster and C. Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 206–255, Theorem 6.

† I have made use of the sort of set defined here in other connections. See my paper *A property which characterizes continuous curves*, Proceedings of the National Academy, vol. 11 (1925), pp. 725–728. Also see this Bulletin, vol. 32 (1926), p. 218, paper 4.

H is identical with that maximal connected subset of $M-M\times C_1-M\times C_2$ determined by H, then H together with those points of M in C_1+C_2 which are limit points of H, will be called a *maximal set* $K(C_1,C_2)M$ or, for the sake of brevity, a set $K'(C_1, C_2)M$.

DEFINITION. If M is a regular point set, P a point of M and C a circle enclosing P, then a *branch of M with respect to P and C* is a set $K'(P, C)M$.

THEOREM 7. *There does not exist, in the plane, a regular, connected, unbounded point set which remains connected upon the omission of any bounded connected subset.*

PROOF. Suppose there does exist such a set. Denote it by M.

1. If C is any circle, and P a point of M within C, say for convenience at the center of C, then I shall show that there exist infinitely many distinct branches of M with respect to P and C such that any two of these are, except for P, mutually separated.

Let S be the region of M which is determined by P and C. Since M is regular, there exists a circle T with center at P such that all points of M interior to T lie in S. Denote the set of all points of M interior to T by T_1.

The set $S-P$ is not connected. For if it were, then $M-(S-P)$ would, by hypothesis, be connected, which is impossible since this set contains no point of T_1 except P. Then $S-P$ is the sum of two mutually separated sets S_1 and S_2. The sets S_1+P and S_2+P are connected, and by Theorems 2 and 5 are regular. If x is a point of S_1, then by Theorem 4 the set S_1+P contains a set $K(P, x)$ (S_1+P). Denote this set by k_1. The set $M-P$ being unbounded, connected (by hypothesis) and regular, contains a set $K(x, C)$ $(M-P)$. Call this set k_2. Denote that portion of k_2 which is not on C by k_2'. The set k_2' is connected and, since it contains x, is a subset of S_1. Denote the set k_1+k_2 by k. Then k is a set $K(P, C)M$. If to the set k_1+k_2 be added all those points of that maximal connected subset

of $S-P$ determined by x, as well as the limit points of these points which are in M and on C, the resulting set is a set $K'(P, C)M$, which will be denoted by K_1. Denote the set of points $K_1-P-K_1\times C$ by H_1. Then H_1 is a connected subset of S_1.

In a similar way it can be shown that there exists a set H_2 which is a subset of S_2, and which, together with P and its limit points on C that belong to M, forms a set $K'(P, C)M$ which will be denoted by K_2. The sets H_1 and H_2 are mutually separated.

(a) If K_1-P and K_2-P are not mutually separated, their sum forms a connected set N.

(b) If K_1-P and K_2-P are mutually separated, let the set of all points of K_i $(i=1, 2)$ which lie between and on the circles C and T be denoted by B_i. The sets B_1 and B_2 are closed in $M-P$ and mutually exclusive. Hence, by Theorem 4, there exists a set $K(B_1, B_2)$ $(M-P)$ which is bounded. Denote this set by N_1. Then $(K_1-P)+(K_2-P)+N_1$ is a bounded connected subset of $M-P$ which will be denoted by N.

In either case (a) or (b), then, there exists a circle G concentric with C, whose radius is greater than the radius of C, and which encloses a connected subset, N, of $M-P$, which contains K_1-P and K_2-P. If g is the region of M determined by P and G, then $g-P$ is the sum of two mutually separated sets g_1 and g_2, and g_1+P and g_2+P are connected and regular. As $N+P$ is connected and lies within G, it is clear that it is a subset of g. Hence N is a subset of g_1+g_2 and being connected is a subset of one of the sets g_1, g_2, say g_1. As above, we can show that M contains a set k_3 which is a branch of M with respect to P and G, and which, except for P and its points on G, is a maximal connected subset of g_2. If h_3 denotes that portion of k_3 which is not on G, then the set h_3+P is regular, by Theorem 6, and hence contains a set $K'(P, C)M$ which will be denoted by K_3. It is clear that the sets K_2 and K_3 are, except for P, mutually separated. Denote the set $K_3-P-K_3\times C$ by H_3.

By means of Theorem 4, it can be shown that there exists a circle G_1 of radius greater than the radius of G, such that the sets $K_i - P$ $(i=1, 2, 3)$ all lie in a connected subset, F, of M, which lies within G_1. If q is the region of M determined by P and G_1, then $q-P$ is the sum of two mutually separated sets, q_1 and q_2, one of which, say q_1, contains F. Then a subset K_4 of q_2+P may be found which is a set $K'(P, C)M$.

Continuing as indicated above, the existence of an infinite sequence of sets $K_2, K_3, K_4, \cdots$, such that for every positive integer $n>1$, K_n is a set $K'(P, C)M$, and such that any two of these sets are, except for P, mutually separated, is established.

2. Consider in particular the sets K_2, K_3, K_4, and K_5. For each i, $(i=2, 3, 4, 5)$, let A_i be a point of K_i on C. Two of these points must separate the other two on C; say A_2 and A_3 separate A_4 and A_5 on C. As $M-(K_4+K_5)$ is connected, by hypothesis, and regular since K_4+K_5 is closed in M, it follows that there exists, by Theorem 4, a bounded set $K(A_2, A_3)$ $(M-K_4-K_5)$ which, together with the set of points K_2+K_3-P, is a bounded connected subset V of $M-P$. There exists a circle E concentric with C, which encloses V and contains no limit points of it.

Just as the existence of branches of M with respect to P and C were established, it can be shown that there exists an infinite set $K_i(A_j)$, $(i=1, 2, 3, \cdots, j=4, 5)$, of branches of M with respect to A_j and E. From the definition of a branch of M, it is clear that the connected set $V+K_4+K_5$ must lie wholly in one branch of M with respect to A_4 and E, say in $K_1(A_4)$, and in one branch of M with respect to A_5 and E, say $K_1(A_5)$.

As the set V contains no limit points of the set of points $K_4+K_5+K_2(A_4)+K_2(A_5)$, every point of it is the center of a circle which neither encloses any point of the latter set nor of E, nor has any point in common with either. The sum of the interiors of all such circles is a connected domain and this domain contains an arc t_1 from A_2 to A_3. Clearly t_1 and $K_4+K_5+K_2(A_4)+K_2(A_5)$ are mutually separated.

The points sets K_2+K_3 and $K_2(A_4)+K_2(A_5)$ are mutually separated. Denote by U the set of points consisting of $K_2(A_4)+K_2(A_5)$ together with its limit points. Then K_2+K_3 and U are mutually exclusive. Let $C(A_2)$ and $C(A_3)$ be circles with centers at A_2 and A_3, respectively, and enclosing no point of U. If a_i $(i=2, 3)$ is a point of H_i lying within $C(A_i)$, there exists an arc b_i joining A_i and a_i which lies entirely within $C(A_i)$, and except for A_i lies wholly within C. As H_2+H_3+P is a connected subset of C containing a_2 and a_3 but no point of U, there exists, by Theorem H of my paper *On a certain type of connected set which cuts the plane*,* an arc b_1 which joins a_2 and a_3, contains no point of U, and lies wholly within C. Clearly the continuous curve consisting of the arcs b_1, b_2, and b_3 contains an arc t_2 which joins A_2 and A_3, lies except for these points entirely within C, and contains no point of U. Similarly, since K_4+K_5 and t_1 are mutually separated, there exists an arc t_3 joining A_4 and A_5, lying except for these points entirely within C, and having no point in common with t_1.

By the corollary to Theorem D of the paper referred to in the preceding paragraph,† there exists a simple closed curve J which is a subset of t_1+t_2 and separates the plane between A_4 and A_5. However, J has no points in common with either $K_2(A_4)$, $K_2(A_5)$ or E, and yet the sum of these three sets is a connected set containing A_4 and A_5. Thus a contradiction is established and the theorem is proved.

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* To appear in the Proceedings of the International Mathematical Congress at Toronto. Theorem H of this paper is the following: Let G be a bounded domain, K any closed set of points and N a connected subset of G which contains no points of K. Then every pair of distinct points of K are the end-points of an arc which lies in G and contains no points of K.

† The corollary referred to here is the following: If A and B separate C and D on a simple closed curve K, AB and CD are arcs joining A,B and C,D, respectively, and lying, except for their end-points, interior to K, and t is an arc from A to B that contains no points of CD, then there exists a simple closed curve J which is a subset of $AB+t$, such that C is interior to J and D exterior to J, or vice versa.

A THEOREM OF FROBENIUS ON QUADRATIC FORMS*

BY PHILIP FRANKLIN

1. *Introduction.* A real quadratic form, in n variables,

$$\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i x_j, \qquad (a_{ij} = a_{ji}), \tag{1}$$

of rank r can always be reduced to the form

$$\sum_{i=1}^{r} c_i x_i^2, \tag{2}$$

by a real non-singular linear transformation.† The number of positive coefficients, P, and the number of negative coefficients, N, in (2) are each independent of the particular reduction used. The difference $P-N=s$ is called the signature of the form. Of the invariants P, N, $r=P+N$, s, only two are independent and any two of them form a complete system of invariants of (1) under non-singular linear transformations.

A form (1) of rank r is said to be *regularly arranged* if the x_i are so numbered that in the set

$$A_0 = 1, \; A_1 = a_{11}, \; A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \cdots, \tag{3}$$

$$A_r = \begin{vmatrix} a_{11} & \cdots & a_{1r} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{r1} & \cdots & a_{rr} \end{vmatrix},$$

$A_r \neq 0$, and no two consecutive A_i are zero. The following theorem is usually given for determining the invariants.

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* Presented to the Society, December 29, 1926.

† For the definitions and theorems quoted in this section see M. Bôcher, *Introduction to Higher Algebra*, New York, 1919, pp. 144–147; or H. Weber, *Lehrbuch der Algebra*, Braunschweig, 1895, vol. 1, pp. 255–265.

Every quadratic form may be regularly arranged, and when so arranged the signature is equal to the number of permanences minus the number of variations of sign in the sequence of A's, the A's which are zero being given signs at random.

In certain cases where the form is not regularly arranged, so that consecutive A's in the set (3) vanish, an analogous theorem holds, as Frobenius has shown.* His discussion is quite complicated. In this note we give a simple proof of the usual theorem, as well as the principal case noticed by Frobenius, based on the properties of the secular equation.

We also prove the following theorem relative to the determination of the invariants of a quadratic form from its principal minors.

If, for any quadratic form, we construct the set

$$(4)\qquad B_0 = 1,\ B_1 = \sum_i a_{ii},\ B_2 = \sum_{i,j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}, \cdots, \quad B_n = |a_{ij}|,$$

the sums being so taken as to include each principal minor of proper order once, the index of the last non-zero term in the set is the rank, r, and the number of permanences of sign in the sequence of non-zero terms is P, so that the signature is $s = 2P - r$.

2. *The Secular Equation.* We quote here a few well known theorems† concerning the equation

$$(5)\qquad L_n(x) = \begin{vmatrix} a_{11} - x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - x & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - x \end{vmatrix} = 0,$$

which we shall need for our argument. Let $L_i(x)$ denote the leading i-rowed minor of $L_n(x)$, and form the sequence

* G. Frobenius, *Ueber das Trägheitsgesetz der quadratischen Formen*, Journal für Mathematik, vol. 114 (1895), pp. 187–232.

† Weber, loc. cit., vol. 1, pp. 276–279.

$$(6) \qquad L_n(x), \ -L_{n-1}(x), \ \cdots, (-1)^{n-1}L_1(x), \ (-1)^n.$$

The set (6) form a Sturm sequence for equation (5), in the sense that

If $a \leqq b$ are two numbers neither of which makes any term of the series (6) *vanish, the excess of the number of variations of sign in the series* (6) *for $x=a$ over that for $x=b$ equals the number of real roots x between a and b, a q-fold root counting as q roots.*

Each of the equations

$$(7) \qquad L_i(x) = 0, \qquad (i = 1,2, \cdots, n)$$

has all its roots real, and the roots of the $(i+1)$th equation are separated by those of the ith.

3. *Application to the Classical Criteria.* Consider any quadratic form (1) of rank r. It is always possible to reducs it to an expression of the type of (2) by an *orthogonal* tranеformation on the n variables x_i.* Such a transformation will reduce the form

$$(8) \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_ix_j - k \sum_{i=1}^{n} x_i^2$$

to the form

$$(9) \qquad \sum_{i=1}^{r} c_ix_i^2 - k \sum_{i=1}^{n} x_i^2 .$$

The values of k which make (9), and hence (8), degenerate, are zero, counted $n-r$ times, and the r numbers c_i. Consequently these are the roots of equation (5), formed corresponding to (8). From the form of (6), we see that for $x=0$, the sequence becomes

$$(10) \qquad A_n, \ -A_{n-1}, \ \cdots, \ (-1)^{n-1}A_1, \ (-1)^nA_0.$$

Now apply the theorem quoted in §2, taking for a a positive number less than any positive root of any of the equations (7), and for b, $+\infty$. The terms of the series (6) for $x=a$ cor-

* Bôcher, loc. cit., pp. 171–173.

responding to non-zero terms in the series (10) will have the same signs as these latter, while the terms corresponding to zero terms will have definite signs. As the signs of this series for $x=a$ may be obtained from those of (10) by giving the zero terms appropriate signs, and as for $x=+\infty$ all the terms of (6) have the same sign, we see that the number of positive c_i, of P, is equal to the number of variations of sign in the series (10) when the zero terms are given proper signs. Finally, since changing alternate signs of a sequence replaces permanences by variations and vice versa, we have the following theorem.

THEOREM I. *If, for any quadratic form* (1) *we set up the sequence*

$$A_0, A_1, \cdots, A_n, \tag{11}$$

and give the zero terms proper signs, the number of permanences of sign will be P.

If, in particular, $A_r \neq 0$, from the separation property of the equations (7), and the fact that $L_r(x)=0$ has now no zero root, while $L_n(x)=0$ has an $(n-r)$-fold zero root, we see that the equation $L_{r+j}(x)=0$ has precisely j zero roots. Thus as x increases through zero, the first $n-r+1$ terms of the series (6) cause $n-r$ variations to be lost. As these account for the $n-r$ zero roots, no additional variations are lost in the rest of the sequence. Also the "proper signs" for the first $n-r$ terms of (10) give rise to no variations, and hence to no permanences in (11). This gives the following theorem.

THEOREM II. *If, for any quadratic form* (1) *of rank r we set up the sequence*

$$A_0, A_1, \cdots, A_r, \tag{12}$$

where $A_r \neq 0$, *and give the zero terms proper signs, the number of permanences of sign will be P.*

COROLLARY 1. *If the form is regularly arranged, i. e. no two consecutive A's in* (12) *vanish, the proper signs may be chosen at random.*

For, if $A_i=0$, while $A_{i-1}\neq 0$, $A_{i+1}\neq 0$, $L_{i-1}(x)=0$ and $L_{i+1}(x)=0$ have no zero roots, and as their roots separate those of $L_i(x)=0$, this equation has a single zero root. Thus $L_i(x)$ changes sign as we pass through zero, and as no variations are lost, A_{i-1} and A_{i+1} must have opposite signs.

COROLLARY 2. *If in* (12) *at most two consecutive A's vanish, if the A's adjacent to the pair of zeros have like signs, the proper signs should lead to one permanence from this set, while if the adjacent A's have unlike signs, the proper signs should lead to two permanences from this set.*

From the separation property, the two adjacent zeros correspond to single roots. Thus the signs given to the two zeros must be such that, if both are changed, the number of permanences is not affected, which necessitates that we have, essentially,

$$+ \mp \pm + ; \qquad + \pm \pm - ;$$

or their negatives.

Considerations similar to those used to prove the corollaries may be used to restrict the possible proper signs if three or more consecutive A's vanish, but these considerations by no means determine the signs completely. In fact, if three consecutive A's vanish, the signature can not always be determined from the set (12). Thus for the form*

$$ax_2^2 + 2ax_2x_3 + 2ax_1x_4 + 2ax_3^2, \tag{13}$$

the sequence (12) is 1, 0, 0, 0, $-a^4$, and the signs are independent of the sign of a. However, if we renumber the variables $y_1=x_2$, $y_2=x_3$, $y_3=x_1$, $y_4=x_4$, the form becomes

$$ay_1^2 + 2ay_1y_2 + 2ay_3y_4 + 2ay_2^2, \tag{14}$$

* Frobenius, loc. cit., p. 199.

and the sequence (12) is $1, a, a^2, 0, -a^4$, so that the form is now regularly arranged, and by corollary 1, $P=3$, if a is positive, while $P=1$ if a is negative.

4. *A New Criterion.* We saw in § 3 that the invariants P and N of the form (1) could be characterized as the number of positive and negative roots, respectively, of the equation (5). The criteria we developed in § 3 were based on a modified Sturm sequence for this equation. As the equation has all its roots real, we may determine the signs of the roots by Descartes' rule as the number of variations of sign in the coefficients of the equation. On expanding (5), we find

$$(15)\quad (-x)^n B_0 + (-x)^{n-1} B_1 + \cdots + (-x) B_{n-1} + B_n = 0,$$

where the B's are given by

$$(16)\quad B_0 = 1,\ B_1 = \sum_i a_{ii},\ B_2 = \sum_{i,j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}, \cdots,$$
$$B_n = |a_{ij}|,$$

the sums being so taken as to include each principal minor of proper order once. Thus for (1), P is the number of variations of sign of the coefficients of (15), or permanences of sign of the sequence (16). Since (15) has $n-r$ zero roots, $B_r \neq 0$, while all the B's following this one vanish. Collecting these results, we have the following theorem.

THEOREM III. *If, for any quadratic form* (1) *we set up the sequence*

$$(17)\qquad B_0, B_1, \cdots, B_n,$$

the index of the last non-zero term in the set will be the rank, r, and the number of permanences of sign in the sequence of non-zero terms will be P, so that the signature is $s=2P-r$.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

ON THE PARTITIONS OF A GROUP AND THE RESULTING CLASSIFICATION*

BY J. W. YOUNG

1. *Introduction.* I propose the following definition: A *partition* of *any* group G is any class $[H]$ of subgroups of G such that every element other than the identity of G is contained in one and only one H. The concept thus defined arose very naturally in connection with the problem of defining a general algebra without introducing the idea of a field.† After this paper was written Professor G. A. Miller called my attention to the fact that he had considered the partitions of a group (though not under the present terminology) in a paper presented to the Society in 1906.‡ His methods and point of view are, however, entirely different and only one of his theorems duplicates a result of the present paper.

Even though the results of the present paper find their immediate application in the domain of general algebras, it seemed desirable to publish them in a separate paper. The concept itself and the process defined in §2 below for the determination of the so-called primitive partition of a group would appear to be of possible value in the general theory of groups and the resulting classification of all groups into types would seem to possess elements of interest. This is due especially to the fact that results in what may be

* Presented to the Society, September 9, 1926.

† See a paper (not yet published), *On the definition of the Scorza-Dickson algebras*, presented to the Society at New York, February 26, 1927; abstract, this Bulletin, vol. 33 (1927), p. 265. The linear systems of order one in an algebra as defined in Dickson, *Algebras and their Arithmetics* (University of Chicago Press, 1923), pp. 9, 10, form a partition of the addition group of the algebra.

‡ G. A. Miller, *Groups in which all the operators are contained in a series of subgroups such that any two have only identity in common*, this Bulletin, vol. 12 (1905–06), p. 446.

called the *general* theory of groups are comparatively rare, while the literature of special classes of groups (finite, continuous, etc.) is enormous. In what follows we place no restrictions on the group G beyond what is implied in the definition of any group.*

2. *The Classification of Groups into Types. The Primitive Partition.* Let $G_{1,s}$ be the cyclical subgroup of G generated by the element $s(\neq 1)$ of G. $G_{1,s}$ then consists of all the positive and negative integral powers of s and $1=s^0$. If there exists a finite integer $m>1$ such that $s^m=1$, s is of finite order and the smallest such integer is called the order of s. Let $G_{1,(s)}$ be any cyclical subgroup of G containing s. Let $G_{2,s}=\{[G_{1,(s)}]\}$ be the subgroup of G generated by all the groups $G_{1,t}$ containing s, i. e., the smallest subgroup of G containing all the $G_{1,(s)}$. Evidently $G_{2,s}$ is uniquely determined by s. In general let $G_{k+1,s}$ be the subgroup of G generated by all the $G_{k,t}$ containing s $(k=1, 2, \cdots)$. Let $[G_k]$ be the class of all subgroups $[G_{k,s}]$ as s ranges over G.

We thus obtain for every s of G a sequence of subgroups $G_{1,s}, G_{2,s}, \cdots$ such that *every* $G_{k,s}$ $(s\neq 1,\ k>1)$ *contains the preceding* $G_{k-1,s}$:

$$G_{1,s} \leqq G_{2,s} \leqq \cdots \leqq G_{k,s} \leqq \cdots .$$

We also obtain a sequence of classes of subgroups $[G_1]$, $[G_2]$, $\cdots$, such that every element of G is contained in one or more subgroups G_k for every value of $k=1, 2, \cdots$, and such that every $[G_k]$ $(k>1)$ contains the preceding class $[G_{k-1}]$ in the sense that every $G_{k-1,s}$ is contained in $G_{k,s}$. We indicate this by writing $[G_{k-1}]\leqq[G_k]$; $[G_{k-1}]=[G_k]$, if and only if, for every s we have $G_{k-1,s}=G_{k,s}$. With this notation we then have

* A class of elements, $s, t, \cdots$, is said to form a group with respect to a law of combination (indicated by juxtaposition) if the following conditions are satisfied: (1) the product st shall be a uniquely defined element of G for every ordered pair s, t of G; (2) for every three elements of G, $(st)u=s(tu)$; (3) there exists in G an element 1, called the *identity*, such that for every s of G, $s1=1s=s$; (4) for every s of G there exists an *inverse* s' such that $ss'=1$.

$$[G_1] \leqq [G_2] \leqq \cdots \leqq [G_k] \leqq \cdots .$$

If there exists a finite integer n such that for every s of G

$$G_{n,s} = G_{n+1,s} = \{[G_{n,(s)}]\},$$

$G_{n,(s)}$ must consist of the single group $G_{n,s}$, i. e., every $(s \neq 1)$ of G is in one and only one $G_{n,s}$; or, if s and t are any two elements of G $(s \neq 1,\ t \neq 1)$, then either $G_{n,s} = G_{n,t}$ or $G_{n,s}$ and $G_{n,t}$ have only the identity in common. Hence we may state the following theorem.

THEOREM 1. *If there exists an n such that $[G_n] = [G_{n\pm 1}]$, then two distinct G_n's have only the identity in common; and conversely.*

To prove the converse, we observe that if two distinct G_n's have only the identity in common and s $(\neq 1)$ is any element of G, $[G_{n,(s)}]$ consists of the single group $G_{n,s}$.

Now let n be the smallest integer for which $[G_n] = [G_{n+1}]$. Our sequence $[G_k]$ then becomes

$$[G_1] < [G_2] < \cdots < [G_k] = [G_{k+1}] = \cdots ;$$

and for every s there exists a smallest n_s such that

$$G_{1,s} < G_{2,s} < \cdots < G_{n_s,s} = G_{n_s+1,s} = \cdots .$$

Clearly we have $n_s \leqq n$, for every s.

Under this hypothesis our sequence $[G_k]$ terminates, in a sense, with $[G_n]$ and *this class is a partition of* G.

If, on the other hand, there exists no finite n such that $[G_n] = [G_{n+1}]$, there must be in G an s such that $G_{1,s} < G_{2,s} < \cdots < G_{k,s} < \cdots$, $k = 1, 2, \cdots$. G must then contain a smallest subgroup $G_{\omega,s}$ containing all the $G_{k,s}$ for all finite values of k. $G_{\omega+1,s}$ is then defined as the group generated by all the $G_{\omega,t}$ containing s, and the process previously defined leads to sequences of groups

$$G_{\omega,s} \leqq G_{\omega+1,s} \leqq \cdots \leqq G_{2\omega,s} \leqq \cdots \leqq G_{\omega^2,s} \leqq \cdots ,$$

and to sequences of classes of groups

$$[G_\omega] \leqq [G_{\omega+1}] \leqq \cdots \leqq [G_{2\omega}] \leqq \cdots \leqq [G_{\omega^2}] \leqq \cdots .$$

If N be the smallest integer, finite or transfinite, for which $[G_N]=[G_{N+1}]$, the group G is said to be of *type* N.* Theorem 1 and the results obtained since may now be stated as follows.

THEOREM 2. *If G is a group of type N two distinct G_N's have only the identity in common, and conversely. The class $[G_N]$ is then a partition of G.*

In a group of type N the partition $[G_N]$ is evidently uniquely determined. It may be called the *primitive partition* of G. The primitive partition may, of course, consist simply of G itself. A partition containing more than one group may be called a *proper partition*. It is evident from the way in which the primitive partition of a group is obtained, that unless the primitive partition is proper, a group cannot have any proper partition. In fact, the following theorem is obvious:

THEOREM 3. *If $[H]$ is any partition of a group G of type N and $[G_\Lambda]$ is the primitive partition of G, then for every s in G we have $G_{N,s}\leqq H_s$.*†

Indeed, we have $H_s=\{[G_{N,t}]\}$, as t ranges over H_s.

3. *The Partitions of an Abelian Group.* If an infinite abelian group G contains elements of finite order, the latter form a subgroup F, invariant in G. The quotient group G/F then contains no elements of finite order, and G may be considered as the direct product of F and G/F.

We consider first an abelian group none of whose elements is of finite order.

THEOREM 4. *Any two cyclical subgroups $G_{1,s}$ and $G_{1,t}$ of an abelian group G with no elements of finite order which have a common element $(\neq 1)$ are contained in a cyclical subgroup $G_{1,u}$ of G.*

* This classification for larger values of N is purely formal. I do not know whether groups of type N exist even for all finite values of N.

† H_s denotes the H containing s.

Let $s \neq t$, and let $s^m = t^n$ $(\neq 1)$ be the common element. If either m or n equals 1, the theorem is proved. Moreover m and n cannot be equal; for, the relation $s^m = t^m$ would imply $(s^{-1}t)^m = 1$ and, since $s^{-1}t \neq 1$, this would imply that G contains an element of finite order. We may assume that m and n are prime to each other; for, the relations $m = km'$, $n = kn'$ $(k > 1)$, would imply $(s^{-m'}t^{n'})^k = 1$, which leads to $s^{m'} = t^{n'}$. Two integers, x, y, therefore, exist such that $mx + ny = 1$. Now, let $t^x s^y = u$. We then have

$$s = s^{mx}s^{ny} = t^{nx}s^{ny} = u^n ;$$

$$t = t^{mx}t^{nx} = t^{mx}s^{my} = u^m.$$

This proves the theorem.

As an immediate corollary we observe that *if any finite number of cyclical subgroups of an abelian group G with no elements of finite order have an element $(\neq 1)$ in common, there exists a cyclical subgroup of G which contains them all.* We may now prove the following theorem.

THEOREM 5. *In any abelian group G contains no elements of finite order, $[G_2]$ is a partition of G.*

To prove the theorem it is sufficient to show that if two subgroups $G_{2,s}$ and $G_{2,t}$ have an element $c(\neq 1)$ in common they coincide. $G_{2,s}$ is the smallest group containing all the $G_{1,(s)}$. Since c is in $G_{2,s}$, c is the product of a finite number of elements each of which is an element of some $G_{1,(s)}$. By the corollary of Theorem 4, this set of $G_{1,(s)}$ are all contained in a $G_{1,(s)}$. In other words, there exists a $G_{1,p}$ containing both c and s. Similarly, there exists a $G_{1,q}$ containing both c and t. Let t_1 $(\neq 1)$ be any element of $G_{2,t}$. Reasoning as before proves the existence of a $G_{1,u}$ containing t, c, and t_1. The groups $G_{1,p}$ and $G_{1,u}$, have c in common $(c \neq 1)$; there exists, therefore, a $G_{1,v}$ containing t, c, t_1, and s. $G_{1,v}$ is, therefore, a $G_{1,(s)}$ and hence t_1 is contained in $G_{2,s}$. Every element of $G_{2,t}$ is then an element of $G_{2,s}$. Similarly, it follows that every element of $G_{2,s}$ is an element of $G_{2,t}$.

COROLLARY. *Any abelian group containing no elements of finite order is of type* $\leqq 2$.

Now, let G be any abelian group all of whose elements are of finite order. Let G contain an element s_1 of order pq where p and q are prime numbers (not excluding the possibility $p=q$); let $s_1^q = s$, whence $s^p = 1$; and let t, of order m, be any other element of G.

We consider, first, the case where m contains a prime factor different from p. Let p^α be the highest power of p contained in m, so that $m = p^\alpha m'$, where m' (>1) is prime to p. Then $t^{p^\alpha} = t_1$ is of order m', and there exists an n such that $m'n \equiv 1 \pmod{p}$. Then,

$$(s_1 t_1)^{qm'n} = (s_1^q)^{m'n} = s^{m'n} = s;$$

and, hence, G_{1,s_1} and $G_{1,s_1 t_1}$ contain s. Therefore,

$G_{2,s}$ contains s_1 and $s_1 t_1$ and, hence, s_1 and t_1;
$G_{2,t}$ contains t_1;
G_{3,t_1} contains s and t;
and $G_{4,s}$ contains t.

We consider, next, the case where $m = p^\alpha$, $\alpha > 1$. Let $t^p = t_1$; then, $(st)^p = t_1$; and

$G_{1,t}$ contains t_1;
$G_{1,st}$ contains t_1;
G_{2,t_1} contains t and st, and, hence, s;
and $G_{3,s}$ contains t.

We consider, finally, the case $m = p$. Let $s_1^p = s'$. Then

G_{1,s_1} contains s';
$G_{1,s_1 t}$ contains s';
$G_{2,s'}$ contains s_1 and $s_1 t$, and hence, t.

Hence, in every case, if G contains an element of order pq, there exists an element s (called s' in the last case), such that $G_{4,s} = G$. Hence no group G containing an element of order pq has a proper partition. If, however, all the elements of G are of the same prime order p, the cyclical subgroups of G evidently form a partition. Hence, we have the following theorem.

THEOREM 6. *If an abelian group all of whose elements are of finite order has a proper partition all of its elements are of the same prime order p; and every abelian group containing more than p elements and all of whose elements are of the same prime order p has a proper partition* $[G_1]$.

COROLLARY 1. *An abelian group of finite order has a proper partition if, and only if, it is of order* $p^m (m>1)$ *and of type* $(1, 1, \cdots, 1)$.*

COROLLARY 2. *An abelian group all of whose elements are of finite order is of type* $\leqq 4$.

Now let G be an abelian group containing elements of finite order and also elements of infinite order. G is then the direct product of a group F containing only elements of finite order and a group I containing no elements of finite order. Let s be any element of I and let t, of order m, be any element of F. Then we have $(st)^m = s^m$, so that $G_{1,st}$ and $G_{1,s}$ have s^m in common. G_{2,s^m} then contains s and st, and hence s and t. Therefore, $G_{3,s}$ contains t; i. e., $G_{3,s}$ contains F. The class of all $[G_{3,s}]$ as s ranges over I all have F in common, therefore, and hence any $G_{4,t}$, where t is in F, contains all the elements of I and also of F. Hence we have $G_{4,t} = G$. We have, therefore, the following theorem.

THEOREM 7. *An abelian group containing elements of finite and also of infinite order has no proper partition.*

In connection with previous results we have the following result.

COROLLARY. *Every abelian group is of type* $\leqq 4$.

We may combine the results of Theorems 5, 6, and 7 as follows.

THEOREM 8. *If an abelian group has a proper partition, either it contains no elements of finite order or all its elements are of the same prime order. In any such group G, the class of subgroups* $[G_2]$ *is a partition.*

* This corollary was proved by G. A. Miller, loc. cit., p. 448.

4. *L-Partitions.* In connection with the contemplated application to general algebras and possibly in other connections the following concept is important. Let $[H]$ be any partition of a group G and let $K=\{H_{s_1}, H_{s_2}, \cdots, H_{s_k}\}$ be the group generated by any finite number of H's. If any H_p having an element $(\neq 1)$ in common with any K is entirely contained in K, the partition $[H]$ is called an *L-partition.*

We first observe the following theorem.

THEOREM 9. *The primitive partition of an abelian group G all of whose elements are of the same prime order p is an L-partition.*

For, the partition consists of the cyclical subgroups $[G_1]$ of order p. If $\{G_{1,s_1}, G_{1,s_2}, \cdots, G_{1,s_k}\}$ contains an element t of a cyclical subgroup we may take this subgroup to be $G_{1,t}$, and we should have $t=s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_k^{\alpha_k}$. But all the powers of t are then contained in $\{G_{1,s_1}, G_{1,s_2}, \cdots, G_{1,s_k}\}$.

We seek next the condition that the primitive partition of an abelian group G containing no elements of finite order be an L-partition. Let $[H]$ be a proper partition of G, let s $(\neq 1)$ be any element of G and let t $(\neq 1)$ be any element of G not in H_s. Let $s^{-1} t^k = u$ (k any integer) and consider the group $\{H_s, H_u\}$. Clearly $H_s \neq H_u$; for, the assumption that u is in H_s would imply $H_s = H_t$. Since $\{H_s, H_u\}$ has $t^k = su$ in common with H_t, if $[H]$ is an L-partition $\{H_s, H_u\}$ must contain t. Hence, there must exist elements s_1, u_1 of H_s and H_u, respectively, such that $t = s_1 u_1$. This gives $t^k = s_1^k u_1^k = su$, whence we have $s_1^k s^{-1} \cdot u^k u^{-1} = 1$. Since H_s and H_u have only identity in common, this gives $s_1^k = s$. Since s was any element in G and k was any integer, it follows that, *if G admits an L-partition, every element of G must have a kth root in G for every integer k.*

We now prove that this necessary condition is also sufficient in order that a proper primitive partition of G be an L-partition. Let now $[H]$ be a proper primitive partition of G, and let $K=\{H_{s_1}, H_{s_2}, \cdots, H_{s_k}\}$ be the subgroup

generated by any finite number of H's. Let $t'(\neq 1)$ be an element common to K and any other H_t, so that we have $t' = s_1' s_2' \cdots s_k'$, where s_i' is in H_{s_i}. Let t be any other element of H_t. Then, by Theorem 5, there exists an element u of H_t and integers m and n such that $t = u^m$ and $t' = u^n$. Under the condition imposed on G there exist in G elements v_i, such that $v_i^n = s_i'$, for every i from 1 to k, and each of these v_i are, of course, in H_{s_i}. We have, then,

$$t' = s_1' s_2' \cdots s_k' = (v_1 v_2 \cdots v_k)^n = u^n.$$

But in a group G containing no elements of finite order the last equality implies $u = v_1 v_2 \cdots v_k$. Hence, u is in K and, hence, t is in K. We have, therefore

THEOREM 10. *The necessary and sufficient condition that a proper primitive partition of an abelian group G containing no elements of finite order be an L-partition is that every element of G have a kth root in G for every integer k.*

DARTMOUTH COLLEGE

ANALYTIC FUNCTIONS WITH ASSIGNED VALUES*

BY PHILIP FRANKLIN

1. *Introduction.* The question of determining when the values of a function at a denumerably infinite set of points in a finite region determine a function analytic in this region, and if so the power series for the function in question, has recently been raised by Professor T. H. Hildebrandt.† It is well known that if the function in question exists, it is uniquely determined.‡ The usual proof gives a process for determining the coefficients in the power series, in which

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* Presented to the Society, February 26, 1927.

† This Bulletin, vol. 32 (1926), p. 552.

‡ W. F. Osgood, *Funktionentheorie*, vol. 1 (1912), p. 337.

the expression for the nth coefficient involves n successive limits. We shall set up expressions involving single limits which determine the coefficients, and hence give a simpler process for obtaining the power series.

If the sequence be formed whose nth term is the nth root of the expression giving the nth coefficient, the existence and boundedness of the limits appearing in this sequence constitute a necessary and sufficient condition that the process leads to an analytic element. To insure that this element leads to a function of the kind sought, certain obvious conditions must be added.

2. *Necessary Conditions.* Consider a denumerably infinite set of points in a closed two-dimensional* region of the z plane. Let a function, $w=f(z)$, which is analytic in this region, be given. If its values at the points of the set are determined, we wish to see how the function may be recovered from these values. Since the given point set is infinite, and in a closed region, it has at least one limit point in the region. For simplicity, take one such limit point as the origin, designate some sequence of points in the set approaching the origin as a limit by $Z_1, Z_2, \cdots$, and the corresponding values of the function $f(z)$ by $W_1, W_2, \cdots$. These values alone uniquely determine the function.†

In a neighborhood of the origin, we have

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots . \tag{1}$$

Consequently, we may write

$$c_0 = \lim_{z \to 0} f(z), \quad \cdots \quad c_n = \lim_{z \to 0} \frac{f(z) - \sum_{k=0}^{n-1} c_k z^k}{z^n} . \tag{2}$$

* This is not an essential requirement, since a function analytic in all the points of a one-dimensional region is necessarily analytic in some closed two-dimensional region including this region.

† Osgood, loc. cit.

If we let z approach zero through the sequence Z_i, we have

$$c_n = \lim_{i\to\infty} \frac{W_i - \sum_{k=0}^{n-1} c_k Z_i^k}{Z_i^n}. \tag{3}$$

While these formulas determine the coefficients successively, if we write them directly in terms of the given quantities, we see that the formula for the nth derivative involves n repeated limits. To find simpler formulas, involving single limits, we consider the expression

$$A_n = \frac{\begin{vmatrix} f(z_1) & z_1^{n-1} & \cdots & z_1 & 1 \\ f(z_2) & z_2^{n-1} & \cdots & z_2 & 1 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ f(z_{n+1}) & z_{n+1}^{n-1} & \cdots & z_{n+1} & 1 \end{vmatrix}}{\begin{vmatrix} z_1^n & z_1^{n-1} & \cdots & z_1 & 1 \\ z_2^n & z_2^{n-1} & \cdots & z_2 & 1 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ z_{n+1}^n & z_{n+1}^{n-1} & \cdots & z_{n+1} & 1 \end{vmatrix}}. \tag{4}$$

This is suggested by the fact* that, for certain real functions $f(z)$, when the z_i in this expression close down on a point z_0, the limiting value of this expression is $f^{(n)}(z_0)/n!$. As the method of proof for the real case is based on Rolle's theorem, which has no simple analog for functions of a complex variable, we must here proceed otherwise.

Let us replace the functions $f(z_i)$, which occur in the numerator of (4), by their series expansions, as given by (1). In any circle $0 \leqq |z| \leqq R$ *inside* the circle of convergence of (1), these series converge uniformly, so that for values of z_i in this circle we may write

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* See the author's paper, *Osculating curves and surfaces*, Transactions of this Society, vol. 28 (1926), p. 402.

$$(5)\qquad A_n = \sum_{k=0}^{\infty} c_k \frac{\begin{vmatrix} z_1^k & z_1^{n-1} & \cdots & z_1 & 1 \\ z_2^k & z_2^{n-1} & \cdots & z_2 & 1 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ z_{n+1}^k & z_{n+1}^{n-1} & \cdots & z_{n+1} & 1 \end{vmatrix}}{\begin{vmatrix} z_1^n & z_1^{n-1} & \cdots & z_1 & 1 \\ z_2^n & z_2^{n-1} & \cdots & z_2 & 1 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ z_{n+1}^n & z_{n+1}^{n-1} & \cdots & z_{n+1} & 1 \end{vmatrix}}.$$

Of the determinants in the numerator, those in the terms for which $k \leqq n-1$ vanish. For $k \geqq n$, the numerator is divisible by the product of all the differences of the z_i, i. e., the Vandermonde determinant in the denominator. The quotient, S_{k-n}, is a homogeneous symmetric function of the z_i of the $(k-n)$th degree, and is readily found to be that in which all terms of this degree appear, each with coefficient unity. Consequently, if the z_i are in the circle $0 \leqq z \leqq r < R/n$, we shall have

$$(6)\qquad S_{k-n} \leqq (|z_1| + |z_2| + \cdots + |z_n|)^{k-n} \leqq (nr)^{k-n}.$$

Thus the series for A_n given in (5) is equivalent to

$$(7)\qquad A_n = c_n + \sum_{k=n+1}^{\infty} c_k S_{k-n}.$$

This shows that the series for $|A_n - c_n|$ is dominated by

$$(8)\qquad \sum_{k=n+1}^{\infty} |c_k| (nr)^{k-n} = nr \sum_{k=n+1}^{\infty} |c_k| (nr)^{k-n-1}.$$

But, for z in the circle $0 \leqq z \leqq R$, the series (1) as well as all the series obtained by termwise differentiation converge absolutely and uniformly.* In particular, from the $(n+1)$th derived series, for $z = R$, we find

$$(9)\qquad \sum_{k=n+1}^{\infty} k(k-1) \cdots (k-n) |c_k| R^{k-n-1} = M.$$

* Osgood, loc. cit., pp. 97–98, and p. 335.

Comparing this with (8), we see that

$$|A_n - c_n| \leqq nr\,M. \tag{10}$$

This shows that when r approaches zero, A_n approaches c_n.

Evidently r will approach zero provided the z_i close down to the origin. We note that if the limit point were z_0, not the origin, and we replaced z_i by $(\bar{z}_i - z_0)$ and $f(z_i)$ by $f(\bar{z}_i)$ in (4), this expression would remain of the same form as at present, with bars on the z's. Since $c_k = f^{(k)}(0)/k!$, we may state the following theorem.

THEOREM I. *If $f(z)$ is an analytic function, and $z_1, z_2, \cdots, z_n$ are n distinct points closing down on z_0, we shall have*

$$\lim A_n = f^{(n)}(z_0)/n!,$$

where A_n is given by (4) *or*

$$\sum_{j=1}^{n} \frac{f(z_j)}{(z_j - z_1)\cdots(z_j - z_{j-1})(z_j - z_{j+1})\cdots(z_j - z_n)}. \tag{11}$$

3. *Sufficient Conditions.* Theorem I suggests the following process for constructing the analytic function, if such exist, determined by the values on a denumerable point set. Select a sequence of points having a single limit point, take n consecutive points of this sequence as the z_i of Theorem I, and form the A_n, and their limits, c_n. This gives the series (1), which determines an analytic element provided its radius of convergence is not zero, i. e., that the maximum limit of $|c_n|^{1/n}$ is finite.* This gives the following theorem.

THEOREM II. *If the values of a function are given at a sequence of points having a single limiting point, and the A_n of Theorem* I *be formed from n consecutive points of the sequence, the existence of the limits*

$$\lim A_n^{1/n},$$

* E. J. Townsend, *Functions of a Complex Variable*, 1915, p. 232.

and the boundedness of these limits as a set, constitute a necessary and sufficient condition that

$$\sum_{n=1}^{\infty} (\lim A_n) z^n$$

yield an analytic element.

COROLLARY. *If the values are given at any point set in a closed region, the conditions that they determine a function analytic in this region are* (1) *that for some subsequence the conditions of Theorem* II *are met,* (2) *that the resulting element is capable of analytic extension over the entire region, and* (3) *that the resulting function takes the correct values at all the given points.*

The third condition here seems stringent, but when it is recalled that if the value of the function at a single point of the region is changed, the required analytic function will no longer exist, it becomes evident that no essentially milder conditions will suffice.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

AN APPLICATION OF ANALYSIS SITUS TO STATISTICS*

BY HAROLD HOTELLING

1. *Introduction.* The theoretical distribution curve of correlation coefficients obtained by sampling under any particular conditions may or may not extend to the limits $r = \pm 1$. Whether it does extend to either of these limits, and if so its order of contact with the r-axis, are determined by features of the problem which will be shown to be essentially topological. These properties of the curve are independent of any influence which some members of the sample may exert upon others, provided this does not amount to complete determination. They are, moreover, to some extent independent of the nature and existence of any correlation between the variates in the population from which samples are drawn, and indeed of the distribution in this population of the variates. Finally, they are independent of heterogeneity in the population. First it will be well to notice the relation of these ideas to *time series*.

2. *Time Series.* It is well known that the correlation coefficient and other statistical measures do not have their usual significance when computed from observations ordered in time. On account of the lack of independence of successive observations, the known sampling distributions and probable error formulas are inapplicable, and no adequate substitutes have been discovered. Economic and social statisticians and meteorologists thus labor under a serious handicap which is largely absent from biometric work.

A contribution toward the removal of this handicap has recently been made by G. Udny Yule in a presidential address to the Royal Statistical Society.† He gives the

* Presented to the Society, San Francisco Section, October 30, 1926.

† *Why do we sometimes get nonsense correlations between time series?* Journal of the Royal Statistical Society, vol. 89 (1926), pp. 1–69.

name *conjunct series* to sequences whose finite differences of any definite order may be considered independent, and inquires as to the distribution of correlation coefficients between conjunct series. Failing to discover a mathematical solution, he resorts to experiment, and gives a frequency distribution of 600 correlation coefficients between conjunct series obtained by cumulating in sets of ten the results of drawing numbers at random from a box. He fits the frequency distribution by an empirical curve, but with the surmise that this does not represent the true function. The correctness of this surmise now appears from the fact that the curve has vertical tangents at the ends of the range, whereas, if the conditions of the experiment be slightly idealized in order to yield an analytic curve (or any curve having a sufficient number of derivatives), this will be horizontal at the ends, and further, will have second-order contact with the axis. Yule's data as presented in his Figure 17 may indeed suggest contact with the axis.

A somewhat different class of questions arises in examining the conformity of a historical variable to a differential equation. Consider for example equations of the form

$$\frac{d^m x}{dt^m} = a + bx,$$

which occur in connection with population estimates, growth curves and studies of periodicity. Certain logical advantages, which are dwelt on elsewhere, attach to the following procedure. From a sequence of observed values of x the values of the mth derivatives at the corresponding times are estimated by means of finite differences. The differential equation then becomes a regression equation: a and b may be determined by least squares, and the correlation coefficient used to test the goodness of fit. But even if the differential equation were without validity and the values of x or of $d^m x/dt^m$ mere random numbers, the correlation coefficients thus obtained would not have the distribution nor probable error ordinarily attributed to r. For here one

set of quantities is derived by manipulation of the other rather than by independent observation.

These and many similar outstanding problems in time series seem to be of great mathematical difficulty, and it is likely that investigators will often have to depend for assurance of the validity of their results upon rough ideas derived from experiments with cards and dice.

3. *Geometric Relations.* Letting the observations $x_1, x_2, \cdots, x_n$ be cartesian coordinates of a point P, the operation of replacing them by deviations from their mean is equivalent to projecting P orthogonally upon the hyperplane

$$x_1 + x_2 + \cdots + x_n = 0.$$

Let the projected point be projected radially from the origin O upon the hypersphere

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

This represents a mere change of units for the observations, which does not alter their correlation with any other sequence. If the observations are independent random drawings from a normally distributed aggregate, they are represented by a point Q taken at random on the hypersphere in such a way that the element of probability is proportional to the element of $(n-2)$-dimensional volume. But since we are not requiring the drawings to be independent, random nor from a normally distributed aggregate, we assume only that the probability that the representing point will be found within a distance δ of a point Z of the hypersphere, divided by δ^{n-2}, approaches a continuous positive function of Z as δ approaches zero.

If Q and S are two points on the hypersphere the coefficient of correlation between the corresponding sequences equals $\cos QOS$, as is well known. If S is determined from Q in a definite manner, this method of determination defines a transformation T of the hypersphere into itself or a part of itself, and we are interested in the amount of motion involved. The correlation is now a function of

position; the hypersphere will be marked out with hypersurfaces of constant correlation, and it is the volume between such hypersurfaces multiplied by a continuous positive function, that determines our frequency curve of sample correlations. The case of independent selection of the points may be regarded as the special and extreme case in which the whole sphere is transformed into one point.

4. *Transformations of a Hypersphere.* A topological study of transformations of an m-dimensional sphere into itself has been made by J. W. Alexander,* who has proved, among other things, that a one-to-one continuous transformation of a sphere into itself always has at least one fixed point if the transformation preserves sense and the sphere is of even dimensionality, or if the transformation reverses sense and the dimensionality is odd.

In a transformation T of a sphere corresponding to a transformation of a statistical series, a fixed point corresponds to a correlation of $+1$. To a correlation of -1 corresponds a fixed point of a new transformation consisting of T followed by a transference R of each point to its diametrically opposite point. This transference R preserves sense of orientation if the sphere is of an odd number of dimensions, but reverses sense if the number of dimensions is even. Alexander's theorem quoted above may therefore be interpreted as follows in terms of the curve of distribution of the correlation coefficient r between a series and its transformed series. Let n be the number of observations; the hypersphere is then of $n-2$ dimensions. If n is even and T preserves sense of orientation, it has a fixed point. The distribution curve for r therefore extends to $+1$. If n is even and T reverses sense, TR has a fixed point, so that the distribution curve extends to -1. If n is odd and T reverses sense the curve extends to $+1$. Only if n is odd and T preserves sense is it possible for the distribution curve to stop short of both extremes, for in this case only neither

* Transactions of this Society, vol. 23 (1922), pp. 89–95.

T nor TR is compelled to have an invariant point. These statements may be summarized in the following table.

	n even	n odd
T reverses sense	For some samples $r=-1$	For some samples $r=+1$
T preserves sense	For some samples $r=+1$	The curve may or may not extend to $r=\pm 1$.

5. *Correlation with Permuted Series.* As an illustration consider the correlation between series $x_1, x_2, \cdots, x_n$ and $x_1', x_2', \cdots, x_n'$ connected by the permutation.

$$T: \qquad x_1' = x_2,\ x_2' = x_3,\ \cdots,\ x_{n-1}' = x_n,\ x_n' = x_1.$$

This is closely related to what Yule (loc. cit.) has called the serial correlation of first order. If n is odd the substitution represents in n-space a rotation, the determinant being $+1$, and therefore preserves sense. If n is even, sense is reversed. The same statements hold for the hyperplane

$$x_1 + x_2 + \cdots + x_n = 0,$$

which is transformed into itself; for each of the regions into which this hyperplane divides the n-space goes into itself. They must therefore hold for the unit sphere in this hyperplane. The situation is described by the upper left and lower right compartments of the table above. If n is even we have, in fact, the sequence

$$1, -1, 1, -1, \cdots, 1, -1$$

which has a correlation -1 with its transformed sequence.

If n is odd, we shall show that for this transformation the frequency curve for r does not extend to either of the extreme values ± 1; there is a maximum value less than unity which the correlation cannot exceed numerically. If this were not true it would follow from continuity considerations that, for T or for TR, a fixed point on the $(n-2)$-dimensional sphere could be found. Indeed the distance from a point to its transform is a function varying continuously over a

finite region, and must therefore take a minimum value. Suppose, then, that a sequence of real numbers $x_1, \cdots, x_n$ can be found which is perfectly correlated with its transformed sequence. We can then find real numbers ρ and μ such that

$$\rho x_i + \mu = x_i' = x_{i+1}, \qquad (i = 1,2, \cdots, n-1),$$

$$\rho x_n + \mu = x_n' = x_1.$$

These are homogeneous linear equations for determining $x_1, \cdots, x_n$ and μ, and have a matrix which, for $n=5$, is

$$\begin{Vmatrix} \rho & -1 & 0 & 0 & 0 & 1 \\ 0 & \rho & -1 & 0 & 0 & 1 \\ 0 & 0 & \rho & -1 & 0 & 1 \\ 0 & 0 & 0 & \rho & -1 & 1 \\ -1 & 0 & 0 & 0 & \rho & 1 \end{Vmatrix}.$$

The determinant of the first n columns equals $\rho^n - 1$, which, since n is odd, vanishes only for $\rho = 1$. But if $\rho = 1$ the determinant of the last n columns equals n, as is seen by adding to the last row all the others. Hence the rank of the matrix is n. Hence all solutions of the equations are proportional to one solution. But one solution is $\mu = 1-\rho$, $x_1 = x_2 = \cdots = x_n = 1$, so that the x's must all be equal for every solution. No such point lies on our hypersphere; indeed the definition of the correlation coefficient presupposes that the x's are not all equal.

The permutation, related to the second serial correlation,

$$x_1' = x_3,\ x_2' = x_4, \cdots, x_{n-1}' = x_1,\ x_n' = x_2$$

supplies an example of a transformation which always preserves sense.

6. *Singular Transformations.* For continuous transformations of the sphere into a part of itself, the distribution curve for r must extend to both extremes. For, in Alexander's terminology, the index is zero, and there must therefore be

an invariant point. In this manner it may be shown that (as is also evident otherwise) a sequence may have a correlation of either 1 or -1 with its differences of any order.

7. *Dimensionality determines Order of Contact.* A further property of the distribution of r can be inferred from the dimensionality of the locus of invariant points. Let this locus π be of p dimensions. Let $u_1, u_2, \cdots, u_{n-2}$ be coordinates on the hypersphere such that $u_{p+1}, u_{p+2}, \cdots, u_{n-2}$ are zero at all points of π and which on the coordinate lines vary as the distance from π, apart from infinitesimals of higher order. We shall assume enough regularity in π and T so that the equations of the transformation may be written

$$u_1' - u_1 = a_{1,p+1}u_{p+1} + a_{1,p+2}u_{p+2} + \cdots + a_{1,n-2}u_{n-2} + \eta_1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$u_{n-2}' - u_{n-2} = a_{n-2,p+1}u_{p+1} + a_{n-2,p+2}u_{p+2} + \cdots + a_{n-2,n-2}u_{n-2} + \eta_{n-2},$$

where $\eta_1, \cdots, \eta_{n-2}$ are functions which vanish to the second or higher order with $u_{p+1}, \cdots, u_{n-2}$ and where the coefficients a_{ij} are either constants or functions of $u_1, \cdots, u_p$ and are not all identically zero. This condition on T will be satisfied for example if in a neighborhood of every point of π the transformation T can be approximated arbitrarily closely by a linear transformation. The transformation T is not restricted to be one-to-one and the matrix of the approximating linear transformation may be of any rank. In an extreme case every point might go into a point of π. The absence from the right-hand members of the equations of terms not involving $u_{p+1}, \cdots, u_{n-2}$ is due merely to the fact that all the expressions $u_i' - u_i$ must vanish on π.

In the right-hand members put $u_i = sv_i$ $(i = p+1, p+2, \cdots, n-2)$, where s is the distance from π. Since not all the a_{ij}'s are zero, at least one of the equations will have a term of the first degree in s.

The distance k from a point to its transform differs only by terms of higher order from the square root of a homo-

geneous positive definite quadratic form in $u_1' - u_1$, $u_2' - u_2$, $\cdots$, $u_{n-2}' - u_{n-2}$, so that we may write

$$k = as + bs^2 + \cdots,$$

the coefficients a, b, $\cdots$ depending on the direction from π, and a being positive everywhere except possibly in isolated directions, where it may vanish. By integrating over the manifold for which k is constant we find, denoting the mean value of s by ρ,

$$k = \alpha\rho + \beta\rho^2 + \cdots,$$

where α, β, $\cdots$ are constants and $\alpha > 0$. This step is justified by the fact that the manifold is closed and everywhere close to π if k is small, which follows from the positive definite character of the quadratic form in $u_1' - u_1$, $\cdots$, $u_{n-2}' - u_{n-2}$ giving the fixed number k^2.

Since $\alpha \neq 0$, we may invert the last series, obtaining

$$\rho = Ak + Bk^2 + \cdots.$$

Now π is the uniform limit of a family of closed non-intersecting $(n-3)$-dimensional loci, on each of which the distance k from a point to its transform is constant. For π, $k = 0$. The $(n-3)$-dimensional volume of one of these loci divided by the p-dimensional volume of π is a function of ρ whose Maclaurin expansion begins with a constant multiple of ρ^{n-p-3}. When this volume is expressed in terms of k its expansion will therefore begin with a term in k^{n-p-3}. The $(n-3)$-dimensional volume is proportional to the ordinate of the frequency curve of k. Since k, when small, differs only by infinitesimals of higher order from $(1-r^2)^{1/2}$ and therefore from $(1-r)^{1/2}$, it follows that the frequency for r, when expanded about $r = 1$, will begin with a term in $(1-r)^{(n-p-3)/2}(dk/dr)$, that is, in $(1-r)^{(n-p-4)/2}$. In this way the dimensionality of the locus of invariant points, which is a topological feature, determines the order of contact of the frequency curve with the axis at the upper extreme. In like manner the dimensionality of the locus for TR determines the order of contact at $r = -1$. If the locus con-

sists of several parts, p refers to the part having the greatest number of dimensions.

8. *Parameter Forms.* The transformation T may be considered not as a single transformation but as depending on parameters $t_1, t_2, \cdots, t_k$ which vary continuously. A frequency distribution of probability $\phi(t_1, t_2, \cdots, t_k)$ is then to be assumed for the parameters. Suppose the dimensionality p of the invariant locus is the same for all values of the parameters. The probability of a correlation between r and $r+dr$ in a sample of n, divided by dr, will be given by an expansion in powers of $1-r$ beginning with a constant times $\phi(t_1, \cdots, t_k)\ (1-r)^{(n-p-4)/2}$, provided the parameters have the particular values $t_1, \cdots, t_k$. Without this proviso, the probability is found by integration with respect to $t_1, \cdots, t_k$ over their entire field. Since r is constant with respect to this integration, and since ϕ is everywhere positive, it appears that the order of contact of the frequency curve for r with the axis is the same as before.

9. *Correlation between Variates in general.* This enables us to apply the result of §7 to the correlation between variates of which one is not fully determined by the other. Let the point on the $(n-2)$-dimensional sphere determined by the n values in a sample of the first variate be Q, and let $\phi(t_1, t_2, \cdots, t_{n-2})dt_1\ dt_2 \cdots dt_{n-2}$ be the probability that Q shall have coordinates differing from $t_1, \cdots, t_{n-2}$ respectively by less than $dt_1, \cdots, dt_{n-2}$. We now regard T as a transformation of the entire sphere into the point Q, which thus constitutes the invariant locus. Hence $p=0$ and the expansion in powers of $1-r$ begins with $(1-r)^{(n-4)/2}$. Contact is therefore of order $(n-2)/2$. If T be preceded by the transformation R of every point to its diametrical opposite it appears that the order of contact at $r=-1$ is the same.

These results agree with the known distribution*

* R. A. Fisher, *On the influence of rainfall on the yield of wheat at Rothamsted*, Philosophical Transactions of the Royal Society, vol. 213B (1923), p. 92. A more general distribution is given by Fisher in Biometrika, vol. 10 (1915).

$$\frac{1}{\pi^{1/2}}\frac{\Gamma[(n-1)/2]}{\Gamma[(n-2)/2]}(1-r^2)^{(n-4)/2}$$

of correlations in samples of n drawn from a normally distributed aggregate in which the variates are uncorrelated.

When, as in Yule's experiment mentioned in §2, $n=10$, contact will of course be of second order.

10. *Extension to Multiple Correlation.* To extend the results of the last section to multiple correlation, let the q points representing the independent variates on the hypersphere S_{n-2} which is the intersection of

$$S_{n-1}: \qquad x_1^2 + x_2^2 + \cdots + x_n^2 = 1,$$

and

$$V_{n-1}: \qquad x_1 + x_2 + \cdots + x_n = 0,$$

determine with the origin a hyperplane V_q within V_{n-1}. Let S_{q-1} be the hypersphere of $q-1$ dimensions in which V_q meets S_{n-2}. The multiple correlation coefficient R will be unity if the point representing the dependent variable lies on S_{q-1}. Here $p=q-1$; the series at $R=1$ will therefore begin with a multiple of $(1-R)^{(n-q-3)/2}$.

The other end of the range of the multiple correlation coefficient is at $R=0$. On S_{n-2} the locus of points for which $R=0$ is of $n-q-2$ dimensions. The $(n-3)$-dimensional volume at distance R from this locus is proportional to $R^{(n-3)-(n-q-2)}=R^{q-1}$, apart from higher powers of R. As a special case we have Fisher's formula, given in a slightly different form in the 1923 paper cited,

$$\frac{\Gamma[(n-1)/2]}{\Gamma[(n-q-1)/2]\Gamma(q/2)}R^{q-1}(1-R^2)^{(n-q-3)/2}$$

for the distribution of R in random samples from uncorrelated material, a formula which for $q=1$ reduces to that for r.

STANFORD UNIVERSITY

A GENERALIZATION OF RECURRENTS

BY MORGAN WARD

1. *Introduction.* It is well known that if

$$\phi(x) = \sum_{r=0}^{\infty} \phi_r x^r, \qquad \psi(x) = \sum_{s=0}^{\infty} \psi_s x^s$$

are two singly infinite series, then the coefficients in the expansion of $\phi(x)/\psi(x)$, $\log \phi(x)$, $e^{\phi(x)}$ can all be expressed as determinants in the quantities ϕ_r, ψ_s. These expressions are called *recurrents* and have been used by several writers* to evaluate determinants involving the binomial coefficients, Bernoulli numbers, etc.

In the present paper, the analogous results are given for the quotient of two *doubly* infinite series, and the logarithm and exponential of a doubly infinite series. The extension to m-tuply infinite series is briefly sketched in §8.

It is believed the expressions obtained are new; there is no reference to any such work in the four volumes of Muir's *History*. We assume throughout that all the series involved are absolutely convergent, so that the derangements and multiplications employed are justified. As a matter of fact, we are dealing essentially with infinite sets of quantities A_{rs}, B_{rs}, C_{rs}, $\cdots$, $(r, s, = 0, 1, 2, \cdots)$; the "variables" which appear in the series are merely convenient carriers for their coefficients.

We shall use, wherever convenient, the convention employed by writers on relativity for summations, namely,

$$U_{rs}x^r y^s,$$

which is taken to mean

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} U_{rs}x^r y^s,$$

the summations being understood.

* Muir's *History*, vols. II, III, IV, Chapters on recurrents.

2. *Degree and Rank.* Given a doubly infinite series

$$U(x,y) = U_{rs}x^r y^s, \tag{1}$$

we shall invariably write U in the order

$$U_{00} + U_{10}x + U_{01}y + U_{20}x^2 + U_{11}xy + U_{02}y^2 + \cdots,$$

or as

$$U(x,y) = \sum_{l=0}^{\infty} \sum_{k=0}^{l} U_{l-k,k}x^{l-k}y^k. \tag{2}$$

We define

$$l = (l-k) + k, \qquad u_{lk} = \frac{l(l+1) + 2(k+1)}{2},$$

as the *degree* and *rank* respectively of the coefficient $U_{l-k,k}$. Hence the degree of a coefficient is the degree of the term it multiplies. The rank of a coefficient is simply its place in the series (2). For since from (2) there are $l+1$ terms of degree l, the coefficient U_{l0} appears in the $[(1+2+3+\cdots+l)+1]$st place, that is,

$$u_{l0} = \frac{l(l+1)}{2} + 1.$$

The coefficient $U_{l-k,k}$ is k terms to the right of U_{l0}, so that its rank is

$$\frac{l(l+1)}{2} + 1 + k = \frac{l(l+1) + 2(k+1)}{2} = u_{lk}.$$

Thus for U_{rs}, the degree is $r+s$, and the rank is

$$u_{rs} = \frac{(r+s)(r+s+1) + 2(s+1)}{2}. \tag{3}$$

Moreover, it follows from the meaning of rank, that given any positive integer n, the equation

$$n = u_{rs} \tag{4}$$

determines a unique pair of non-negative integers r, s, and hence a unique coefficient U_{rs} in the series (2). Let k be any integer not greater than $r+s$. Then, by (3),

$$u_{r+s-k,k} = \frac{(r+s)(r+s+1)+2(k+1)}{2};$$

hence

$$u_{r+s-k,k} + s - k = \frac{(r+s)(r+s+1)+2(s+1)}{2} = u_{rs}.$$

In particular

$$u_{r+s,0} + s = u_{rs}, \qquad s \leqq r+s. \tag{5}$$

3. *Coefficients for a Product.* If

$$A(x,y) = A_{qr}x^q y^r,$$

$$B(x,y) = B_{st}x^s y^t,$$

then we know that

$$A(x,y) \cdot B(x,y) = C_{uv}x^u y^v,$$

where

$$C_{uv} = \sum_{\sigma=0}^{u} \sum_{\tau=0}^{v} A_{u-\sigma,v-\tau}B_{\sigma\tau}. \tag{6}$$

4. *Coefficients for a Quotient.* Consider now

$$P(x,y) = P_{uv}x^u y^v,$$

$$Q(x,y) = Q_{qr}x^q y^r,$$

and let

$$\frac{P(x,y)}{Q(x,y)} = Z(x,y) = Z_{st}x^s y^t, \tag{7}$$

where the coefficients Z_{st} are to be determined.

First, we can assume $Q_{00} \neq 0$. For, if $Q_{00} = Q_{10} = Q_{01} = \cdots = 0$, $Q_{ij} \neq 0$, multiply both sides of (7) by $x^i y^j$, replacing $Q(x, y)/x^i y^j$ by $Q'(x, y)$ and $x^i y^j Z(x, y)$ by $Z'(x, y)$ with $Z_{00}' = Z_{10}' = Z_{01}' = \cdots = 0$, $Z_{ij}' = Z_{00}$. We then have a new equality of the same form as (7) with $Q_{00}' = Q_{ij} \neq 0$. Thus $P(x, y) = Q(x, y)Z(x, y)$, or, by (6),

$$P_{uv} = \sum_{\sigma=0}^{u} \sum_{\tau=0}^{v} Q_{u-\sigma,v-\tau}Z_{\sigma\tau}, \qquad (u,v = 0,1,2,\cdots). \tag{8}$$

It may be noted in passing, that just as recurrents are related to difference equations with constant coefficients, so may (7), if $Q(x, y)$ be a polynomial, be looked upon as a linear partial difference equation with constant coefficients to determine Z_{uv}.

Let us introduce the symbol

$$(\lambda_{r-k,k}\ ;\ u - r + k,\ v - k),$$

defined by the relations

$$(9)\quad \begin{cases} \lambda_{r-k,k} = \dfrac{r(r+1) + 2(k+1)}{2}, \\ (\lambda_{r-k,k}\ ;\ u - r + k, v - k) = 0, \text{ if } r > u + k, \text{ or } k > v, \\ \qquad = Q_{u-r+k,v-k}, \text{ if } k \leqq v \text{ and } r \leqq u + k. \end{cases}$$

Then (8) may be written

$$(10)\quad \sum_{r=0}^{u+v} \sum_{k=0}^{r} (z_{r-k,k}\ ;\ u - r + k, v - k) Z_{r-k,k} = P_{uv},$$

$$(p_{uv} = 1, 2, 3, \cdots).$$

For $\lambda_{r-k,k} = z_{r-k,k}$, by definition of rank in (3). Moreover setting $r-k=\sigma$, $k=\tau$ in (10), we see that every term that occurs in (8) occurs in (10), and conversely.

Finally, by virtue of (9) we may replace (10) by

$$(11)\quad \sum_{r=0}^{n} \sum_{k=0}^{r} (z_{r-k,k}\ ;\ u - r + k, v - k) Z_{r-k,k} = P_{uv},$$

$$(p_{uv} = 1, 2, 3, \cdots),$$

where n is any integer $\geqq u+v$. Take $n = z_{ij}$ and consider the set of $n = p_{ij}$ equations (11), in the n unknowns Z_{00}, Z_{10}, Z_{01}, $\cdots$, Z_{ij},

$$(12)\quad \begin{cases} Q_{00} Z_{00} = P_{00}, \\ Q_{01} Z_{00} + Q_{00} Z_{10} = P_{10}, \\ \cdots\cdots\cdots\

Since $Q_{00} \neq 0$, we have, solving for Z_{ij} by determinants,

$$(13) \qquad Q_{00}^{n} Z_{ij} = \begin{vmatrix} Q_{00} & 0 & 0 & \cdots & 0 & P_{00} \\ Q_{10} & Q_{00} & 0 & \cdots & 0 & P_{10} \\ Q_{01} & 0 & Q_{00} & \cdots & 0 & P_{01} \\ Q_{20} & Q_{10} & 0 & \cdots & 0 & P_{20} \\ Q_{11} & Q_{01} & Q_{10} & \cdots & 0 & P_{11} \\ Q_{02} & 0 & Q_{01} & \cdots & 0 & P_{02} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ Q_{ij}, (2\,;\, i-1, j+1), (3, i-2, j+2) & \cdot & \cdot & \cdot & & P_{ij} \end{vmatrix},$$

where (13) is constructed on the following scheme. The elements in the sth column $(s < n)$ consist of $s-1$ zeros, then the coefficients of $Q(x, y)$ of degree zero, one, two, three, $\cdots$, in their proper order, the groups of coefficients of the same degree being separated by r' zeros, where $s = z_{r'-k', k'}$ determines r'. In fact, we see from (11), that the elements in the sth column are given by the expression

$$(14) \qquad (s\,;\, u - r' + k', v - k'),$$

where r' and k' are determined from the equation

$$z_{r'-k', k'} = s,$$

in accordance with (4), and (u, v) has the successive values

$$(15) \quad (0,0), (1,0), (0,1), (2,0), (1,1), \cdots, (u,v), \cdots, (i,j),$$

$n = \lambda_{ij}$ in number, (u, v) appearing in the λ_{uv}th place in (15). Now the $\sigma+1$ terms (uv) of constant degree $u+v=\sigma$, $(0 \leqq \sigma \leqq i+j)$, appear in the order

$$(16) \qquad (\sigma, 0), (\sigma - 1, 1), \cdots, (\sigma - k, k), \cdots, (0, \sigma).$$

If n is replaced u by $\sigma - v$, (14) becomes

$$(s\,;\, \sigma - r' + k' - v, v - k'),$$

so that for $v=0, 1, 2, \cdots, \sigma$, we have the values of (14) for the sequence (16). But this expression vanishes, by (9), unless

(a) $$v-k' \geqq 0,$$

(b) $$\sigma-r'+k'-v \geqq 0.$$

Thus the first k' terms of (16) yield k' zeros. Replace v by $r+k'$, where to satisfy (a) and (b) $0 \leqq r \leqq \sigma-r' \geqq 0$. We thus obtain from the next $\sigma-r'+1$ terms of (15),

$$(s\,;\,\sigma-r',0),(s\,;\,\sigma-r'-1,1),\cdots,(s\,;\,0,\sigma-r'),$$

or by (9),

$$Q_{\sigma-r',0},\quad Q_{\sigma-r'-1,1},\quad \cdots,\quad Q_{0,\sigma-r'},$$

the coefficients of $Q(x, y)$ of degree $\sigma-r'$ in their proper order. The remaining terms of (16) produce $\sigma+1-k'-(\sigma-r'+1)=r'-k'$ zeros.

Since the sequence of degree $\sigma+1$ following (16) produces k' zeros followed by

$$Q_{\sigma+1-r',0},\quad Q_{\sigma-r',1},\quad \cdots,\quad Q_{0,\sigma+1-r'},$$

we see that the coefficients of degree $\sigma-r'=0, 1, 2, 3, \cdots$ are separated by r' zeros, as stated. Q_{00} appears when $\sigma-r'=0$. From (a) and (b),

$$v=k',\qquad u=\sigma-v=r'-k'.$$

Hence Q_{00} appears in the $\lambda_{r'-k',k'}$ or the sth place in the column; i. e., in (13), the elements Q_{00} lie along the main diagonal. The last column in (13) consists of the elements $P_{00}, P_{10}, P_{01}, \cdots, P_{ij}$ in order of rank.

5. *Final Coefficients in the sth Column.* There is some doubt about the last few elements in the sth column, but this is obviated as follows. Take $s=z_{i+j-\tau,\tau}$, $(0 \leqq \tau \leqq j-1)$, i. e., consider the $(n-j)$th, $(n-j-1)$th, $\cdots$, $(n-1)$th columns of (13).

We have then $s=n-j+\tau$ so that the sth column contains $n-j+\tau$ zeros, Q_{00}, followed by $i+j$ zeros by our results in §4. But since there are only n elements in the column, Q_{00} is followed by $j-\tau-1$ zeros, since $j-\tau-1$ is always

less than $i+j$. Hence we can reduce (13) to a determinant of the $(n-j)$th order multiplied by Q_{00}^{j} to some power, for the j columns just considered consist entirely of zeros save along the main diagonal where Q_{00} appears.

Now $n=z_{ij}$ and $z_{i+j,0}+j=n$, by (5). Hence, setting $n-j=\nu$, we have

$$n-j=z_{i+j,0}=\frac{(i+j)(i+j+1)}{2}+1=\nu.$$

The elements in the νth row are now

$$Q_{ij},(2\,;\,i-1,j+1),(3\,;\,i-2,j+2),\cdots,(\nu-1\,;\,i-\nu+2,j+\nu-2),P_{ij}$$

so that the sth column terminates with

$$(s\,;\,i-s+1,j+s-1)$$

and in the $(\nu-1)$th row the elements are

$$(z_{r-k,k}\,;\,k-r,\ i+j-1-k)=\delta_{r+1,k+i}Q_{0,i+j-1-r},\qquad (r,k=0,1,2,\cdots,i+j-1),$$

where δ_{uv} is the Kronecker symbol.

Thus we have

$$Q_{0,i+j-1},0,\ Q_{0,i+j-2},0,0,\ Q_{0,i+j-3},0,0,0,\ \cdots,\ Q_{00},P_{0,i+j-1},$$

so that our evaluation of Z_{ij} gives us

$$(17)\quad Q_{00}^{\nu}Z_{ij}=\begin{vmatrix} Q_{00} & 0 & \cdots & P_{00}\\ Q_{10} & Q_{00} & \cdots & P_{01}\\ Q_{01} & 0 & \cdots & P_{10}\\ Q_{20} & Q_{10} & \cdots & P_{20}\\ Q_{11} & Q_{01} & \cdots & P_{11}\\ Q_{02} & 0 & \cdots & P_{02}\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ Q_{1,i+j-2} & \cdot & \cdots & \cdot\\ Q_{0,i+j-1} & 0 & \cdots & \cdot\\ Q_{ij} & (2\,;\,i-1,j+1), & \cdots & P_{ij} \end{vmatrix},$$

where

$$v = \frac{(i+j)(i+j+1)}{2} + 1.$$

6. *Expression of the Z's as Recurrents.* There remains still one more simplification; the quantities $Z_{i+j,0}, Z_{0,i+j}$ can be expressed as recurrents. For we obtain from (7), §4, by the ordinary multiplication rule

$$P_{uv} = \sum_{t=0}^{u} \sum_{s=0}^{v} Q_{ts} Z_{u-t,v-s}, \quad (u,v = 0,1,2,3, \cdots). \tag{18}$$

This result may be written

$$P_{uv} - R_{uv} = \sum_{t=0}^{u} Q_{t0} Z_{u-t,v}, \qquad (u = 0,1,2,3, \cdots), \tag{19}$$

where

$$R_{uv} = \sum_{t=0}^{u} \sum_{s=1}^{v} Q_{ts} Z_{u-t,v-s}, \tag{20}$$

so that

$$R_{u0} = 0,$$

$$R_{u1} = \sum_{t=0}^{u} Q_{t1} Z_{u-t,0},$$

$$R_{u2} = \sum_{t=0}^{u} Q_{t1} Z_{u-t,1} + \sum_{t=0}^{u} Q_{t2} Z_{u-t,0},$$

etc.

The formula (19) gives for $u = 0, 1, 2, 3, \cdots, i+j$, the set of $i+j+1$ equations

$$\begin{aligned}
Q_{00}\ Z_{0v} &= P_{0v} - R_{0v},\\
Q_{10}\ Z_{0v} + Q_{00}\ Z_{1v} &= P_{1v} - R_{1v},\\
Q_{20}\ Z_{0v} + Q_{10}\ Z_{1v} + Q_{00}\ Z_{2v} &= P_{2v} - R_{2v},\\
&\vdots\\
Q_{i+j,0} Z_{0v} + Q_{i+j-1,0} Z_{1v} + Q_{i+j-2,0} Z_{2v} + \cdots & \\
+ Q_{00} Z_{i+j,v} &= P_{i+j,v} - R_{i+j,v},
\end{aligned}$$

so that

$$(21)\qquad Q_{00}^{i+j+1} Z_{i+j,v} = \begin{vmatrix} Q_{00} & 0 & \cdots & P_{0v} - R_{0v} \\ Q_{10} & Q_{00} & \cdots & P_{1v} - R_{1v} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ Q_{i+j,0} & & \cdots & P_{i+j,v} - R_{i+j,v} \end{vmatrix}.$$

In particular

$$(22)\qquad Q_{00}^{i+j+1} Z_{i+j,0} = \begin{vmatrix} Q_{00} & 0 & \cdots & P_{00} \\ Q_{10} & Q_{00} & \cdots & P_{10} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ Q_{i+j,0} & & \cdots & P_{i+j,0} \end{vmatrix}.$$

which gives the required expression for $Z_{i+j,0}$ as a recurrent. From symmetry the expression for $Z_{0,i+j}$ is derived from (22) by simply interchanging the subscripts of all the terms in (21).

We may observe that, having obtained the quantities Z_{u0} by (22), we know R_{u1}, so that we can calculate the quantities $Z_{i+j-1,1}$ by means of (21). Proceeding thus step by step, we can finally calculate Z_{ij}.

There are a number of relations among the determinants (17), (22). For example, suppose we interchange x and y in the equation (7),

$$\frac{P(x,y)}{Q(x,y)} = Z(x,y).$$

The effect is merely to interchange the subscripts of the coefficients throughout. Hence in (17), we can interchange the subscripts of Z_{ij}, the Q's and the P's and obtain an expression for Z_{ji}, ν, the order of the determinant, being unaffected by the process. Again, we may write (7) as

$$\frac{P(x,y)}{Z(x,y)} = Q(x,y),$$

so that we can interchange the roles of the Q's and Z's in (17).

7. *Final Expressions for the Z's.* The first eleven coefficients in the development of

$$\frac{P(x,y)}{Q(x,y)} = Z_{00} + Z_{10}x + Z_{01}y + \cdots + Z_{03}y^3 + \cdots$$

are

$$Z_{00} = Q_{00}^{-1}P_{00},$$

$$Z_{10} = Q_{00}^{-2}\begin{vmatrix} Q_{00} & P_{00} \\ Q_{10} & P_{10} \end{vmatrix}, \qquad Z_{10} = Q_{00}^{-2}\begin{vmatrix} Q_{00} & P_{00} \\ Q_{01} & P_{01} \end{vmatrix},$$

$$Z_{20} = Q_{00}^{-3}\begin{vmatrix} Q_{00} & 0 & P_{00} \\ Q_{10} & Q_{00} & P_{10} \\ Q_{20} & Q_{10} & P_{20} \end{vmatrix}, \quad Z_{02} = Q_{00}^{-3}\begin{vmatrix} Q_{00} & 0 & P_{00} \\ Q_{01} & Q_{00} & P_{01} \\ Q_{02} & Q_{01} & P_{02} \end{vmatrix},$$

$$Z_{11} = Q_{00}^{-4}\begin{vmatrix} Q_{00} & 0 & 0 & P_{00} \\ Q_{10} & Q_{00} & 0 & P_{10} \\ Q_{01} & 0 & Q_{00} & P_{01} \\ Q_{11} & Q_{01} & Q_{10} & P_{11} \end{vmatrix},$$

$$= Q_{00}^{-4}\begin{vmatrix} Q_{00} & 0 & 0 & P_{00} \\ Q_{01} & Q_{00} & 0 & P_{01} \\ Q_{10} & 0 & Q_{00} & P_{10} \\ Q_{11} & Q_{10} & Q_{01} & P_{11} \end{vmatrix},$$

$$Z_{30} = Q_{00}^{-4}\begin{vmatrix} Q_{00} & 0 & 0 & P_{00} \\ Q_{10} & Q_{00} & 0 & P_{10} \\ Q_{20} & Q_{10} & Q_{00} & P_{20} \\ Q_{30} & Q_{20} & Q_{10} & P_{30} \end{vmatrix},$$

$$Z_{03} = Q_{00}^{-4}\begin{vmatrix} Q_{00} & 0 & 0 & P_{00} \\ Q_{01} & Q_{00} & 0 & P_{01} \\ Q_{02} & Q_{01} & Q_{00} & P_{02} \\ Q_{03} & Q_{02} & Q_{01} & P_{03} \end{vmatrix},$$

$$Z_{21} = Q_{00}^{-7}\begin{vmatrix} Q_{00} & 0 & 0 & 0 & 0 & 0 & P_{00} \\ Q_{10} & Q_{00} & 0 & 0 & 0 & 0 & P_{10} \\ Q_{01} & 0 & Q_{00} & 0 & 0 & 0 & P_{01} \\ Q_{20} & Q_{10} & 0 & Q_{00} & 0 & 0 & P_{20} \\ Q_{11} & Q_{01} & Q_{10} & 0 & Q_{00} & 0 & P_{11} \\ Q_{02} & 0 & Q_{01} & 0 & 0 & Q_{00} & P_{02} \\ Q_{21} & Q_{11} & Q_{20} & Q_{01} & Q_{10} & 0 & P_{21} \end{vmatrix},$$

$$Z_{12} = Q_{00}^{-7}\begin{vmatrix} Q_{00} & 0 & 0 & 0 & 0 & 0 & P_{00} \\ Q_{01} & Q_{00} & 0 & 0 & 0 & 0 & P_{01} \\ Q_{10} & 0 & Q_{00} & 0 & 0 & 0 & P_{10} \\ Q_{02} & Q_{01} & 0 & Q_{00} & 0 & 0 & P_{02} \\ Q_{11} & Q_{10} & Q_{01} & 0 & Q_{00} & 0 & P_{11} \\ Q_{20} & 0 & Q_{10} & 0 & 0 & Q_{00} & P_{20} \\ Q_{12} & Q_{11} & Q_{02} & Q_{10} & Q_{01} & 0 & P_{12} \end{vmatrix}.$$

8. *Quotient of two m-tuply Infinite Series.* The same method can be applied to the development of the quotient of two triply, or indeed of two m-tuply infinite series. We need only to generalize the formulas for degree and rank of §2, for product of two series in §3 and to introduce a symbol corresponding to the $(\lambda_{r-k,k};\ u-r+k,\ v-k)$ of §4 to obtain the analog of (17); the analog of (21) is obtained with equal ease. Thus for the m-tuply infinite series,

$$A(x_1, \cdots, x_m) = A_{i_1 i_2 \cdots i_m} x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m},$$

$i_1+i_2+\cdots+i_m$ is the *degree* of the coefficient A_i above and

$$(23)\quad a_{i_1 i_2 \cdots i_m} = \sum_{r=1}^{m} \binom{i_r + i_{r+1} + \cdots + i_m + m - r}{m - r + 1} + 1$$

is its *rank*, when A is written so that the terms of degree $0, 1, 2, \cdots, r, r+1, \cdots$ succeed each other in order, and the terms of degree r are arranged in alphabetic order.

For the product of two such series, we have the formula

$$A(x_1, \cdots, x_m) \cdot B(x_1, \cdots, x_m) = C(x_1, \cdots, x_m),$$

where

$$(24) \quad C_{j_1, \cdots, j_m} = \sum_{r=0}^{j} A_{j_1 - r_1, j_2 - r_2, \cdots, j_m - r_m} B_{r_1, r_2, \cdots, r_m}.$$

If we define $Z(x_1, \cdots, x_m)$ by

$$(25) \quad \frac{P(x_1, \cdots, x_m)}{Q(x_1, \cdots, x_m)} = Z(x_1, \cdots, x_m),$$

then our new symbol is

$$\Delta_{sj} = (\lambda_{sj}\,;\, j_1 - s_1 + s_2, j_2 - s_2 + s_3, \cdots, j_{m-1} - s_{m-1} + s_m, j_m - s_m),$$

defined by $\Delta_{sj} = 0$ if $j_r - s_r + s_{r+1}$ is negative for any r between 0 and $m+1$, and

$$(26) \quad \Delta_{sj} = Q_{j_1 - s_1 + s_2, \cdots, j_m - s_m},$$

if $j_r - s_r + s_{r+1}$ is positive for every r between 0 and $m+1$, and by convention $s_{m+1} = 0$. But our final results in the general case are completely obscured by the symbolism introduced to express them.

9. *Expansion of a Logarithm.* We can readily obtain the expansion of

$$(27) \quad \log Q(x, y) = Z(x, y)$$

where

$$Q(x, y) = Q_{qr} x^q y^r, \qquad Q_{00} \neq 0,$$
$$Z(x, y) = Z_{st} x^s y^t.$$

For, operating on (27) with

$$x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y},$$

we obtain a result of the form

$$\frac{P(x,y)}{Q(x,y)} = W(x,y),$$

where

$$W_{st} = (s+t)Z_{st}, \qquad P_{uv} = (u+v)Q_{uv},$$

by Euler's theorem on homogeneous functions and our convention as to the order of an infinite series.

Thus $Z_{00} = \log Q_{00}$; the other coefficients are derived from our previous expressions by replacing Z_{ij} by $Z_{ij}/(i+j)$ and P_{uv} by $(u+v)Q_{uv}$.

10. *Expansion of an Exponential.* For $e^{Q(x,y)}$, a slightly different procedure is necessary. Let

$$(28) \qquad \begin{cases} e^{Q(x,y)} = W(x,y), \\ \theta = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}. \end{cases}$$

We shall have

$$(29) \qquad \begin{cases} Q(x,y) = Q_{qr}x^q y^r, & \theta Q = (q+r)Q_{qr}x^q y^r, \\ W(x,y) = W_{st}x^s y^t, & \theta W = (s+t)W_{st}x^s y^t. \end{cases}$$

Then

$$(30) \qquad \theta e^Q = \theta Q e^Q = (\theta Q)\cdot W = \theta W.$$

Hence, by (6),

$$(31) \quad (u+v)W_{uv} = \sum_{\sigma=0}^{u}\sum_{\tau=0}^{v}(u+v-\sigma-\tau)Q_{u-\sigma,v-\tau}W_{\sigma\tau}.$$

Now as in §4 introduce a symbol

$$(\lambda_{r-k,k}\,;\, u-r+k, v-k)$$

defined as in (9), with one modification, namely, while

$$\lambda_{r-k,k} = \frac{r(r+1)+2(k+1)}{2}$$

and

$$(\lambda_{r-k,k}\,;\, u-r+k, v-k) = 0, \quad \text{if } r > u+k, \text{ or } k > v,$$
$$= (u+v-r)Q_{u-r+k,v-k},$$
$$\text{if } k \leqq v \text{ and } r \leqq u+k,$$

we have

$$(\lambda_{r-k,k}\,;\, u-r+k, v-k) = -(u+v),$$

for $k = v$ and $r = u+v$. Then (31) may be written in the form

$$(32) \qquad \sum_{r=0}^{u+v} \sum_{k=0}^{r} (W_{r-k,k}\,;\, u-r+k, v-k) W_{r-k,k} = 0,$$

just as (10) was equivalent to (8) in §4. Also $(\lambda_{00}; 0, 0) = 0$, but from (28) we see that

$$e^{Q_{00}} = W_{00} = -P_{00}, \text{ say.}$$

Thus (28) becomes equivalent to (10) if we replace in each equation P_{uv} by 0 for $u+v>0$ and $(\lambda_{uv}; 0, 0)$ by $-(u+v)$, instead of by Q_{00}. We thus obtain the following set of w_{ij} equations for W_{ij}:

$$\begin{aligned}
-W_{00} &= P_{00},\\
1\cdot Q_{10}W_{00} - 1\cdot W_{10} &= 0,\\
1\cdot Q_{01}W_{00} + 0\cdot W_{10} - 1\cdot W_{01} &= 0,\\
2\cdot Q_{20}W_{00} + 1\cdot Q_{10}W_{10} + 0\cdot W_{01} - 2W_{20} &= 0,\\
2\cdot Q_{11}W_{00} + 1\cdot Q_{01}W_{10} + 1\cdot Q_{10}W_{01} + 0\cdot W_{20} - 2W_{11} &= 0,\\
2\cdot Q_{02}W_{00} + 0\cdot W_{10} + 1\cdot Q_{01}W_{01} + 0\cdot W_{20} &\\
+ 0\cdot W_{11} - 2W_{02} &= 0,\\
\cdots\cdots\cdots

The determinant of this system is

$$(-1)(-1)^2(-2)^3(-3)^4\cdots$$
$$(-i-j+1)^{i+j}(-i-j)$$
$$=(-1)^{w_{ij}}\,1^2\cdot 2^3\cdot 3^4\cdots(i+j-1)^{i+j}(i+j)^j.$$

Just as before, if we solve for W_{ij}, we can reduce the determinant we obtain corresponding to (12) to one of the νth order,

$$\nu=\frac{(i+j)(i+j+1)}{2}+1.$$

But we can also develop this expression with respect to its last row which is P_{00}, 0, 0, $\cdots$, 0, obtaining a determinant of the $(\nu-1)$st order with a factor $(-1)^{\nu-1}$. Hence our final form for W_{ij} is

$$1^2\cdot 2^3\cdot 3^4\cdots(i+j-1)^{i+j}W_{ij}$$
$$=-\begin{vmatrix} Q_{10} & -1 & 0 & 0 & 0 & \cdots & 0\\ Q_{01} & 0 & -1 & 0 & 0 & \cdots & 0\\ 2Q_{20} & Q_{10} & 0 & -2 & 0 & \cdots & 0\\ 2Q_{11} & Q_{01} & Q_{10} & 0 & -2 & \cdots & 0\\ 2Q_{02} & 0 & Q_{01} & 0 & 0 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot\\ & & & & & & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -(i+j-1)\\ (i+j)\,Q_{ij} & (2,i-1,j+1) & & & & \cdots & \end{vmatrix}.$$

As before, we can interchange subscripts of all the Q's to obtain W_{ji}. We can also express $W_{i+j,0}$ and $W_{0,i+j}$ as recurrents; thus

$$(i+j-1)!W_{i+j,0} = -\begin{vmatrix} Q_{10} & -1 & & & 0 \\ 2Q_{20} & Q_{10} & -2 & & 0 \\ 3Q_{30} & 2Q_{20} & Q_{10} & -3 & 0 \\ \vdots & & & & \vdots \\ & & & & -(i+j) \\ (i+j)Q_{i+j,0} & \cdots & & & Q_{10} \end{vmatrix}$$

with a similar expression for $W_{0,i+j}$.

11. *Conclusion.* It hardly seems necessary to give numerical examples of these expansions. As in the case of recurrents, from expressions of such generality any desired example may be derived by a mere substitution of numbers for letters in the general formulas. The quotient of two polynomials, the reciprocal of a series or a polynomial, for example, are included as special cases.

It appears from the expressions for Z_{21} and Z_{12} in §7, that a further immediate reduction of the order of the determinants (17) is sometimes possible; but to explicate this reduction in the general case would be to mar the simplicity and symmetry of our developments.

In conclusion I should like to thank Professor E. T. Bell for criticism and suggestions in the writing of this paper.

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A CORRECTION

In the paper by H. W. March, *The Heaviside operational calculus*, this Bulletin, vol. 33(1927), on page 312, in the line following equation (2), change "*negative*" to "*positive*."

APPELL ON TENSOR CALCULUS

Traité de Mécanique Rationnelle. Vol. V: *Eléments de Calcul Tensoriel, Applications Géométrique et Mécanique.* By Paul Appell with the collaboration of René Thiry. Paris, Gauthier-Villars, 1926. vi+198pp.

This work of Appell on the absolute differential calculus and its applications forms the 5th volume of his *Traité de Mécanique Rationnelle* and is intended as the first part of a treatment of the mechanics of the theory of relativity. Yet it would seem that the present is no time for the appearance of such a text. The uncertain position of the general theory of relativity as shown by the unsuccessful attempts of Einstein and others to develop a field theory of electromagnetic phenomena, together with the new views, beginning with Heisenberg and ending with Schrödinger, which have lately been advanced, show that we do not yet stand on solid ground. It may be necessary to make substantial changes in the mathematical foundations of relativity theory. As a consequence it is better to regard this last book of Appell merely as an elementary exposition of some work in pure mathematics without regard to applications.

One feature of the book which deserves considerable adverse criticism is the lack of any *real* application to mechanics. A few of the simpler equations of mechanics have been written in the tensor notation on the basis of the three-dimensional euclidean geometry. But no applications to mechanics of the more general mathematical developments have been made.

The first chapter is devoted to certain fundamental theorems on linear and quadratic forms, and as such is introductory to the tensor calculus proper; an especially small type of printing is employed. A treatment of these fundamental theorems is usually omitted by writers on this subject. I regard this chapter as a valuable addition to the book.

Owing to the all too many books which have appeared in recent years on the theory of relativity, as well as the increasing number of mathematical papers written on the basis of the tensor calculus, there does not remain much room for originality in the second chapter which is devoted to the elements of this subject. However, on account of the importance of this chapter for the later developments, it may be well to consider it in some detail. Going out from the two fundamental requirements governing the equations of transformation of *a system of functions* (f), see §20, the system of functions known as a tensor is introduced as a simple system satisfying the fundamental requirements. After this the rules of tensor algebra are laid down, the fundamental quadratic differential form $g_{\alpha\beta}dx^{\alpha}dx^{\beta}$ is produced, and the differential equations of geodesics are developed in the ordinary manner. Then follows a cumbersome treatment of covariant differentiation of a tensor on the basis of the artificial and foreign idea of a comparison of tensors at different points of the manifold. In reality, co-

variant differentiation of a tensor has its natural origin in the determination of the conditions of integrability of the equations of transformation of the components of the tensor itself and those of the fundamental tensor $g_{\alpha\beta}$. Next the Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta}$ is deduced together with certain identities satisfied by it. The chapter closes with the derivation of certain well known formulas.

Much of the work in this chapter, and in the rest of the book for that matter, could be simplified very much by the systematic use of normal coordinates, and in particular by a consideration of the relationship of the process of covariant differentiation to this system of coordinates. In this connection there arise the "extensions" of a tensor which in general are distinct from its covariant derivatives and which have important theoretical applications. For example, to give an almost trivial case, the extensions of the fundamental tensor $g_{\alpha\beta}$ may be used to derive the complete set of identities of the Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta}$ from which *all* other identities satisfied by this tensor can be deduced by algebraic processes. Appell has nothing to say as to whether the identities (27) p. 58 constitute all the identities satisfied by the tensor $R_{\alpha\beta\gamma\delta}$, and I take it that he is not quite sure of himself on this point since the identities (27) are not all algebraically independent among themselves.

Now a remark regarding notations! It is somewhat helpful in calculation to write the Christoffel three-index symbols of the second kind in the inverted form $\left\{ {}^{i}_{\alpha\ \beta} \right\}$ or to denote them by a symbol such as $\Gamma^{i}_{\alpha\ \beta}$ in order that the summation of indices may be in conformity with the general rule (see §47, p. 47). For the purpose of emphasizing the operation of covariant differentiation Appell, following Galbrun, denotes, for example, by $\Delta_k A_r$ the covariant derivative of the vector A_r (see §47, p. 50). I do not regard this as a desirable notation. The objections to the cumbersome symbol $\sum$ to denote a summation likewise apply to the Δ notation, and in operations involving repeated covariant differentiation, where a number of Δ's appear before a tensor, these amount to nothing more than a succession of stumbling blocks in the way of ready calculation. What is needed is a notation for covariant differentiation such as $A_{r/k}$ or $A_{r,k}$ which will bring this operation into evidence but in as slight a manner as possible.

Beginning with the third chapter, the book will probably take on an added interest for most readers. The euclidean space of three dimensions is discussed with reference to general curvilinear coordinates. The formulas for area and volume are brought in, and an interesting geometrical interpretation of the covariant and contravariant components of a tensor of the first order is given. To this chapter also belong the mechanical formulas which I have already mentioned.

The fourth and fifth chapters deal with the euclidean and Riemannian spaces of n dimensions respectively. The fourth chapter contains the generalization of the geometrical portion of the third, and in addition a consideration of subspaces of the euclidean space, parallel displacement of vectors, etc. The proof of the sufficiency of the condition $R_{\alpha\beta\gamma\delta}=0$ for the space to be flat is very straightforward analytically and as good as I have ever come across. However, it is not quite clear just what is meant

by $\Delta_k a^i$ in the generalization of the first formulas of Serret-Frenet in §85, since a^i is a function of a single parameter and not of the independent variables (x), as is necessary for the process of covariant differentiation. Certainly the fifth chapter containing the treatment of Riemann space by the method of Levi-Civita, which consists in regarding this space as a subspace of a euclidean space of a larger number of dimensions, is one of the most interesting in the book. But from the standpoint of physical application, it is the intrinsic point of view essentially which is demanded.

For the most part, the sixth chapter is devoted to the presentation of the affine and metric manifolds of H. Weyl. The method of development is not that used by Weyl in *Raum, Zeit, Materie*, but one which has been used so extensively by Cartan in which a plane manifold is associated with each point of the given continuum. A portion of the chapter is also given to the spaces of torsion of Cartan and there is a short paragraph on the geometry of Eddington. This latter, however, gives a very incomplete account of the views of Eddington and it would seem that a theory which has had such a stimulating influence in this field could have more space allotted to its description.

My objection to the work of Cartan is that it consists in a large measure of a mere superabundance of geometrical terminology; and this applies even after its analytical structure has been improved by use of the tensor calculus. Its lack of content is evident when once the pure invariant theoretic viewpoint is adopted.

Finally, there is a chapter in small type on non-euclidean geometries as treated by Cayley, which might, perhaps, have been made more interesting and colorful by more verbal detail.

I believe, however, that the book will be very helpful to students wishing to gain a first insight into this mathematical subject, in spite of the criticisms which I have made, since as a whole it is written very clearly. It should prove quite satisfactory for use as an elementary treatise as intended, in fact, by its author. But this is no more than might be expected, for the genius of Appell as a clear and lucid expositor is recognized by everyone.

T. Y. Thomas

SHORTER NOTICES

Mathematical and Physical Papers, 1903–13. By Benjamin Osgood Peirce. Harvard University Press, 1926. 444 pp.

The publication of the collected papers of those who have made important contributions to science is always a valuable undertaking, and it is gratifying to have this volume which includes most of Professor Peirce's publications during the last ten years of his life. Most of the papers in this volume are concerned with the magnetic properties of iron. Professor Peirce devoted much time to the study of this subject, and his work is marked by the great care he took in his experiments and by his thoroughgoing analysis of the details of the problems involved; it is an admirable example of the best type of experimental work.

There are also included in this volume four papers of more mathematical interest, dealing with problems in the theory of the potential.

At the end of the volume there is a bibliography of all of Professor Peirce's publications. Of the 56 papers listed there, 21 are included in this volume.

E. P. Adams

Les Équations de la Dynamique de l'Éther. By Henri Eyraud. Paris, Blanchard, 1926.

Some years ago Cartan generalised Weyl's idea of space by supposing that the coefficient Γ^i_{pq} which occurs in the equations defining parallel displacement of a vector, changes, by an amount A^i_{pq} when the suffixes p and q are interchanged.

The new quantities A^i_{pq} can be regarded as the components of a tensor which is called the torsion of the space of affine connection. This tensor plays an important part in Eyraud's geometrical theory of the ether, for its contracted components represent the vector potential of the electromagnetic field.

When the torsion is zero and in this case only can we, by a suitable change of coordinates, annul at any given point all the components Γ^i_{pq}.

Eyraud shows that the space which supports electromagnetic actions is projectively Riemannian so that there is always a Riemann space having the same isotropic lines (rays of light) and the same geodesic lines (orbits for free motion) as the ether itself. It is Hamilton's principle which leads him to a knowledge of the true geodesic character of the ether. It gives besides the electromagnetic equations of Maxwell, the two corresponding systems of gravitational equations, and the second system of gravitational equations expresses precisely that the ether is applicable projectively and conformally on a Riemann space. He claims to have shown that the energy of the electromagnetic field and an electronic energy are the only types of energy both in the ether and inside the atoms.

H. Bateman

The Fundamental Concepts of Physics in the Light of Modern Discovery. By Paul R. Heyl. Baltimore, Williams and Wilkins, 1926. xii+112 pp.

This pleasing little book consists of a six-page foreword and three lectures on the changing bases of the fundamental concepts of physics. The first chapter deals with the eighteenth century, which is characterized as the century of materialism. The nineteenth century, treated in Chapter II, is called the century of correlation. In the third chapter is an account of fundamental physical concepts as they are conceived in the twentieth century. This century is characterized as the century of hope. The book is not such as should have a detailed review in a mathematical journal, but it is one of distinct interest to the mathematician who wishes to think of a part of his science in relation to the fundamental questions of physics with which certain domains of mathematics are intimately connected. It is a stimulating little volume.

R. D. Carmichael

Elementare Reihenlehre. By Hans Falckenberg. Berlin, Walter de Gruyter, 1926. 136 pp.

This small volume of the well known Sammlung Göschen begins with a chapter on the fundamental properties of real numbers. In particular, the author shows how to perform the four fundamental operations on irrational as well as rational numbers. Irrational numbers are introduced by means of the well known Dedekind cut.

A brief chapter follows, dealing with finite series and their sums. Here are found many useful formulas relating to finite arithmetic series of any order, and finite geometric series. Here also are found the binomial theorem, the polynomial theorem, and a discussion of the binomial coefficients.

In Chapter III, the fundamentals of the theory of infinite series are treated very fully. Beginning with sequences, the subject is developed by a succession of carefully worded definitions, theorems, and rules, including, of course, the usual well known tests for convergence. This portion of the book is arranged for ready reference by means of blackface marginal notations.

As the title indicates, the book deals only with the elementary portions of the theory of series. The topic of uniform convergence, for example, is not treated. The author does, however, include in the last chapter some of the more fundamental properties of power series. The convergence of the binomial series is discussed; also the hypergeometric series. Finally the elementary properties of the functions defined by the exponential series and the sine and cosine series are derived; the inversion of series is discussed; and the inverses of the exponential function and the trigonometric functions are studied.

Altogether, there is crowded into the 136 pages of this little work a surprising amount of useful material in an attractive and easily accessible form.

Louis Ingold

An Introductory Course of Mathematical Analysis. By Charles Walmsley. Cambridge, University Press, 1926.

This book, which commences with a preface by Dr. W. H. Young, contains an elementary but rigorous presentation of the theory of irrational numbers, limiting processes, infinite series, and definite integrals. The prominence given to inequalities, and more especially to inequation, is an interesting feature of the book. The treatment of the elementary functions is particularly good and the matter is well arranged. The index of symbols at the end of the book is a welcome addition. The book contains a number of suitable examples.

H. Bateman

La Théorie de la Relativité. Vol. II. By M. von Laue. Translated from the second German edition by Gustave Letang. Paris, Gauthier-Villars, 1926. xvi+312pp.

This French translation of a standard work will be of value to those of us, and we are many, who find French easier to read than German. Those familiar with the literature in this field need no introduction to von Laue. The field covered by this volume is the generalised theory of relativity; accompanied by very good introductions to tensor analysis (Chapter II) and non-euclidean geometry (Chapter IV). While the volume is by no means a popular exposition, the historical development of the subject has been attractively presented as an integral part of the more formal derivation of the theory.

C. N. Reynolds, Jr.

Aufgaben zur synthetischen Geometrie. By K. Kommerell. Leipzig, Teubner, 1925. 136 pp.

This is the first of a series devoted to solutions of questions in mathematics and in analytic mechanics set by members of the faculties of the University of Tübingen and of the Technische Hochschule at Stuttgart for the state examinations of candidates for positions as teachers in secondary schools.

The present volume is a collection of 113 problems in projective geometry, and the field is covered with rather surprising evenness. The questions are often of considerable difficulty, the solutions are comprehensive, and the extensive comments throw much light on the relations between the problems and the general theory. There are many references to original papers. Not infrequently the author comments frankly on the ease or difficulty of a question and on its value as an examination question.

For a first course in projective geometry about a third of the questions are suitable for assignment to students or for examinations. Certain others would be suitable for supplementing the ordinary material of the course.

B. H. Brown

NOTES

Professor J. R. Kline, of the University of Pennsylvania, has been appointed an associate editor of the Transactions of this Society.

The April, 1927, number of the Transactions of this Society (volume 29, No. 2) contains the following papers: *Singular case of pairs of bilinear, quadratic, or Hermitian forms*, by L. E. Dickson; *Triads of ruled surfaces*, by A. F. Carpenter; *Certain uniform functions of rational functions*, by E. P. Starke; *On rejection to infinity and exterior motion in the restricted problem of three bodies*, by B. O. Koopman; *A connected and regular point set which has no subcontinuum*, by R. L. Wilder; *Meromorphic functions with addition or multiplication theorems*, by J. F. Ritt; *Real functions with algebraic addition theorems*, by J. F. Ritt; *Concerning continua in the plane*, by G. T. Whyburn; *Extremals and transversality of the general calculus of variations problem of the first order in space*, by Jesse Douglas; *A figuratrix for double integrals*, by P. R. Rider; *Manifolds with a boundary and their transformations*, by Solomon Lefschetz; *On sets of functions of a general variable*, by L. L. Dines.

Two new volumes of the Society's Colloquium Series will appear in the summer of 1927: *Potential Theory*, by G. C. Evans, and *Algebraic Arithmetic*, by E. T. Bell. Orders for these volumes may be sent to the Society's office now, and the books will be sent when they appear; the price is $2.00 each (to members, $1.50). If desired, the members may have them charged on their bill of January, 1928. Two other volumes are in preparation and are expected to appear shortly: *The New Differential Geometry*, by L. P. Eisenhart, and *Dynamical Systems*, by G. D. Birkhoff.

The first issue of the American Journal of Mathematics in its new form, under the joint auspices of The Johns Hopkins University and this Society, as announced on page 1 of this volume of this Bulletin, has appeared as Number 1 of Volume 49. In addition to the Board of Editors previously announced, Professors E. T. Bell and F. D. Murnaghan are associate editors. This number contains the following papers: *Stability and the equations of dynamics*, by G. D. Birkhoff; *Quaternary quadratic forms representing all integers*, by L. E. Dickson; *Reduction formulas for the number of representations of integers in certain quadratic forms*, by E. T. Bell; *Generalization of certain theorems of Bohl, (second paper)*, by F. H. Murray; *Applications of the determinant and permanent tensors to determinants of general class and allied tensor functions*, by C. M. Cramlet; *Transformations leaving invariant certain partial differential equations of physics*, by R. D. Carmichael; *Transformations leaving invariant the heat equations of physics*, by J. A. Goff; *The application of fractional operators to functional equations*, by H. T. Davis; *Classification of quadrics in hyperbolic space*, by James Pierpont.

The following persons have been appointed associate editors of the Annals of Mathematics: as representing this Society, Professors D. C.

Gillespie, W. L. Hart, and R. E. Langer; as representing the Mathematical Association of America, Professors W. B. Ford, and Anna Pell-Wheeler.

The History of Science Society, in collaboration with committees of the American Mathematical Society, the Mathematical Association of America, the American Astronomical Society, and the American Physical Society, proposes to commemorate the 200th anniversary of the death of Sir Isaac Newton at a meeting to be held at Columbia University, November 25–26, 1927. Professor D. E. Smith is chairman of the program committee; the representatives of the American Mathematical Society and the Mathematical Association of America on this committee are professors R. C. Archibald, E. W. Brown, and Florian Cajori.

The Royal Academy of Sciences of Holland has awarded the Lorentz medal for distinguished service in the field of physics to Professor Max Planck, of the University of Berlin.

The Class of Physical Sciences of the Royal Academy of Bologna announces the following subjects for its Adolfo Merlani prize in mathematics: (1) to study by a direct method the problem of the extremals of the curvilinear integral $\int_C F(x, y; x', y'; x'', y'')dt$; (2) to present a monograph on conformal representation. Manuscripts or printed works published not earlier than 1927 on either subject, should be sent to the Secretary of the Class of Physical Sciences of the Academy before December 31, 1928. Competition is not limited to Italians.

Cambridge University announces the award of the following prizes: Smith's prizes to S. Goldstein, of St. John's College, and W. V. D. Hodge, of St. John's College, for essays entitled, respectively, *Mathieu functions* and *Linear systems of plane algebraic curves of any genus*; Rayleigh prizes to D. Burnet, of Clare College, and C. A. Meredith, of Trinity College, for essays entitled respectively, *Electric radiation over the earth's surface* and *Some theorems on infinite cardinals*. The subject announced for the Adams prize for 1927–28 is the following: *The variations of the earth's magnetic field in relation to electric phenomena in the upper atmosphere and on the earth*; a contribution to the theory of the origin of the various phenomena is desired.

The following have been elected foreign correspondents of the Reale Istituto Lombardo, Milan: Professors E. Borel and J. Hadamard, of Paris; Professors D. Hilbert and E. Landau, of Göttingen; Professor A. von Brill, of Tübingen.

Professor E. T. Bell, of the California Institute of Technology, has been elected a member of the National Academy of Sciences.

Professor G. D. Birkhoff, of Harvard University, has been elected an honorary member of the Edinburgh Mathematical Society.

The University of Lemberg has conferred an honorary doctorate on Professor R. A. Millikan, of the California Institute of Technology.

Professor E. W. Brown, of Yale University, has accepted the invitation of the Society to deliver its fifth Josiah Willard Gibbs Lecture, in connection with the meetings of the Society and the American Association for the Advancement of Science at Nashville, in December, 1927.

Professor Constantin Carathéodory, of the University of Munich, has been appointed visiting professor at Harvard University for the second half of the academic year 1927–28.

The following awards of Guggenheim fellowships for the coming academic year in the fields of mathematics and mathematical physics are announced: Dr. C. H. Eckert, of the California Institute of Technology, for research in quantum theory; Professor Philip Franklin, of the Massachusetts Institute of Technology, for the study of integral equations and orthogonal functions; Professor G. E. Gibson, of the University of California, for the study of the theory of band spectra; Dr. W. V. Houston, of the California Institute of Technology, for the study of quantum mechanics as applied to the explanation of spectra; Dr. F. C. Hoyt, of the University of Chicago, for research in quantum theory and its relation to radiation and atomic structure; Professor V. F. Lenzen, of the University of California, for the study of statistical mechanics; Professor M. S. Vallarta, of the Massachusetts Institute of Technology, to study the connection between Schrödinger's wave mechanics and the Einstein theory of relativity; Professor H. S. Vandiver, of the University of Texas, for research on Fermat's last theorem and the laws of reciprocity in the theory of algebraic numbers.

Professor Eugenie M. Morenus of Sweet Briar College has been awarded the Anna B. Brackett Memorial Fellowship by the American Association of University Women.

Dr. O. Mader has been appointed professor of mechanics at the Munich Technical School.

Dr. J. E. Lennard-Jones, of the University of Bristol, has been promoted to a professorship of theoretical physics.

The following have been appointed docents in European universities: Dr. Richard Brauer, in mathematics, at the University of Königsberg; Dr. Georg Feigl, in mathematics, at the University of Berlin; Dr. Pasqual Jordan, in theoretical physics, at the University of Göttingen; Dr. Kornel Lanczos, in theoretical physics, at the University of Frankfurt a. M.; G. Mammana and E. Raimondi, in algebraic analysis and rational mechanics, respectively, at any Italian universities of their choice.

Princeton University has established a professorship in memory of Charles A. Young, who was professor of astronomy at the University from 1887 to 1908. Professor H. N. Russell, director of the Princeton Observatory, has been made the first incumbent.

Yale University plans to honor the memory of Josiah Willard Gibbs by the establishment of a Gibbs Fund of $250,000, the income of which will be devoted to the departments of chemistry, physics, and mathematics.

The list of doctorates conferred by American Universities during 1926, as published in the May–June issue of this Bulletin, should have included the following:

Christine Ladd-Franklin, Johns Hopkins, February, *On the algebra of logic.*

G. M. Merriman, Cincinnati, June, *Sufficient conditions for the summability of double series, with applications to the double Fourier series.*

The following graduate courses in mathematics are announced or the academic year 1927–1928.

Brown University.—By Professors R. G. D. Richardson and C. R. Adams: Introduction to advanced analysis.—By Professors J. D. Tamarkin and C. R. Adams: Seminary course.—By Professor A. A. Bennett. Advanced algebra.—By Professor J. D. Tamarkin: Theory of differential equations.

University of California.—By Professor Florian Cajori: Seminar on foundations of the calculus, I; Seminar on history of algebra, II; The teaching of mathematics in secondary schools, I.—By Professor B. A. Bernstein: Logic of algebra, I; Logic of Geometry, II; Seminar on foundations of mathematics, II.—By Professor M. W. Haskell: Line geometry, I; Theory of continuous groups.—By Professor D. N. Lehmer: Algebraic curves and surfaces, II; Theory of Numbers, II.—By Professor Pauline Sperry: Differential geometry, I; Projective differential geometry, II.—By Professor J. H. McDonald: Functions of a complex variable; Advanced analytic mechanics.—By Professor T. M. Putnam: Special analytic functions, I; Partial differential equations, II.—By Professor Frank Irwin: Integral equations, I; Galois theory of equations, II.—By Professor C. A. Noble: Calculus of variations.

University of Chicago.—By Professor E. H. Moore: General analysis, I, II, III, IV, V; Seminar on foundations of mathematics.—By Professor H. E. Slaught: Differential equations; Definite integrals.—By Professor G. A. Bliss: Calculus of variations and Riemannian geometry; Partial differential equations; Algebraic functions; Boundary value problems in the calculus of variations; Thesis work in analysis.—By Professor L. E. Dickson: Topics in algebra; Topics in the theory of numbers; Advanced topics in algebra and the theory of numbers; Thesis work in algebra and theory of numbers.—By Professor E. P. Lane: Analytic projective geometry; Space curves and ruled surfaces; Solid analytic geometry; Differential geometry.—By Professor A. C. Lunn: Statistics and probability; Vector analysis; Dyadics and crystal physics; Fourier series and Bessel functions; Units and dimensions; Vector analysis in Riemannian-Einstein space.—By Professor M. I. Logsdon: Advanced analytic geometry; Algebraic geometry; Algebraic invariants.—By Pro-

fessor W. D. MacMillan: Analytic mechanics I, II; Celestial mechanics; Dynamics of rigid bodies.—By Professor L. M. Graves: Vectors, matrices and quaternions; Functions of a complex variable; Functions of a real variable.—By Professor R. W. Barnard: Limits and series.—By Professor Walter Bartky: Modern theories of differential equations, I, II; Vector analysis; Electrodynamics; Quantum mechanics. In connection with all advanced courses students may register for Reading and Research.

University of Cincinnati.—By Professor Harris Hancock: Advanced calculus; Theory of algebraic numbers; Thesis work in algebraic numbers.—By Professor Louis Brand: Mechanics of deformable bodies; Electrodynamics.—By Professor C. N. Moore: Theory of functions of a real variable; Course in reading and research.—By Professor R. E. Hundley: Advanced technical mechanics.—By Professor C. A. Garabedian: Elasticity, II; Thesis work in elasticity.—By Professor W. C. Osterbrock: The differential equations of engineering.—By Professor I. A. Barnett: Integral equations.—By Professor Meyer Salkover: Quantum theory and atomic structure.

Columbia University.—By Professor T. S. Fiske: Elementary exposition of modern advances in mathematical science; Theory of functions.—By Professor E. Kasner: Seminar in differential geometry.—By Professor W. B. Fite: Differential equations.—By Professor J. F. Ritt: Elliptic functions.—By Professor G. A. Pfeiffer: Principles and scope of geometry.—By Dr. B. O. Koopman: Mathematical theory of deformable media, with applications to hydro-dynamics and elasticity; Partial differential equations of mathematical physics.

Cornell University.—By Professor J. I. Hutchinson: Theory of functions of a complex variable.—By Professor Virgil Snyder: Projective geometry.—By Professor F. R. Sharpe: Hydro-dynamics and elasticity.—By Professor Arthur Ranum: Theory of numbers.—By Professor W. A. Hurwitz: Differential equations of mathematical physics.—By Professor W. B. Carver: Theory of finite groups.—By Professors Hutchinson and Carver: Elementary differential equations.—By Professor D. C. Gillespie: Principles of mechanics.—By Professor C. F. Craig: Modern higher algebra.—By Mr. P. A. Fraleigh: Advanced calculus.—By Dr. B. F. Kimball: Differential geometry of curves and surfaces.—By Mr. H. Poritsky: Graphical and mechanical computations; Wave motion.—By Mr. H. C. Shaub: Advanced analytic geometry.

Harvard University.—By Professor W. F. Osgood: Functions of real variables, I.—By Professor J. L. Coolidge: Higher geometry.—By Professor E. V. Huntington: Mathematical methods in statistics.—By Professor G. D. Birkhoff: Problem of three bodies.—By Professor W. C. Graustein: Introduction to modern geometry; Advanced calculus, I; Projective geometry.—By Professor Marston Morse: Theory of potential II; Functions of real variables, II.—By Professor H. W. Brinkman: Advanced calculus, II; Theory of numbers.—By Dr. H. M. Stone: Probability

calculus, II; Theory of numbers.—By Dr. M. H. Stone: Probability; Analytical theory of heat, problems in elastic vibrations; Expansion problems connected with ordinary differential equations.—By Dr. M. S. Demos: Calculus of variations. Professor Morse will conduct a fortnightly seminar in the theory of functions. Courses of research are offered by Professor Osgood in the theory of functions, by Professor Coolidge in geometry, by Professor Huntington in postulate-theory, by Professor Birkhoff in the theory of differential equations, by Professor Graustein in geometry, by Professor Morse in analysis situs, by Professor Walsh in analysis, by Professor Brinkmann in the theory of groups, and by Dr. Stone in expansion problems.

University of Illinois.—By Professor E. J. Townsend: Functions of a complex variable.—By Professor G. A. Miller: Theory of groups. —By Professor J. B. Shaw: Linear algebra.—By Professor R. D. Carmichael: Linear differential equations.—By Professor A. Emch: Algebraic geometry.—By Professor A. R. Crathorne: Theory of probability.—By Professor O. C. Hazlett: Theory of numbers.

University of Iowa.—By Professor H. L. Rietz: Actuarial theory; Statistics; Seminar in actuarial science and statistics.—By Professor E. W. Chittenden: Advanced calculus; Functions of a real variable. —By Professor R. P. Baker: Theoretical mechanics; Higher algebra.—By Professor J. F. Reilly: Partial differential equations; Mathematics of finance; Finite differences.—By Professor Roscoe Woods; Advanced analytic geometry; Solid analytic geometry.—By Professor C. C. Wylie: Celestial mechanics.—By Dr. L. E. Ward: Functions of a complex variable. —By Dr. N. B. Conkwright: Differential equations; Theory of equations.

Johns Hopkins University.—By Professor F. Morley: Inversive By Professor A. Cohen: Theory of functions; Differential equations and advanced calculus.—By Professor F. D. Murnaghan: Hydro dynamics; Elasticity; Tensor analysis; New quantum mechanics.—By Professor J. R. Musselman: Projective geometry.

Massachusetts Institute of Technology.—By Professor F. S. Woods: Advanced calculus.—By Professor D. P. Bartlett: Least squares and probability.—By Professor C. L. E. Moore: Theoretical aeronautics; Advanced wing theory; Rigid dynamics.—By Professor H. B. Phillips: Vector analysis; Theory of the gyroscope.—By Professor G. Rutledge: Theory of functions; Modern algebra.—By Dr. D. J. Struik: Differential geometry.—By Professor N. Wiener: Fourier series and integral equations.

University of Michigan.—By Professor J. L. Markley: Studies in the differential and integral calculus.—By Professor J. W. Glover: Theory of probability; Finite differences; Advanced mathematical theory of interest and life contingencies.—By Professor W. B. Ford: Advanced calculus with especial reference to Fourier series and harmonic analysis; Infinite series and products; Advanced differential equations.— By Professor L. C. Karpinski: Higher algebra; Teachers' seminary in

mathematics; History of mathematics.—By Professor Peter Field: Advanced mechanics; Theory of the potential; Vector analysis.—By Professor T. R. Running: Graphical methods; Empirical formulas; Differential equations for chemical engineers; Mathematical theory of heat conduction. —By Professor J. W. Bradshaw: Descriptive geometry; Projective geometry.—By Professor T. H. Hildebrandt: Theory of functions of a real variable.—By Professor C. E. Love: Infinite processes; Differential equations.—By Professor H. C. Carver: Mathematical theory of statistics; Advanced mathematical theory of interest and life contingencies.—By Professor L. A. Hopkins: Celestial mechanics.—By Professor V. C. Poor: Elements of elasticity and hydrodynamics.—By Professor C. J. Coe: Analytic mechanics; Integral equations.—By Professor L. J. Rouse: Advanced calculus; Fourier series and harmonic analysis.—By Professor W. W. Denton: Advanced calculus; Elements of mechanics; Partial differential equations of physics.—By Professor N. H. Anning: Differential equations; Theory and use of mathematical instruments.—By Professor J. A. Shohat: Selected topics in analysis.—By Professor J. A. Nyswander: Higher algebra; Algebraic theory; Modern theory of differential equations. —By Professor G. Y. Rainich: Quadratic forms and quadratic numbers; Theory of functions of a complex variable; Differential geometry; Mathematics of relativity.—By Professor R. L. Wilder: Analysis situs.—By Mr. D. K. Kazarinoff: Projective geometry for engineers; Calculus of variations; Mathematical theory of aerofoils; Advanced stability.—By Mr. O. J. Peterson: Solid analytic geometry.—By Mr. S. E. Field: Differential equations.

Ohio State University.—By Professor H. W. Kuhn: Theory of equations; Ordinary differential equations.—By Professor S. E. Rasor: Functions of a complex variable.—By Professor H. Blumberg: Introduction to modern mathematics; Point sets; Problems in analysis.—By Professor C. C. Morris: Theory of probability; Advanced statistics.—By Professor J. H. Weaver: Advanced euclidean geometry; Advanced calculus.—By Professor C. C. MacDuffee: Theory of numbers; Theory of algebraic numbers; Linear algebras.—By Professor A. D. Michal: Fourier series; Partial differential equations; Calculus of variations.—By Professor Grace Bareis: Synthetic projective geometry; Advanced analytic geometry.—By Professor C. T. Bumer: Finite differences; Actuarial theory; Vector analysis.

Princeton University.—By Professor H. B. Fine: Theory of elimination, (first term).—By Professor L. P. Eisenhart: Differential geometry; Riemannian geometry.—By Professor O. Veblen: Seminary.—By Professor J. H. M. Wedderburn: Linear algebras, (second term).—By Professor S. Lefschetz: Analysis situs; Functions of a complex variable.—By Professor J. W. Alexander: Functions of a real variable; Partial differential equations

Rice Institute.—By Professor G. C. Evans: Differential and integral equations; General dynamics and relativity.—By Professor L. R. Ford: Theory of functions of a complex variable; Algebraic functions and their

integrals.—By Professor H. E. Bray: Theory of functions of a real variable. —By Dr. A. H. Copeland: Finance: Statistics; Probability.

University of Texas.—By Professor M. B. Porter: Analytic functions.—By Professor R. L. Moore: Point sets and continuous transformations; Research in point-set theory.—By Professor E. L. Dodd: Infinite processes; Research in probability.—By Professor H. J. Ettlinger: Theory of elasticity; Research in differential equations.—By Professor P. M. Batchelder: Relativity.—By Professor A. E. Cooper: Continuous groups.

Yale University.—By Professor James Pierpont: Theory of functions of a complex variable; Differential geometry, I.—By Professor P. F. Smith: Geometrical transformations and continuous groups.—By Professor E. W. Brown: Advanced mechanics.—By Professor W. R. Longley: Approximation methods.—By Professor W. A. Wilson: Functions of real variables.—By Professor E. J. Miles: Calculus of variations, I.—By Professor J. K. Whittemore: Differential geometry, II.—By Professor J. I. Tracey: Analytic geometry, I.—By Dr. L. T. Moore: Higher Algebra. —By Dr. H. M. Gehman: Plane analysis situs.—By Dr. C. A. Shook: Potential theory and Laplace's equations; Hydromechanics.

Professor A. A. Bennett, of Lehigh University, has been appointed professor of mathematics at Brown University.

Dr. H. W. Brinkmann, of Harvard University, has been promoted to an assistant professorship of mathematics.

Professor J. I. Coolidge, of Harvard University, has been appointed exchange professor to France; he will lecture at the Sorbonne on algebraic plane curves.

Professor Tobias Dantzig, of the University of Maryland, lectured during the academic year 1926–27 at the Bureau of Standards on the mathematical theory of elasticity.

Professor W. C. Eells, of the department of applied mathematics at Whitman College, has been appointed associate professor of education at Stanford University.

Professor Tomlinson Fort, of Hunter College, has been appointed head of the department of mathematics at Lehigh University.

Dr. H. M. Gehman, of Yale University, has been promoted to an assistant professorship of mathematics.

Professor R. L. Green, of Stanford University, has retired.

Dr. C. M. Huber, of Rutgers University, has been promoted to an assistant professorship of mathematics.

Dr. L. S. Hulburt, collegiate professor of mathematics at Johns Hopkins University, has retired.

Assistant Professor M. H. Ingraham, of Brown University, has accepted a professorship of mathematics at the University of Wisconsin.

Associate Professor O. D. Kellogg, of Harvard University, has been promoted to a full professorship of mathematics.

Professor C. J. Keyser, of Columbia University, has retired, as Adrain professor emeritus.

Assistant Professor R. E. Langer, of Brown University, has accepted a professorship of mathematics at the University of Wisconsin.

Dr. Harry Levy has been appointed assistant professor of mathematics at the University of Illinois.

Associate Professor I. Maizlich has been promoted to a full professorship at Centenary College.

Professor E. D. Meacham has been made assistant dean of the College of Arts and Sciences at the University of Oklahoma.

Dr. A. D. Michal has been appointed assistant professor of mathematics at the Ohio State University.

Professor J. J. Nassau, of Case School of Applied Science, has been granted a year's leave of absence, beginning next September, to study at Cambridge University.

Dr. Oystein Ore, of the University of Oslo, has been appointed assistant professor of mathematics at Yale University.

Assistant Professor G. E. Raynor, of Wesleyan University, has been appointed associate professor of mathematics at the University of Oklahoma.

Associate Professor W. D. Reeve, of Teachers' College, Columbia University has been promoted to a full professorship of mathematics.

Dr. D. E. Richmond has been appointed assistant professor of mathematics at Williams College.

Assistant Professor J. F. Ritt has been promoted to an associate professorship of mathematics at Columbia University.

Dr. C. A. Shook has been promoted to an assistant professorship of mathematics at Yale University.

Assistant Professor J. D. Tamarkin, of Dartmouth College, has been appointed assistant professor of mathematics at Brown University.

Mr. H. S. Thurston, of Brown University, has been appointed professor of mathematics at Acadia University.

Associate Professor W. A. Wilson has been promoted to a full professorship at Yale University.

The following appointments to instructorships in mathematics are announced:

Brown University, Mr. A. O. Hickson;

Harvard University, Dr. M. S. Demos, Dr. M. H. Stone of Columbia University;

University of Iowa, Mr. John Stehn;

Yale University, Mr. H. T. Engstrom, Mr. T. H. Rawles.

Dr. A. H. Bucherer, professor of mathematical physics at the University of Bonn, died April 16, 1927, at the age of sixty-two.

Dr. Franz Mertens, formerly professor of mathematics at the University of Vienna, is dead.

M. Daniel Berthelot, professor of physics at the University of Paris, has died, at the age of sixty-two.

Mr. H. B. Goodwin, formerly examiner in nautical astronomy at the Royal Naval College, Greenwich, died February 24, 1927, at the age of seventy-nine.

Professor R. F. Borden, of George Washington University, died March 15, 1927. Professor Borden had been a member of this Society since 1920.

Dr. F. H. Loud, emeritus professor of mathematics and astronomy at Colorado College, has died, at the age of seventy-five.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

AGOSTINI (A.) e BORTOLOTTI (E.). Esercizi di geometria analitica. Volume 2. Bologna, Zanichelli, 1926.

BALSER (L.). Sphärische Trigonometrie. Kugelgeometrie in konstruktiver Behandlung. (Mathematisch-Physikalische Bibliothek.) Leipzig, Teubner, 1927. 52 pp.

BARNETT (I. A.). Plane analytic geometry. New York, Wiley, 1926. 9+269 pp.

BERWICK (W. E. H.). Integral bases. (Cambridge Tracts, No. 22.) London, Cambridge University Press, 1927. 95 pp.

BIERBERBACH (L.). Einführung in die konforme Abbildung. 2te, neubearbeitete Auflage. (Sammlung Göschen.) Berlin, de Gruyter, 1927. 131 pp.

BLOCH (A.). Les fonctions holomorphes et méromorphes dans le cercle-unité. (Mémorial des Sciences Mathématiques, No. 20.) Paris, Gauthier-Villars, 1926. 60 pp.

BORTOLOTTI (E.). See AGOSTINI (A.).

BREUSCH (F.). Ziele und Wege des Unterrichts in Mathematik und exakten Wissenschaften. I: Mathematik. Karlsruhe, G. Braun, 1926. 97 pp.

BUCHHOLZ (H.). Das Problem der Kontinuität. (Neue Psychologische Studien, herausgegeben von F. Krueger, 3ter Band, 1tes Heft.) München, Beck'sche Verlags-Buchhandlung, 1927. 132 pp.

BURALI-FORTI (C.). Geometria analitico-proiettiva. 2a edizione. Torino, Petrini, 1926. 9+290 pp.

ČECH (E.). See FUBINI (G.).

COOLIDGE (J. L.). Einführung in die Wahrscheinlichkeitsrechnung. Deutsche Ausgabe von F. M. Urban. Leipzig, Teubner, 1927. 10+212 pp.

LE CORBEILLER (P.). Contribution à l'étude des formes quadratiques à indéterminées conjuguées. Toulouse, Librairie de l'Université, 1926. 179 pp.

DALGLEISH (I. S.). The lightning graphs. Series I. London, Crosby Lockwood, 1926.

ENRIQUES (F.). Courbes et fonctions algébriques d'une variable. Traduit par M. Legaut. Paris, Gauthier-Villars, 1926. 592 pp.

FORSYTH (A. R.). Calculus of variations. Cambridge, University Press, and New York, Macmillan, 1927. 22+656 pp.

FUBINI (G.) e ČECH (E.). Geometria proiettiva differenziale. Tomo 2. Bologna, Zanichelli, 1927. Pp. 389–794.

GANET (M.). Les systèmes d'équations aux dérivées partielles. (Mémorial des Sciences Mathématiques, No. 21.) Paris, Gauthier-Villars, 1927. 56 pp.

GRASSMANN (H.). Projektive Geometrie der Ebene unter Benutzung der Punktrechnung dargestellt. Band 2: Ternäres. Teil 2. Leipzig, Teubner, 1927. 16+522 pp.

HADAMARD (J.). Cours d'analyse professé à l'Ecole Polytechnique. Tome 1. Paris, Hermann, 1927. 8+624 pp.

HERRMANN (A.). Das Delische Problem. Die Verdopplung des Würfels. (Mathematisch-Physikalische Bibliothek.) Leipzig, Teubner, 1927. 58 pp.

KRUEGER (F.). See BUCHHOLZ (H.).

LAGRANGE (R.). Calcul différentiel absolu. (Mémorial des Sciences Mathématiques, No. 19.) Paris, Gauthier-Villars, 1926. 39 pp.

——See NÖRLUND (N. E.).

LEGAUT (M.). See ENRIQUES (F.).

LEVI-CIVITA (T.). The absolute differential calculus (calculus of tensors). Edited by E. Persico. Authorized translation by Miss M. Long. London, Blackie, 1927. 16+450 pp.

LIETZMANN (W.). Über die Beurteilung der Leistungen in der Schule. Psychologisches. Leipzig, Teubner, 1927. 117 pp.

LONG (M.). See LEVI-CIVITA (T.).

DE LOOR (B.). De hoofdstelling van de algebra van intuisionisties standpunt. (Dissertation.) Amsterdam, Swets en Zeitlinger, 1925. 8+63 pp.

MANNOURY (G.). Mathesis en mystiek. Amsterdam, Wereldbibliotheek, 1925. 116 pp.

NIEWENGLOWSKI (B.). Cours de géométrie analytique. Tome IV: Application des quaternions à la géométrie analytique. Paris, Gauthier-Villars, 1927. 212 pp.

NÖRLUND (N. E.). Leçons sur les séries d'interpolations. Rédigées par R. Lagrange. (Collection Borel.) Paris, Gauthier-Villars, 1926.

PERRON (O.). Algebra. Band I: Die Grundlagen. Band II: Theorie der algebraischen Gleichungen. Berlin, de Gruyter, 1927. 8+308+8+243 pp.

PERSICO (E.). See LEVI-CIVITA (T.).

ROTHE (R.). Höhere Mathematik. 2te Auflage. Teil I: Differentialrechnung und Grundformeln der Integralrechnung. Leipzig, Teubner, 1927. 6+186 pp.

RUNGE (C.). Vektoranalysis. 2te Auflage. Band 1. Leipzig, Hirzel, 1926. 8+195 pp.

RUSSELL (B.). See WHITEHEAD (A. N.).

URBAN (F. M.). See COOLIDGE (J. L.).

DE VRIES (H.). Historische Studien. I. Groningen, Noordhoff, 1926. 192 pp.

WEATHERBURN (C. E.). Differential geometry of three dimensions. Cambridge, University Press, 1927. 12+268 pp.

WHITEHEAD (A. N.) and RUSSELL (B.). Principia mathematica. 2d edition. Volume II. Cambridge, University Press, 1927. 32+742 pp.

YOUNG (L. C.). The theory of integration. (Cambridge Tracts, No. 21.) London, Cambridge University Press, 1927. 53 pp.

PART II. APPLIED MATHEMATICS

Ainsley (T. L.). Magnetism and the deviation of the compass in iron ships. New edition. South Shields, Ainsley, 1926. 165 pp.

Andrade (E.). The structure of the atom. 3d edition, revised and enlarged. London, Bell, 1927. 18+750 pp.

Auerbach (F.) und Hort (W.), herausgegeben von. Handbuch der physikalischen und technischen Mechanik. In 7 Bänden. Band 1, Lieferung 1, und Band 3. Leipzig, Barth, 1927. 10+306+10+468 pp.

Barrow (F. L.). See Pippard (A. J. S.).

Behm (W.). Welteis und Weltentwicklung. Leipzig, Voigtländer, 1926.

Berger (R.). Die Schalltechnik. Braunschweig, Vieweg, 1926. 4+115 pp.

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Chwolson (O. D.). Lehrbuch der Physik. 2te, verbesserte und vermehrte Auflage. Band 4, Abteilung 2, von G. Schmidt. 48 pp.

Clatworthy (J. P.). An introduction to mechanics. 2 volumes. London, Methuen, 1926. 9+207+7+225 pp.

Compton (A. H.). X-rays and electrons. New York, Van Nostrand, 1926. 15+403 pp.

Foerster (M.). Die Grundzüge des Eisenbetonbaues. 3te, verbesserte und vermehrte Auflage. Berlin, Springer, 1926. 582 pp.

de Fontviolant (B.). Résistance des matériaux, analytique et graphique. Tome II. Paris, Baillière, 1927. 728 pp.

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Gay (L.). Les mathématiques du chimiste. Paris, Hermann, 1926. 208 pp.

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Hoppe (E.). Geschichte der Physik. Braunschweig, Vieweg, 1926. 536 pp.

Hort (W.). See Auerbach (F.).

Jordan (C.). Statistique mathématique. Avec une préface de M. d'Ocagne. Paris, Gauthier-Villars, 1927. 8+344 pp.

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Knight (C. K.). Advanced life insurance, with illustrations of the principles and practices of actuarial science. New York, Wiley, 1926. 11+426 pp.

Krey (H.). Erddruck, Erdwiderstand und Tragfähigkeit des Baugrundes. Berlin, Springer, 1926.

Lamla (E.). Grundriss der Physik für Naturwissenschaftler, Mediziner und Pharmazeuten. Berlin, Springer, 1925.

Langworthy (F.). Reflections on the structure of the atoms. London, Watts, 1926. 11+260 pp.

VON LAUE (M.) und VON MISES (R.). Stereoskopbilder von Kristallgittern. Berlin, Springer, 1926. 43 pp.+24 plates.

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MANES (A.), herausgegeben von. Fortbildungsvorträge. Teil II: Lebensversicherung und sonstige Personenversicherung. Berlin, E. S. Mittler, 1926.

VON MISES (R.). See VON LAUE (M.).

MOREUX (T.). Pour comprendre la mécanique. Paris, Doin, 1926. 260 pp.

NÖLKE (F.). Entwicklung im Weltall; kosmogonische Probleme und Hypothese. Hamburg, Henri Grand, 1926.

D'OCAGNE (M.). See JORDAN (C.).

OTTENHEIMER (J.). Balistique intérieure. Paris, Colin, 1926. 200 pp.

PADFIELD (W. W.). Stresses in space frames. London, Association of Engineering and Shipbuilding Draftsmen, 1926.

PIPPARD (A. J. S.) and BARROW (F. L.). The stress analysis of bow girders. London, H. M. Stationery Office, 1926. 4+27 pp.

PLANCK (M.). Physikalische Gesetzlichkeit im Lichte neuerer Forschung. Leipzig, Barth, 1926. 48 pp.

REY (A.). Le retour éternel et la philosophie de la physique. Paris, Flammarion, 1927. 316 pp.

SCHEEL (K.). See GEIGER (H.).

SCHMID (H.). Statische Probleme des Tunnel- und Druckstollenbaues und ihre gegenseitigen Beziehungen. Berlin, Springer, 1926.

SCHMIDT (G.). See CHWOLSON (O. D.).

SCHRÖDINGER (E.). Abhandlungen zur Wellenmechanik. Leipzig, Barth, 1927. 9+169 pp.

SMEKAL (A.). Allgemeine Grundlagen der Quantenstatistik und Quantentheorie. Leipzig, Teubner, 1926. 6+362 pp.

STEPHAN (P.). Die technische Mechanik des Maschineningenieurs. Band 5: Die Statik der Fachwerke. Berlin, Springer, 1926. 140 pp.

TARN (A. W.). The student's guide to life assurance in theory and practice. London, Macdonald and Evans, 1926. 307 pp.

WESTPHAL (W.). See GEIGER (H.).

WOOD (E. H.). Textbook of mechanics. New York, Wiley, 1926. 251 pp.

WUNDER (R.). Die kurzen Wellen. Sende und Empfangsschaltungen. (Bibliothek des Radio-Amateurs.) Berlin, Springer, 1926. 108 pp.

THE FIFTY-SECOND REGULAR MEETING OF THE SAN FRANCISCO SECTION

The fifty-second regular meeting of the San Francisco Section of the Society was held at the University of British Columbia on Saturday, June 18, 1927. The total attendance was thirty, including the following fifteen members of the Society:

Bernstein, Biggerstaff, Blichfeldt, Daniel Buchanan, J. W. Campbell, Carpenter, DeCou, Griffin, McAlister, W. E. Milne, Moritz, Mullemeister, Neikirk, F. S. Nowlan, Winger.

Professor Daniel Buchanan was elected temporary chairman. President Klinck of the University of British Columbia welcomed the visiting members. The Section accepted the cordial invitation of Professor Griffin to hold the next Summer meeting at Reed College. After the meeting the visitors were entertained at a luncheon on the Campus followed by a drive through the city of Vancouver.

Titles and abstracts of papers read at the meeting follow. The papers of Professors Bell, Buchanan, Davis, Dines, Moore, Dr. Trjitzinsky, Mr. Gage, Mr. Jerbert, and Dr. Whyburn were read by title. The papers of Miss Deutsch, Miss Haack, Miss Pennock, Miss Rollins, and Mr. So were sponsored and read by Professor Griffin. Mr. Gage's paper was sponsored by Professor Buchanan.

1. Professor E. T. Bell: *Cauchy's cyclotomic function and functional powers.*

This paper has appeared in the July-August number of this Bulletin.

2. Professor E. T. Bell: *Numerical functions of multipartite integers and compound partitions.*

An arithmetic of hypercomplex numbers with r positive integral coefficients is developed. The divisibility properties of functions of such numbers are a logical resultant of those of Dirichlet multiplication and the process of elliptic multiplication and the processes of elliptic functions; they give rise to new aspects of compound partitions.

3. Professor E. T. Bell: *Partition polynomials.*

A class of polynomials depending on all the partitions of a given integer into unequal parts is defined. For special values of the s independent variables in the nth polynomial, the value is the number of representations of n in the form $cx \cdots xx$ where $c, \cdots$ are integers 0; for other special values the nth polynomial is the binary quadratic class number for the negative determinant n.

4. Professor Daniel Buchanan: *The Trojan asymptotic satellites.*

An earlier paper discussed the asymptotic satellites in the vicinity of the equilateral triangle equilibrium points, but the analysis was not valid for certain small ratios of the finite masses, for example, that of Jupiter and the sun. A different treatment of the problem is now given which is valid for certain small ratios only.

5. Professor A. F. Carpenter: *Complete systems of invariants and covariants of a certain system of surfaces.*

An analytic basis for the projective differential geometry of triads of ruled surfaces whose generators are in one-to-one correspondence has been established in a paper soon to appear in the Transactions. As a sequel the present paper develops with reasonable completeness the theory of the invariants and covariants of such a system of surfaces.

6. Professor D. R. Davis: *Characteristic properties of differential equations of the calculus of variations.*

Self-adjoint systems of differential equations of the second order are discussed. For a given system of differential equations of the second order the unique existence of an adjoint system is proved. Necessary and sufficient conditions for a given system of the second order to be self-adjoint are established. Characteristic properties of differential equations of the calculus of variations are then derived. It is here shown that if two equations of the form $H(x, y, z, y', z', y'', z'') = 0$, $K(x, y, z, y', z', y'', z'') = 0$ are to be the Euler equations for the integral $I = \int_{x_1}^{x_2} f(x, y, z, y', z')dx$, they must have their equations of variation $H_y u + H_z v + H_{y'} u' + H_{z'} v' + H_{y''} u'' + H_{z''} v'' = 0$, $K_y u + K_z v + K_{y'} u' + K_{z'} v' + K_{y''} u'' + K_{z''} v'' = 0$, self-adjoint.

7. Professor D. R. Davis: *Integrals whose extremals have given differential equations.*

That the equations of variation of a given system of differential equations of the second order, $H = K = 0$, must be self-adjoint in order that the given equations be the Euler equations for an integral I was established as a necessary condition in the preceding paper. The sufficiency of this property is proved here. Necessary and sufficient conditions for the system $H = K = 0$ to be self-adjoint are given. Also the most general form of the

integrand f for an integral I whose Euler equations are the equations $H=K=0$ is determined.

8. Professor D. R. Davis: *A method of determining an integral I whose extremals are a given family of curves.*

The problem of finding an integral whose extremals are a given family of curves is here treated. Necessary and sufficient conditions for the two given equations of the form $y''=F(x, y, z, y', z')$, $z''=G(x, y, z, y', z')$ to have solutions $y=y(x)$, $z=z(x)$ which represent the extremals for an integral I, of the form expressed in abstract 6, are derived. Also the possibility of determining linear conbinations of the two given equations $y''=F$, $z''=G$, which shall have equations of variation that are self-adjoint, is discussed. As an application of the above theory the following cases are treated: (1) integrals whose extremals are straight lines, (2) integrals whose extremals are semicircles, (3) integrals whose extremals are catenaries.

9. Miss Helen Deutsch: *Note on certain questions of probability.*

This note considers the most probable straight line for a given table of data, and discusses criteria for the line to be unaltered by reversing the selection of the independent variable, also an upper limit for the changes in the constants of the line for given errors in the table. Some curious results of random sampling are also reported.

10. Professor L. L. Dines: *On positive solutions of a system of linear equations.*

This paper presents a simple algorithm for determining whether a given system of linear algebraic equations with real numerical coefficients admits a solution in which each of the unknowns has a *positive* value, and for determining such a solution when one exists. It will appear in the Annals of Mathematics.

11. Professor L. L. Dines: *On completely signed sets of functions.*

A set of functions $f_1(x), f_2(x), \cdots, f_m(x)$, continuous on an interval $a \leqq x \leqq b$, is said to be a *completely signed* set, if there exist 2^m points x_1, $x_2, \cdots, x_{2^m}$, on the interval such that the matrix $\|f_i(x_j)\|$ is a completely signed matrix (as already defined, Annals of Mathematics, vol. 28, p. 42). The author shows that the condition of being completely signed is sufficient (by no means necessary) to ensure the following two properties for the set: (1) every linear combination of the functions changes sign on the interval; (2) there exists a positive continuous function on the interval which is orthogonal to each function of the set. The further remark is made that the notion of a completely signed set may be extended to a set of functions of a general variable. The paper will appear in the Annals of Mathematics.

12. Mr. W. B. Doggett: *On certain caustic surfaces and optical phenomena.*

If the farther half of a spherical flask be silvered, and the flask be nearly filled with water, the observer will see at bottom an apparent bubble or globule of immiscible fluid. Experiment indicates that this is the image of the upper surface of the water, distorted by reflection and refraction; and this conclusion is verified mathematically. Preliminary to this study, there is a discussion of caustic surfaces formed by reflection from spherical and parabolic surfaces.

13. Mr. W. H. Gage: *Asymptotic satellites near the equilibrium point in the isosceles triangle solution of the problem of three bodies (elliptic case).*

If two finite bodies of equal mass move in circles or ellipses about their common center of gravity, then an infinitesimal third body may be initially projected so that it will move in a periodic orbit and remain equidistant from the finite bodies. The paper determines the orbits which are asymptotic to this periodic orbit when the motion of the finite bodies is elliptic.

14. Professor F. L. Griffin: *Points of minimum travel for a distributed population.*

The General Report of the U. S. Census, after defining the center of population, erroneously states that it is the point at which all the people could be assembled with the minimum aggregate travel, going directly. This suggests the problem of determining points of minimum travel for areas of various shapes and for a population whose density varies in an arbitrary manner. Several cases are discussed.

15. Miss Bonnie Haack: *Certain families of curves having a parabola as their envelope.*

This paper formulates the inverse of the envelope problem for families of coplanar curves, and studies certain families of straight lines, circles, and parabolas having a given parabola as their envelope.

16. Mr. A. R. Jerbert: *Projective differential geometry of a system of linear, homogeneous, first-order differential equations.*

This paper studies a system of linear, homogeneous, first-order differential equations in three dependent and one independent variables. A fundamental system of independent solutions can be interpreted as the homogeneous coordinates of three points in a plane. This system of equations defines three curves "to within a projectivity." As the independent variable varies, the points describe three associated curves, corresponding points being those given by the same value of the independent variable. The complete system of invariants of the system is obtained and interpreted as expressing projective properties of the associated curves.

17. Professor R. E. Moritz: *The general solution of a certain Diophantine equation in three unknowns.*

A certain problem in partitions leads to the equations $x-1=kn$, $y=hn$, where k, n, and h are positive integers subject to the condition $(k+1)(n-1)-n^{n-1}(hn+n-1)=0$ but otherwise unrestricted. This paper deals with the general positive integral solutions.

18. Professor R. E. Moritz: *Some new formulas for estimating depreciation of physical properties.*

As is well known, the reducing balance method of computing depreciation results in an excessive reduction of book values during the early years of an organization by creating what amounts to a secret reserve. This paper shows how this objection may be removed by considering the ratio of depreciation as a function of the time and by determining this function with regard to the nature of the property under consideration.

19. Professor R. E. Moritz: *A modification of Glaisher's proof of Sterling's theorem.*

Glaisher's so called elementary proof of Sterling's formula for the approximation of large numbers is open to the objection that it assumes without proof the convergence of a certain double series, and assumes as constant a variable which approaches a constant as a limit. The paper offers an elementary proof of the theorem which is free from these objections.

20. Professor L. I. Neikirk: *A totally discontinuous function.* Preliminary report.

If $f_1(x)$ and $f_2(x)$ are any two functions of x, then the function under discussion will be equal to f_1 at every point of an interval except at an enumerable set of points which are everywhere dense, where it is equal to f_2. Between every two points at which it equals f_1 there is a point at which it equals f_2, and vice versa. An explicit formula is given for this function.

21. Professor F. S. Nowlan: *Arithmetics of rational division algebras of order nine.*

In this paper the integral elements (in the Dickson sense) of a certain rational division algebra D are determined. The basal units of D are given by $y^ix^j(i, j=0, 1, 2)$, where $y^3=\eta\epsilon$ with η the product of integral powers of 7 and rational primes of the form $7m\pm1$ and ϵ is the product of rational primes $7m\pm2$ and $7m\pm3$, with at least one such factor entering to an integral power not divisible by 3. Further, x satisfies the irreducible cyclic equation $w^3+w^2-2w-1=0$. For such value of ϵ of the form $7m\pm1$ there are three sets of integral elements. If, however, ϵ is of the form $7m\pm2$ or $7m\pm3$, then for each of its values there is unique set of integral elements, viz., the set I of all elements with a rational integral coordinates, referred, however, to a transformed set of basal units in case $\eta\neq1$.

22. Professor F. S. Nowlan: *A note on the direct product of a division algebra and a total matric algebra.*

This paper establishes certain theorems concerning an algebra expressible as the direct product of a division algebra and a total matric algebra. In particular, it develops a method for finding the primitive idempotent elements of a total matric algebra.

23. Professor F. S. Nowlan: *A problem related to matric algebras.*

Given the basis of a total matric algebra all similar bases are determined.

24. Miss Dorothory Pennock: *A generalization of the circular and hyperbolic functions of sectorial areas.*

The familiar line definitions are generalized by measuring the sectorial areas from an origin other than the center of the circle or hyperbola. The functions so generalized are studied in detail as to their graphs, periodicity, basic relations, derivatives, integrals, and development in series. A short discussion of similar functions for the parabola is appended.

25. Miss Marguerite Rollins: *Functionally conjugate curves. The case of the harmonic mean.*

A basic reference curve B and a given curve G are selected, and a conjugate curve R is defined as having its radius vector in any direction the harmonic mean of the corresponding radii vectors of B and G. The character of R is studied when B and G are straight lines or circles or certain other conics. When B and G coincide as a single conic, R becomes the polar of the origin.

26. Mr. Hung Shu So: *On the projective determination and classification of conics through given points.*

This paper presents a detailed discussion of the character of conics as determined by given sets of points, with special attention to degenerate forms, and with simple criteria for the selection of points which will determine any desired type of conic, or vice versa.

27. Dr. W. J. Trjitzinsky: *Integrals of non-analytic functions.*

Functions non-analytic in the sense of being non-monogenic do not possess a unique derivative at a point. If C be a closed curve, then under certain conditions $\left|\int_C F(z)dz\right| < 4\sqrt{2}M \cdot A$, where M is the upper bound of the maximum directional derivative on and within C, and A is the area within C. This relation is an analoge of Cauchy's theorem and is proved by using essentially Goursat's method in proving Cauchy's theorem. As

a corollary, we have upper bounds for certain line integrals much closer than the upper bounds given by the ordinary method.

28. Dr. W. J. Trjitzinsky: *Zeros of a function and zeros of its derivatives.*

This paper will appear in full in an early issue of this Bulletin.

29. Dr. W. J. Trjitzinsky: *Relations satisfied by coefficients of periodic solutions.*

Assuming that there is a periodic solution of a linear homogeneous equation with periodic coefficients containing parameters and differentiating the solution with respect to one of the parameters, the derivative is found to be a periodic solution of a certain linear non-homogeneous differential equation. Solving this by the method of variation of parameters and introducing the condition of periodicity, a relation is found satisfied by the coefficients of a periodic solution containing all of them. This method is, in general, complicated, but the difficulty is avoided by using Fite's recent results concerning periodic solutions. The results of the paper were applied to the Mathieu differential equation of the elliptic cylinder and to Hill's equation of the lunar theory.

30. Dr. W. J. Trjitzinsky: *Representation of functions determined by their initial values.* First paper.

Borel, Denjoy, and de la Vallée Poussin have investigated classes of functions more general than the class of analytic functions; they are characterized by the fact that they are determined by the initial values (of the function and its successive derivatives); de la Vallée Poussin gives a practical method for constructing such functions when the initial values are assigned by means of a series $\alpha_1 \cos m_1 x + \alpha_2 \cos m_2 x + \cdots$ where m_i are integers such that $m_{k+1}/m_k > r > 1$ for all k. In this paper it is shown that the functions under consideration are representable by a series at the form $\alpha_1 f(m_1 x) + \alpha_2 f(m_2 x) + \cdots$ where $|f(x)|$, $|f^{(2p)}(x)| < M$ for all p's and all real x's. As a corollary, it is shown that when m_i are restricted in a certain way, an analytic function $F(x)$ is expansible on $(-1, +1)$ in a series $F(x) = \sum_{i=1}^{\infty} \alpha_i f(m_i x)$, where α_i are given by complex integrals. The proof is close to de la Vallée Poussin's proof.

31. Dr. W. J. Trjitzinsky: *Representation of functions determined by their initial values.* Second paper.

It is found that when a function $F(x)$ which is determined by its initial values is represented by the series $\sum_{i=1}^{\infty} \alpha_i f(m_i x)$, the approximation furnished by taking the first n terms of the series is of the order of $1/r^{2n} (r > 1)$. Moreover, it is found that the functions $f(x)$ which are admissible in the above representation may be computed in the form $f(x) = \sum_{j=0}^{\infty} C_j x^{2j}$ where

$C_j=(-1)^j\sum_{k=1}^{\infty}(p_k^{2j}c_k)$ and c_i, p_j are any numbers such that $\sum_{i=0}^{\infty}|c_i|$ converges and $|p_j|\leqq 1$; also, it is found that these functions $f(x)$ satisfy on $(-1, +1)$ relations of the form $f(x)=\sum_{i=1}^{\infty}\alpha_i f(m_i x)$, where α_i are independent of the choice of $f(x)$.

32. Dr. W. J. Trjitzinsky: *Polynomials approximating to functions analytic or quasi-analytic.*

Taking de la Vallée Poussin's representation of functions by means of a series $\sum_{i=1}^{\infty}\alpha_i \cos m_i x$, let us take the first n terms of the series i.e., a trigonometrical polynomial of order m_n which approximates to the function. By Fejér's result it would follow that the maximum of the above approximating polynomial is not greater than m_n times its minimum (within a period), and conversely. In this paper it is shown that under certain slight restrictions the above immediate consequence of Fejér's result can be made more precise (though less immediate); namely, a polynomial of order m_n approximating to any of the functions of the class under consideration has its maximum not greater than n times its minimum ($n<m_n$), and conversely.

33. Dr. G. T. Whyburn: *Some theorems on continuous curves.*

In this paper it is shown that if M denotes any plane continuous curve, the following theorems are true. (1) If G denotes the set of all the cut points $[P]$ of M such that P is an irregular point of some sub continuum of M, then each point of G belongs to some simple closed curve in M, and G is countable. (2) The cut point P of M belongs to no simple closed curve in M if and only if it is an end point of every continuum obtained by adding P to some maximal connected subset of $M-P$. (3) In order that M should fail to separate the plane it is necessary and sufficient that every maximal cyclic curve of M should be a simple closed curve plus its interior. (4) A maximal cyclic curve of M is a simple closed curve if and only if it is the common boundary of two complementary domains of M. (5) If M itself is not the boundary of any domain, than in order that every proper subcontinuum of M should be the boundary of a domain it is necessary and sufficient that M should be the sum of three arcs joining two points A and B of M and having no other common points.

34. Dr. G. T. Whyburn: *Concerning closed and connected subsets of the cut points of continua.*

Let M be a bounded continuum and K any closed and connected subset of the cut points of M. The author has shown (Transactions of this Society, vol. 29, 1927, p. 383) that K is an acyclic continuous curve. In this paper it is shown that: (1) K has only a countable number of end points; (2) if AB is any arc of K then $M-AB$ plus some countable subset T of AB is the sum of the elements of a countable upper semi-continuous collection G' of mutually exclusive continua each of which contains exactly

one point of T; furthermore, if G denotes the collection obtained by adding to G' all those *points* of $AB-T$, then M itself is a *simple continuous arc of elements of* G from the element containing A to the one containing B; (3) $M-K$ plus some countable subset H of K is the sum of the elements of a countable upper semi-continuous collection E' of mutually exclusive continua each of which contains exactly one point of H, and if E denotes the collection obtained by adding to E' all those *points* of $K-H$, then M itself is an *acyclic continuous curve of elements of* E whose end elements are the elements of E' which contain end points of K; (4) M is regular at every point of $K-H$.

35. Professor R. M. Winger: *The equianharmonic cubic and its group.*

The equianharmonic cubic admits, as is well known, a monomial group of order 54, which is a subgroup of the Hesse group. A complete system of invariants of this group is obtained and numerous invariant curves are considered. In particular, the pencil of invariant sextics, which contains several interesting members, is discussed and some new properties of the sextic invariant of the Hesse group are found.

36. Mr. John Biggerstaff: *Arithmetic of nonians.*

The writer finds the set of integral elements of the algebra of nonians according to Dickson's set of properties. Application of the theorem that the product of nonions equals the product of the norms of the factors, gives solutions of diophantine problems involving relations of the third degree.

37. Professor J. W. Campbell: *On the strength of guy wires.* Preliminary report.

A solution is obtained for the restoring couple afforded to a tower by a given pair of symmetric guy wires when the tower leans through a small angle in the plane of the wires.

38. Professor R. L. Moore: *A separation theorem.*

In this paper it will be shown that if T is a totally disconnected closed subset of the boundary of a bounded simply connected domain D and there exists a continuum K containing T and such that $K-T$ is a subset of D then there exists a simple closed curve J containing a subset of T and enclosing a part of $K-T$ and such that $(J+T)-T$ is a subset of $D-(K-T)$.

It is also shown that this proposition does not remain true on the omission of "a part of" before "$K-T$" even though T consists of a simple point on the *outer* boundary of D. This corrects a statement made on page 473 (see Theorem 3) of the author's paper concerning the separation of point sets by curves (Proceedings of the National Academy, vol. 11, 1925).

39. Dr. G. T. Whyburn: *Concerning connected and regular point sets.*

This paper will appear in full in an early issue of this Bulletin.

B. A. Bernstein,
Secretary of the Section

BEQUESTS TO THE SOCIETY

At its meeting held on May 6, 1927, the Board of Trustees of the Society adopted the following resolution:

Resolved, that the Treasurer be, and hereby is, instructed to credit to endowment all bequests made to the American Mathematical Society except those specifically designated for other purposes.

Several members have already made provision for bequests to the funds of the Society. The specific action of the Trustees was taken in view of the suggestion of one of these persons that other members wishing to take similar action might know what disposition would be made of unspecified bequests. For such bequests, the following form, which is sufficient in any state, is suggested:

"I give and bequeath to the American Mathematical Society, a corporation incorporated under the laws of the District of Columbia, the sum of . . . . . dollars."

Still others have written into their wills bequests for specific purposes, such as prizes. Members who desire to make, such bequests are invited to consult the officers of the Society regarding the most urgent needs of mathematics in this country.

W. B. Fite,
Treasurer.

ON THE GEOMETRY OF LINEAR DISPLACEMENT*

BY D. J. STRUIK

1. *Introduction.* The first to consider the displacement of a vector to a point at an infinitesimal distance in a non-euclidean geometry seems to have been L. E. J. Brouwer for the case of space of constant curvature.† Its importance for differential geometry was pointed out for the first time, however, by Levi-Civita (1917. 1),‡ for the case of a hypersurface immersed in an euclidean space, that is, the case of an n-dimensional Riemannian displacement. It was with this paper of Levi-Civita that the new development of differential geometry was started. Weyl (1918. 1 ; 1918. 3) first pointed out the way in which Levi-Civita's results could be generalized and he originated what we call *the Weyl displacement.* Even before Weyl and Levi-Civita, generalizations of analogous character had been suggested by Hessenberg (1916. 1). The geometry of linear displacement was developed in the papers of König (1919. 1 ; 1920. 1), Eddington (1921. 1 ; 1923. 4), Weyl (1918. 1 ; 1918. 3), and Schouten (1922. 1). Schouten gave a general classification of the different geometries defined by linear displacements, which he published later in his book (1924. 9). We adopt in general Schouten's notation.

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* Address delivered at the meeting of this Society in New York, February 26, 1927. The address is elaborated and provided with a bibliography. Papers published in 1927 are not discussed.

† L. E. J. Brouwer, *Het krachtveld der nieteuklidische negatief gekromde ruimten*, Verslagen Akademie Amsterdam, vol. 15 (1906), pp. 75–94 ≡ *The force field of the non-euclidean spaces with negative curvature*, Proceedings Academy Amsterdam, vol. 9 (1906), pp. 116–133. See however also a remark of H. Poincaré in A. Dall' Acqua, *Sulla teoria delle congruenze di curve in una varietà qualunque a tre dimensioni*, Annali di Matematica, (3), vol. 6 (1901), pp. 1–40.

‡ Such references refer to the bibliography at the end of this paper (see p. 558). Independently of Levi-Civita, Schouten obtained analogous results (1918.4).

That euclidean geometry is a special case of Riemannian geometry had been shown by Riemann. It could now be pointed out by König (1919. 1 ; 1920. 1) and Schouten (1923. 7 ; 1924. 9) that affine geometry, the differential geometry of which had been recently developed by Blaschke, Pick, Radon, and others, had the same relation to the affine displacement, introduced by Weyl, as euclidean geometry had to the displacement of Levi-Civita. The question now arose of approaching projective and conformal geometry by the way of linear displacement. Weyl (1921. 2) was successful in this. Cartan, who had given an introduction to the geometry of linear displacement starting from a different point of view in a series of papers (printed in 1922 and 1923), also approached projective and conformal geometries in his own way (papers of 1924). Schouten (1924. 7; 1924. 8) pointed out the relation between Cartan's and König's results.

Another way of attacking the problems of a linear displacement was found by Eisenhart and Veblen (1922. 2). These authors do not start with the displacement, but define it by a set of partial differential equations of the second order representing the paths (geodesic lines) of the displacement. It was especially the affine displacement that was the object of their investigation and of those of T. Y. Thomas and J. M. Thomas (1924–26). They developed further projective and conformal geometries, and also the non-symmetric displacement. One of their results is a theory of differential invariants of these displacements. Another author who studied this part of the theory is Weitzenböck (1920. 2 ; 1922. 13). T. Y. Thomas also gave a new way to introduce projective geometry into the geometry of paths (1925. 13 ; 1926. 11).

Applications of geometries of linear displacements to the theory of relativity were made by Weyl (1918, 1919), Eddington (1921. 1 ; 1923. 4), Einstein (1923. 1 , 1923. 2 ; 1923. 3), and others (see bibliography).

A recent application of the theory of linear displacements to the theory of semi-simple and simple continuous groups was given by Cartan and Schouten (1925. 3).

2. *The n-Dimensional Manifold and Ricci Algebra.* The geometry of linear displacement starts with the assumption of sets of points which constitute a *domain* of an *n-dimensional manifold.* Such a domain will be denoted by X_n. This assumption implies a set of postulates which are discussed by Veblen (1925. 16, pp. 132–136).* Veblen introduces points as undefined terms and asserts that there exists a way of labelling the points in a biunique way by means of ordered sets of n numbers x^ν, $\nu = 1, 2, \cdots, n$. This is equivalent to the *introduction of an n-cell* as a set of points capable of supporting a euclidean geometry. We then get a set of postulates for an arbitrary geometry of linear displacements, by first giving such a set for a euclidean geometry and then postulating that the geometry under consideration be *analytically related* to that of Euclid. That means, that if A be a point of the euclidean space, and R_A be an n-dimensional region containing A, in which the points are uniquely denoted by coordinates y^ν, and R_P an analogous region containing a point P of X_n, in which the relations of geometry (as the $\Gamma^\nu_{\lambda\mu}$) are analytic functions of the coordinates x^ν, there is an analytic relation

$$y^\nu = f(x^\mu), \qquad (\mu, \nu = 1, 2, \cdots, n), \tag{1}$$

with a unique inverse.

When the coordinates x^ν of X_n undergo an analytic transformation with a unique inverse

$$'x^\nu = \phi^\nu(x^\mu), \qquad (\mu, \nu = 1, 2, \cdots, n) \tag{2}$$

the differentials of these coordinates in general suffer a homogeneous linear transformation

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* See also H. Lebesgue, *Sur les correspondances entre les points de deux espaces*, Fundamenta Mathematicae, vol. 2 (1921), pp. 256–285, where other literature (e.g., Cantor, Brouwer) is cited.

$$(3) \qquad {}'dx^\nu = \frac{\partial' x^\nu}{\partial x^\mu} dx^\mu, \qquad \Delta = \left| \frac{\partial' x^\lambda}{\partial x^\mu} \right| \neq 0,$$

and these transformations build up the linear homogeneous or affine group. Hence the infinitesimal region of X_n at a point P can be considered as an infinitesimal affine space E_n in which the differentials can be regarded as representing a set of rectilinear coordinate axes. Thus we can introduce at one point of an X_n the whole algebra of Ricci calculus, which is identical with the theory of invariants and covariants of the linear homogeneous group. Hence we can speak *at one point* of X_n of contravariant, covariant, and mixed quantities (vectors and tensors), the most simple contravariant vector being dx^ν. The three principal invariant operations between these quantities are

(a) transvection (Überschiebung), for example :

$$v_\lambda w^\lambda, \; v_{\lambda\mu\nu} w^\mu, \; R_{\omega\mu\lambda}^{\cdots\quad\omega}, \text{etc.};$$

(b) alternation, for example :

$$v_{[\mu} w_{\lambda]} = \tfrac{1}{2}(v_\mu w_\lambda - v_\lambda w_\mu),$$

$$v_{[\lambda\mu\nu]} = \tfrac{1}{6}(v_{\lambda\mu\nu} - v_{\lambda\nu\mu} + v_{\mu\nu\lambda} - v_{\mu\lambda\nu} + v_{\nu\lambda\mu} - v_{\nu\mu\lambda});$$

(c) mixing, for example :

$$v_{(\mu} w_{\lambda)} = \tfrac{1}{2}(v_\mu w_\lambda + v_\lambda w_\mu),$$

$$v_{(\lambda\mu\nu)} = \tfrac{1}{6}(v_{\lambda\mu\nu} + v_{\lambda\nu\mu} + v_{\mu\nu\lambda} + v_{\mu\lambda\nu} + v_{\nu\lambda\mu} + v_{\nu\mu\lambda}).$$

3. *Linear Displacement.* To begin with, there is nothing to compare the quantities at any point $P(x^\nu)$ with the quantities at another point $Q(x_1^\nu)$. At one point P we can at least compare the lengths of vectors in the same direction (for instance we can define nv^ν as having n times the length of v^ν*), of simple bivectors $v_{[\mu} w_{\lambda]}$ in the same plane direction, etc. But even this fails at different points, and we can only compare

* We shall denote vectors and tensors by their components, and speak e.g. of the *vector* v^ν instead of the *vector* with components v^ν.

the values at P and Q of a given scalar field. In Riemannian geometry we overcome the difficulty by the introduction of a quadratic differential form

$$ds^2 = g_{\lambda\mu}dx^\lambda dx^\mu, \tag{4}$$

which enables us

(a) to define at one point the length of a contravariant vector and the angle between two of them,

(b) to compare these vectors, and hence their lengths and angles at two different points P and P' an infinitesimal distance apart.

This is done by *geodesic parallelism*, introduced by Levi-Civita (1917. 1), by means of which it is possible (see also (Schouten 1918. 4)) to define the *covariant differentials* of a vector field $v^\nu(x^1, \cdots, x^n)$, uniquely defined at P and P':

$$\begin{cases} \delta v^\nu = dv^\nu + \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\} v^\lambda dx^\mu, \\ \delta w_\lambda = dw_\lambda - \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\} w_\nu dx^\mu, \end{cases} \tag{5}$$

where the $\left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\}$ depend on $g_{\lambda\mu}$ and $\partial g_{\lambda\mu}/\partial x^\nu$ as follows:

$$\left\{ \begin{matrix} \lambda\mu \\ \sigma \end{matrix} \right\} = g^{\nu\sigma} \left[\begin{matrix} \lambda\mu \\ \nu \end{matrix} \right] = \frac{1}{2} g^{\nu\sigma} \left(\frac{\partial g_{\lambda\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} \right); \tag{6}$$

$g^{\nu\sigma} = (\text{minor of } g_{\nu\sigma} \text{ in determ. of } g_{\lambda\mu}) \div \text{this determ.}$

When $\delta v^\nu = 0$ or $\delta w_\lambda = 0$, we call the respective vectors at P and P' *parallel*. The formulas (5) define the *Riemannian displacement*.

In an X_n we have no fixed tensor $g_{\lambda\mu}$, but we can still give a definition of parallelism which enables us to retain part of (b). Instead of the $\left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\}$ we therefore introduce, for one set of x^ν, n^3 arbitrary functions $\Gamma^\nu_{\lambda\mu}$ and n^3 arbitrary functions $\Gamma'^\nu_{\lambda\mu}$ and define

$$\begin{cases} \delta v^\nu = dv^\nu + \Gamma^\nu_{\lambda\mu} v^\lambda dx^\mu, \\ \delta w_\lambda = dw_\lambda - \Gamma^\nu_{\lambda\mu} w_\nu dx^\mu, \end{cases} \tag{7}$$

as *covariant differentials* of the vector fields v^ν and w_λ. It is therefore necessary that a transformation of coordinates (3) imply a transformation of the $\Gamma^\nu_{\lambda\mu}$ and $\Gamma'^\nu_{\lambda\mu}$ as follows :

$$(8)\quad \begin{cases} \Gamma'^{\kappa}_{\omega\pi} = \dfrac{\partial x^\lambda}{\partial' x^\omega}\dfrac{\partial x^\mu}{\partial' x^\pi}\dfrac{\partial' x^\kappa}{\partial x^\nu}\Gamma^\nu_{\lambda\mu} + \dfrac{\partial x^\mu}{\partial' x^\pi}\dfrac{\partial' x^\kappa}{\partial x^\nu}\dfrac{\partial}{\partial x^\mu}\dfrac{\partial x^\nu}{\partial' x^\omega}, \\ {}'\Gamma'^{\kappa}_{\omega\pi} = \dfrac{\partial x^\lambda}{\partial' x^\omega}\dfrac{\partial x^\mu}{\partial' x^\pi}\dfrac{\partial' x^\kappa}{\partial x^\nu}\Gamma'^\nu_{\lambda\mu} + \dfrac{\partial x^\mu}{\partial' x^\pi}\dfrac{\partial' x^\kappa}{\partial x^\nu}\dfrac{\partial}{\partial x^\mu}\dfrac{\partial x^\nu}{\partial' x^\omega}, \end{cases}$$

or

$$(9)\quad \begin{cases} \Gamma^\nu_{\lambda\mu}\dfrac{\partial' x^\omega}{\partial x^\nu} = {}'\Gamma^\omega_{\rho\sigma}\dfrac{\partial' x^\rho}{\partial x^\lambda}\dfrac{\partial' x^\sigma}{\partial x^\mu} + \dfrac{\partial^2 x^\omega}{\partial x^\lambda \partial x^\mu}, \\ \Gamma'^\nu_{\lambda\mu}\dfrac{\partial' x^\omega}{\partial x^\nu} = {}'\Gamma'^\omega_{\rho\sigma}\dfrac{\partial' x^\rho}{\partial x^\lambda}\dfrac{\partial' x^\sigma}{\partial x^\mu} + \dfrac{\partial^2 x^\omega}{\partial x^\lambda \partial x^\mu}. \end{cases}$$

When the Γ are transformed in this way, δv^ν and δw_λ are transformed as vectors. The transformations of the covariant and of the contravariant vectors are independent of each other. As the formulas (7) for δv^ν and δw_λ are linear in the dx^ν, the displacement they define is called the *general linear displacement*. It passes into the Riemannian displacement if a tensor $g_{\lambda\mu}$ exists so that

$$(10)\qquad \Gamma^\nu_{\lambda\mu} = \Gamma'^\nu_{\lambda\mu} = \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\},$$

and if this tensor $g_{\lambda\mu}$ is taken as the fundamental tensor (that is, as defining the ds^2 as in (4)). This general linear displacement (7) was first introduced by Schouten (1922. 1 ; see also 1924. 9). Schouten also gave a set of axioms on which this displacement is founded (1924. 9, pp. 63–64). These ideas had been anticipated by König (1919. 1 ; 1920. 1), whose conclusions will be considered in §11.

When $\delta v^\nu = 0$, or $\delta w_\lambda = 0$, we say that the corresponding vector is moved *parallel* to itself in the sense of the defined displacement.

From the covariant differentials we pass to the covariant derivatives by the formulas

$$\delta v^\nu = dx^\mu \nabla_\mu v^\nu, \qquad \delta w_\lambda = dx^\mu \nabla_\mu w_\lambda ; \tag{11}$$

hence

$$\begin{cases} \nabla_\mu v^\nu = \dfrac{\partial v^\nu}{\partial x^\mu} + \Gamma^{\nu}_{\lambda\mu} v^\lambda, \\ \nabla_\mu w_\lambda = \dfrac{\partial w_\lambda}{\partial x^\mu} - \Gamma'^{\nu}_{\lambda\mu} w_\nu. \end{cases} \tag{12}$$

The covariant differentials of quantities of higher degree are obtained by assuming the formula, valid for all quantities ϕ and ψ :

$$\delta(\phi\psi) = (\delta\phi)\psi + \phi\delta\psi, \tag{13}$$

so that, for example,

$$\begin{cases} \delta g_{\lambda\mu} = dg_{\lambda\mu} - \Gamma'^{\alpha}_{\lambda\beta} g_{\alpha\mu} dx^\beta - \Gamma'^{\alpha}_{\mu\beta} g_{\lambda\alpha} dx^\beta, \\ \nabla_\mu v^{\alpha}_{\cdot\beta\gamma} = \dfrac{\partial}{\partial x^\mu} v^{\alpha}_{\cdot\beta\gamma} + \Gamma^{\alpha}_{\kappa\mu} v^{\kappa}_{\cdot\beta\gamma} - \Gamma'^{\kappa}_{\beta\mu} v^{\alpha}_{\cdot\kappa\gamma} - \Gamma'^{\kappa}_{\gamma\mu} v^{\alpha}_{\cdot\beta\kappa}. \end{cases} \tag{14}$$

The covariant differential of a scalar field $p(x^\nu)$ is the ordinary differential

$$\delta p = dp, \qquad \nabla_\mu p = \frac{\partial p}{\partial x^\mu}. \tag{15}$$

4. *The Fields $C^{\cdots\nu}_{\lambda\mu}$ and $S^{\cdots\nu}_{\lambda\mu}$.* The Γ and Γ' are not tensors, as is shown by (9). However

$$C^{\cdots\nu}_{\lambda\mu} = \Gamma^{\nu}_{\lambda\mu} - \Gamma'^{\nu}_{\lambda\mu} \tag{16}$$

has tensor character (Schouten 1922. 1), as is shown by (9), or by the formula

$$\Delta_\mu A^\nu_\lambda = \frac{\partial}{\partial x^\mu} A^\nu_\lambda + \Gamma^{\nu}_{\alpha\mu} A^\alpha_\lambda - \Gamma'^{\alpha}_{\lambda\mu} A^\nu_\alpha = \Gamma^{\nu}_{\lambda\mu} - \Gamma'^{\nu}_{\lambda\mu}, \tag{17}$$

where

$$A^\nu_\lambda \begin{cases} = 0, & \nu \neq \lambda \\ = 1, & \nu = \lambda \quad \text{(not summed)} \end{cases} \tag{18}$$

is the unit tensor (often called the Kronecker symbol and denoted by δ_λ^ν *).

It plays a role in the covariant differentiation of a transvection, for example,

$$\delta(v_\lambda w^\lambda) = (\delta v_\lambda) w^\lambda + v_\lambda \delta w^\lambda - C_{\mu\lambda}^{\cdots\nu} v_\nu w^\lambda dx^\mu. \tag{19}$$

When we consider geometries in which the equation

$$v^\lambda w_\lambda = 0 \tag{20}$$

remains invariant under displacement, we need the following form for $C_{\lambda\mu}^{\cdots\nu}$:

$$C_{\lambda\mu}^{\cdots\nu} = C_\mu A_\lambda^\nu. \tag{21}$$

In this case the *incidence* of the point represented by v^ν, and the affine E_{n-1} represented by w_λ, remains invariant under displacement. If not only incidences are invariant, but also the transvection itself, we have $C_\mu = 0$, or

$$C_{\lambda\mu}^{\cdots\nu} = 0. \tag{22}$$

This case (22) is assumed by most authors. The independence of the displacements in the contravariant and covariant cases, characterized by $C_{\lambda\mu}^{\cdots\nu} \neq 0$, is considered only in the papers of König, Schouten, and Dienes (1924. 24). In the following pages we shall asume in general that (22) is satisfied. For this case Dienes considers the integrals of (12), which lead to functions of lines.

In Schouten's† book (1924. 9) formulas for the case $C_{\lambda\mu}^{\cdots\nu} \neq 0$ are given.

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* For vectors and tensors we always use Latin letters in this paper.

† Schouten also deals with another generalization. He admits a definition of covariant vectors by means of an equation

$$'w_\lambda = \tau^{-1} \frac{\partial x^\nu}{\partial' x^\lambda} w_\nu \; ; \quad w_\lambda = \tau \frac{\partial' x^\nu}{\partial x^\lambda} w_\nu,$$

where $\tau(x^\nu)$ is an arbitrary function. For $\tau = 1$ we get the ordinary definition of a covariant vector. The meaning of this definition is trivial in an E_n, but not in an X_n. He calls it the *alteration of covariant measure.* An analogous alteration of contravariant measure is not possible unless the differentials dx^ν cease to be exact differentials.

We now observe that the expressions

$$S_{\lambda\mu}^{\cdots\nu}=\tfrac{1}{2}(\Gamma_{\lambda\mu}^{\nu}-\Gamma_{\mu\lambda}^{\nu}) \tag{23}$$

have tensor character, as is shown by (9) or by the formula

$$\delta_2 d_1 x^\nu-\delta_1 d_2 x^\nu=2d_1x^\lambda d_2x^\mu S_{\lambda\mu}^{\cdots\nu}. \tag{24}$$

Thus if d_1x^ν is moved parallel to itself (in the sense of the displacement) in the direction d_2x^ν and in the same way d_2x^ν in the direction d_1x^ν we do not get in general a closed quadrilateral. Only when $S_{\lambda\mu}^{\cdots\nu}=0$ (except for quantities of higher order) do we always obtain a closed figure. In the case of $S_{\lambda\mu}^{\cdots\nu}\neq 0$, Eddington speaks of "infinitely crinkled," and Cartan of "torsion." The case $S_{\lambda\mu}^{\cdots\nu}\neq 0$ we will call *non-symmetric*, and the case $S_{\lambda\mu}^{\cdots\nu}=0$ *symmetric*. In this last case

$$\Gamma_{\lambda\mu}^{\nu}=\Gamma_{\mu\lambda}^{\nu}, \tag{25}$$

which expresses the *Christoffel symmetry*. An intermediate case is that in which

$$S_{\lambda\mu}^{\cdots\nu}=S_{[\lambda}A_{\mu]}^{\nu}=(S_\lambda A_\mu^{\nu}-S_\mu A_\lambda^{\nu})/2. \tag{26}$$

Then the quadrilateral of d_1x^ν and d_2x^ν is closed when and only when the directions of d_1x^ν and d_2x^ν lie in the E_{n-1} of S_λ. This case is called *semi-symmetric*. Hessenberg (1916. 1), the first to introduce a geometry belonging to general linear displacement, obtained the non-symmetric case. He then specialized on the symmetric case (1916. 1, pp. 210–211). Afterwards Weyl also came to the symmetric case (1918. 1) in the paper in which he introduced the idea of displacement in the sense in which we use it here. Weyl spoke of an *affine* displacement.

The curl of a vector field w_λ is, in a non-symmetric displacement,

$$\nabla_{[\mu}v_{\lambda]}=\frac{1}{2}\left(\frac{\partial v_\lambda}{\partial x^\mu}-\frac{\partial v_\mu}{\partial x^\lambda}\right)-S_{\lambda\mu}^{\cdots\nu}v_\nu \tag{27}$$

and hence only depends on the alternating part of the $\Gamma_{\lambda\cdot}^{\nu}$.

To every asymmetric displacement belongs, in a univocal way, a symmetric one. For let $\Lambda^{\nu}_{\lambda\mu}$ be the symmetric part of $\Gamma^{\nu}_{\lambda\mu}$:

$$\Lambda^{\nu}_{\lambda\mu} = \tfrac{1}{2}(\Gamma^{\nu}_{\lambda\mu} + \Gamma^{\nu}_{\mu\lambda}), \tag{28}$$

or

$$\Gamma^{\nu}_{\lambda\mu} = \Lambda^{\nu}_{\lambda\mu} + S^{\cdot\cdot\nu}_{\lambda\mu}, \tag{29}$$

then $\Lambda^{\nu}_{\lambda\mu}$ is transformed in the same way (8, 9) as $\Gamma^{\nu}_{\lambda\mu}$, and thus defines a displacement. Also by writing

$$\begin{aligned}\delta v^{\nu} &= dv^{\nu} + \Gamma^{\nu}_{\lambda\mu} v^{\lambda} dx^{\mu} = (dv^{\nu} + \Gamma^{\nu}_{\lambda\mu} v^{\lambda} dx^{\mu}) \\ &\quad + S^{\cdot\cdot\nu}_{\lambda\mu} v^{\lambda} dx^{\mu},\end{aligned} \tag{30}$$

and considering the vector character of $S^{\cdot\cdot\nu}_{\lambda\mu} v^{\lambda} dx^{\mu}$, we see the vector character of

$$*\delta v^{\nu} = dv^{\nu} + \Lambda^{\nu}_{\lambda\mu} v^{\lambda} dx^{\mu}. \tag{31}$$

This remark was made by J. M. Thomas (1926.2).

5. *Introduction of a Metric.* Again taking $C^{\cdot\cdot\nu}_{\lambda\mu} = 0$, we may observe that, in a general linear displacement, the differential of a tensor $g^{\lambda\mu}$ gives an affinor $Q^{\cdot\lambda\mu}_{\mu}$ of third degree :

$$\begin{cases} \delta g^{\lambda\nu} = Q^{\cdot\lambda\nu}_{\mu} dx^{\mu}, \\ \nabla_{\mu} g^{\lambda\nu} = Q^{\cdot\lambda\nu}_{\mu}. \end{cases} \tag{32}$$

Then, differentiating the covariant tensor $g_{\lambda\mu}$ belonging to $g^{\lambda\mu}$, if we assume that its rank is n, we get by (6),

$$\begin{cases} \delta g_{\lambda\mu\nu} = -\, g_{\lambda\alpha} g_{\nu\beta} Q^{\cdot\alpha\beta}_{\mu} dx^{\mu}, \\ \nabla_{\mu} g_{\lambda\nu} = -\, g_{\lambda\alpha} g_{\nu\beta} Q^{\cdot\alpha\beta}_{\mu}. \end{cases} \tag{33}$$

Suppose now, that the $\Gamma^{\nu}_{\lambda\mu}$ can be so chosen that there exists at least one tensor (of rank n) $g_{\lambda\mu}$ so that

$$Q^{\cdot\lambda\nu}_{\mu} = Q_{\mu} g^{\lambda\nu}, \tag{34}$$

hence

$$\begin{cases} \delta g^{\lambda\nu} = Q_{\mu} g^{\lambda\nu} dx^{\mu}, \\ \nabla_{\mu} g^{\lambda\nu} = Q_{\mu} g^{\lambda\nu}. \end{cases} \tag{35}$$

In that case also

$$\delta g_{\lambda\nu} = - Q_\mu g_{\lambda\nu} dx^\mu \tag{36}$$

and we speak of a *conformal* displacement (Schouten 1924.9, p. 72). That this geometry is possible is seen from the general theorem which we will also state for the case of $C_{\lambda\mu}^{\cdot\cdot\nu} \neq 0$, *that it is possible to define in a unique way every set of* $\Gamma_{\lambda\mu}^{\nu}$ *and* $\Gamma_{\lambda\mu}'^{\nu}$, *and hence every linear displacement, by two arbitrary affinor fields* $C_{\lambda\mu}^{\cdot\cdot\nu}$ *and* $S_{\lambda\mu}^{\cdot\cdot\nu}$ *and a field* $Q_\mu^{\cdot\lambda\nu}$ *defined as the derivative of a tensor field* $g^{\lambda\nu}$ *of rank n.*

The proof can be given by the explicit elimination of $\Gamma_{\lambda\mu}^{\nu}$ and $\Gamma_{\lambda\mu}'^{\nu}$ from the equations expressing $C_{\lambda\mu}^{\cdot\cdot\nu}$, $S_{\lambda\mu}^{\cdot\cdot\nu}$ and $Q_\mu^{\cdot\lambda\nu}$ (and $g^{\lambda\nu}$) in terms of the $\Gamma_{\lambda\mu}^{\nu}$ and $\Gamma_{\lambda\mu}'^{\nu}$. The explicit formula for the case $C_{\lambda\mu}^{\cdot\cdot\nu} = 0$ is

$$\Gamma_{\lambda\mu}^{\nu} = \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\} + T_{\lambda\mu}^{\cdot\cdot\nu}, \tag{37}$$

where the Christoffel symbol must be taken as belonging to $g^{\lambda\nu}$ and

$$\left\{ \begin{aligned} T_{\lambda\mu}^{\cdot\cdot\nu} &= \tfrac{1}{2}(g_{\lambda\alpha} Q_\mu^{\cdot\alpha\nu} + g_{\mu\alpha} Q_\lambda^{\cdot\nu\alpha} - g^{\nu\gamma} g_{\mu\alpha} g_{\lambda\beta} Q_\gamma^{\cdot\alpha\beta}) \\ &\quad + S_{\lambda\mu}^{\cdot\cdot\nu} - g^{\nu\beta}(g_{\lambda\alpha} S_{\beta\mu}^{\cdot\cdot\alpha} + g_{\mu\alpha} S_{\beta\lambda}^{\cdot\cdot\alpha}). \end{aligned} \right. \tag{38}$$

This theorem was first given by Schouten (1922. 1; see also 1924. 9, p. 72–74).

If now we take $Q_\mu^{\cdot\lambda\nu} = Q_\mu g^{\lambda\nu}$ in (37) and (38), we can construct a set of $\Gamma_{\lambda\mu}^{\nu}$ which define a conformal displacement.

In a conformal geometry every tensor

$$'g_{\lambda\mu} = \sigma g_{\lambda\mu}, \tag{39}$$

where σ is an arbitrary function of x^ν, has a differential of the form (35) or (36). For we have

$$\left\{ \begin{aligned} \delta' g_{\lambda\nu} &= - dx^\mu \left(\frac{\partial \sigma}{\partial x^\mu} - \sigma Q_\mu \right) g_{\lambda\nu} \\ &= - \left(Q_\mu - \frac{\partial \log \sigma}{\partial x^\mu} \right) g_{\lambda\nu} dx^\mu \\ &= - Q_\mu' g_{\lambda\nu} dx^\mu, \end{aligned} \right. \tag{40}$$

where

$$Q'_\mu = Q_\mu - \frac{\partial \log \sigma}{\partial x^\mu} . \tag{41}$$

We take one of these tensors $g_{\lambda\mu}$ and define by means of it the angle between two directions v^ν, w^ν :

$$\cos\theta = \frac{g_{\lambda\mu} v^\lambda w^\mu}{\sqrt{g_{\lambda\mu} v^\lambda v^\mu} \sqrt{g_{\lambda\mu} w^\lambda w^\mu}} , \tag{42}$$

which is invariant under (39). We then call the tensors (39) the *fundamental tensors* of the displacement. We have proved by (40), that when *one tensor* $g_{\lambda\mu}$ *has a differential of the form* (36), *every tensor* $\sigma g_{\lambda\mu}$ *has a differential of this form.* It can also be shown *that only the fundamental tensors* $\sigma g_{\lambda\mu}$ *possess this property* (Schouten 1924.9, p. 220).

If in particular Q_μ is a gradient vector, there is, by (41), one fundamental tensor $g_{\lambda\mu}$ uniquely determined so that

$$\delta g_{\lambda\mu} = 0. \tag{43}$$

In this case the displacement is called *metrical.* We may remark *that one and only one linear displacement is determined by a field* $S_{\lambda\mu}^{\cdot\cdot\nu}$ *alternating in* λ *and* μ *and a tensor field* $g_{\lambda\mu}$ *of rank* n, *when* $\Gamma^\nu_{\lambda\mu} = S_{\lambda\mu}^{\cdot\cdot\nu}$ *and* $\nabla_\omega g_{\lambda\mu} = 0$. This is a special case of the preceding theorem.

When this tensor $g_{\lambda\mu}$ is taken as fundamental tensor, we can define length and angle by means of (42) and

$$ds^2 = g_{\lambda\mu} dx^\lambda dx^\mu. \tag{44}$$

Length and angle of vectors remain invariant under a metrical displacement, and only under such a displacement, as is easily shown. When in particular $\Gamma^\nu_{\lambda\mu} = \Gamma^\nu_{\mu\lambda}$, the conformal displacement passes into the *displacement of Weyl,* introduced by him (1918. 1, 3 ; see also 1918. 2), and the metrical displacement into the *displacement of Levi-Civita,* (1917. 1), which is the foundation of Riemannian geometry. Indeed, in this case (37) passes into

$$\Gamma^{\nu}_{\lambda\mu} = \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\}. \tag{45}$$

A simple example of a metrical displacement which is not symmetric was given by Cartan (1924.5) and Hessenberg (1925. 24). Take a sphere and a set of meridians on it. Define the displacement by the assumption that a contravariant vector is moved parallel to itself when the angle with the meridians is always the same. Length and angle remain invariant, hence the displacement is metrical. But it is not the displacement of Levi-Civita, hence there must be an affinor $S^{\cdots\nu}_{\lambda\mu} \neq 0$. It can easily be seen, by (66), that this metric on the sphere is even holonomous (see §8). The geodesic lines of this displacement are the loxodromics of the sphere. For other examples see Cartan (1926. 15). For the relation to the parallelism of Clifford see Cartan (1924. 5).

Necessary conditions that a symmetric system of $\Gamma^{\nu}_{\lambda\mu}$ may be of the form (45) or of the form of Weyl geometry were given by Eisenhart-Veblen (1922. 2). Extension of their methods can be found in Schouten (1925. 18) and Veblen-Thomas (1923. 14).

6. *Geometry of Paths.* Two other ways have been pointed out by which to approach the linear displacements (with $C^{\cdots\nu}_{\lambda\mu} = 0$), the geometry of paths developed by Veblen and Eisenhart (1922. 2), and the linear displacements of points instead of vectors developed by Cartan (1923 and later, see for example, 1925. 1). The geometry of paths does not start with the displacement itself, but with the geodesic lines. A geodesic line arises when a contravariant vector or a covariant $(n-1)$-vector moves parallel to itself in its own direction. Hence the equations of a *contravariant geodesic line* are

$$v^{\mu}\nabla_{\mu}v^{\nu} = \alpha v^{\nu}, \tag{46}$$

where v^{ν} means a contravariant vector tangent to the curve $x^{\nu} = x^{\nu}(t)$, and α is a function of the x^{ν}. The equations of a *covariant geodesic line* are

$$v^{\mu}\nabla_{\mu}v_{\lambda_1 \cdots \lambda_{n-1}} = \beta v_{\lambda_1 \cdots \lambda_{n-1}}, \tag{47}$$

where $v_{\lambda_1 \cdots \lambda_{n-1}}$ is a covariant $(n-1)$-vector tangent to the curve, and β is a function of the x^ν. A covariant geodesic line is, however, also a contravariant one, whenever $C_{(\lambda\mu)}^{\cdots\nu} = C_{(\lambda} A_{\mu)}^{\nu}$ (Friedmann-Schouten 1924. 11). Since we suppose $C_{\lambda\mu}^{\cdots\nu} = 0$, we have only to deal with the equation (46), or

$$\frac{d^2x^\nu}{dt^2} + \Gamma_{\lambda\mu}^{\nu} \frac{dx^\lambda}{dt} \frac{dx^\mu}{dt} = \alpha \frac{dx^\nu}{dt}. \tag{48}$$

From (48) we see that the equations of geodesics only depend on the symmetric part $\Lambda_{\lambda\mu}^{\nu}$ of $\Gamma_{\lambda\mu}^{\nu}$ (see end §4). The coefficient depends on the choice of the parameter t. We can make $\alpha = 0$, if we introduce a new parameter l defined, by the equation

$$l = C_1 + C_2 e^{\int \alpha \, dt} dt, \tag{49}$$

with two constants of integration C_1 and C_2. We thus get the equations of the geodesics :

$$\frac{d^2x^\nu}{dl^2} + \Lambda_{\lambda\mu}^{\nu} \frac{dx^\lambda}{dl} \frac{dx^\mu}{dl} = 0. \tag{50}$$

Hence *we have found on a geodesic line a natural scale of measurement determined but for the choice of an origin and a multiplicative factor* (Eisenhart 1922. 5).

The equation (50) is the starting point for the geometry of paths. The integral curves are the natural generalization of the straight lines of euclidean geometry and of the geodesics of Riemannian geometry. To start with this equation, means to consider the geometry of linear displacements as a geometric theory of a particular kind of differential equations of the more general class

$$\frac{d^2x^\nu}{ds^2} = \phi^\nu\left(x, \frac{dx}{ds}\right), \tag{51}$$

where ϕ^ν is an arbitrary function of the x^ν and its derivatives. (For $n = 2$, see Cartan 1924. 4, p. 232.) From this point of

view the study of equations (50) is a continuation of studies started by Pascal and others.*

7. *Normal Coordinates.* It is now possible to define a special set of coordinates at each point of X_n by means of the paths defined by (50). Take a point $P(x_0^\nu)$ and a direction $(dx^\nu/ds)_0 = v_0^\nu$ through it.

Differentiations of (50) with respect to the natural parameter gives

$$\frac{d^3x^\nu}{ds^3} + \Lambda^{\nu}_{\lambda_1\lambda_2\lambda_3}\frac{dx^{\lambda_1}}{ds}\frac{dx^{\lambda_2}}{ds}\frac{dx^{\lambda_3}}{ds} = 0, \tag{52}$$

$$\frac{d^4x^\nu}{ds^4} + \Lambda^{\nu}_{\lambda_1\lambda_2\lambda_3\lambda_4}\frac{dx^{\lambda_1}}{ds}\frac{dx^{\lambda_2}}{ds}\frac{dx^{\lambda_3}}{ds}\frac{dx^{\lambda_4}}{ds} = 0, \tag{53}$$

where

$$\Lambda^{\nu}_{\lambda_1\cdots\lambda_p\omega} = \frac{\partial}{\partial x}\,(\omega\Lambda_{\lambda_1\cdots\lambda_p}) - p\Lambda^{\nu}_{\alpha(\lambda_1\cdots\lambda_{p-1}}\Lambda^{\alpha}_{\lambda_p\omega)}. \tag{54}$$

Then we can obtain the expansion of the coordinates of a geodesic line;

$$\begin{aligned} x^\nu = x_0^\nu + v_0^\nu t &- \frac{1}{2!}(\Lambda^{\nu}_{\lambda\mu})_0 v_0^\lambda v_0^\mu t^2 \\ &- \frac{1}{3!}(\Lambda^{\nu}_{\lambda\mu\nu})^{\lambda}_{0} v_0 v_0^\mu v_0^\nu t^3 - \cdots, \end{aligned} \tag{55}$$

which converges for all values of t if we assume the $\Lambda^{\nu}_{\lambda\mu}$ to possess derivatives of all orders at x_0^ν, $|v_0^\nu|$ to be finite, and

$$|\Lambda^{\nu}_{\lambda_1\cdots\lambda_p}| < G, \tag{56}$$

for all values of p, where G is a given constant.

If we write

$$v_0^\nu t = y^\nu, \tag{57}$$

* E. Pascal, *Grundlagen für eine Theorie der Systeme totaler Differentialgleichungen zweiter Ordnung*, Mathematische Annalen, vol. 54 (1901), pp. 400–416; see also Comptes Rendus, vol. 130 (1900), pp. 645–647.

(56) passes into

$$x - x_0^\nu = y^\nu - \frac{1}{2!}(\Lambda^\nu_{\lambda\mu})_0 y^\lambda y^\mu - \frac{1}{3!}(\Lambda^\nu_{\lambda\mu\nu})_0 y^\lambda y^\mu y^\nu - \cdots . \tag{58}$$

Since the functional determinant $|\partial x^\nu/\partial y^\lambda| \neq 0$, we can solve these equations with respect to y^ν, and we get the convergent expansion

$$\left\{ \begin{aligned} y^\nu &= (x^\nu - x_0^\nu) + \frac{1}{2!}(\Omega^\nu_{\lambda\mu})_0 (x^\lambda - x_0{}^\lambda)(x^\mu - x^\mu_0) \\ &+ \frac{1}{3!}(\Omega^\nu_{\lambda\mu\nu})_0 (x^\lambda - x_0{}^\lambda)(x^\mu - x^\mu)(x^\nu - x_0^\nu) + \cdots , \end{aligned} \right. \tag{59}$$

where

$$\left\{ \begin{aligned} \Omega^\nu_{\lambda\mu} &= \Lambda^\nu_{\lambda\mu}, \\ \Omega^\nu_{\lambda\mu\omega} &= \Lambda^\nu_{\lambda\mu\omega} + 3\Omega^\nu_{\alpha(\lambda}\Lambda^\alpha_{\mu\omega)}, \\ \Omega^\nu_{\lambda_1\lambda_2\lambda_3\lambda_4} &= \Lambda^\nu_{\lambda_1\lambda_2\lambda_3\lambda_4} + 4\{\Omega^\nu_{\alpha(\lambda_1}\Lambda^\alpha_{\lambda_2\lambda_3\lambda_4)} \\ &\quad + \Omega^\nu_{\alpha\beta}\Lambda^{\beta}_{(\lambda_1\lambda_2}\Lambda^\alpha_{\lambda_3\lambda_4)} + \Omega^\nu_{\alpha(\lambda_1\lambda_2}\Lambda^\alpha_{\lambda_3\lambda_4)}\}. \end{aligned} \right. \tag{60}$$

Through (58) and (59) new coordinates are defined in a finite region around $P(x_0^\nu)$, the *normal coordinates*. They are the natural generalization of Riemann's normal coordinates for Riemannian geometry,* and were first introduced by Veblen (1922. 3). Under an arbitrary transformation (4) of the x the y^ν are transformed linearly and homogeneously, as follows from (57), and precisely as the components of a contravariant vector. In an E_n the (y^ν) would represent the radius vector of a point with respect to P, and then the *normal coordinates give a representation of the X_n on the E_n at P.*

The ordinary derivatives of a quantity at P with respect to the normal coordinates at P are components of tensors.

* See also G. D. Birkhoff, *Relativity and Modern Physics*, Cambridge: Harvard University Press, 1923. Birkhoff calls these coordinates *Riemannian* coordinates (p. 118). He uses the term *Normal coordinates* (p. 124) for another purpose.

For example, when $v_{\lambda\mu\nu}$ is an affinor, and $\bar{v}_{\lambda\mu\nu}$ are its components in the system of normal coordinates, then the equations

$$(61)\qquad \begin{cases} w_{\lambda\mu\nu\omega} = \left(\dfrac{\partial \bar{v}_{\lambda\mu\nu}}{\partial y^{\omega}}\right)_0, \\ w_{\lambda\mu\nu\omega_1\omega_2} = \left(\dfrac{\partial^2 \bar{v}_{\lambda\mu\nu}}{\partial y^{\omega_1}\partial y^{\omega_2}}\right)_0 \end{cases}$$

define a set of functions of x^{ν} which satisfy the equations

$$(62)\qquad \begin{cases} w_{\lambda\mu\nu\omega} = \nabla_{\omega} v_{\lambda\mu\nu}, \\ w_{\lambda\mu\nu\omega_1\omega_2} = \nabla_{\omega_1}\nabla_{\omega_2} v_{\lambda\mu\nu}. \end{cases}$$

The first derivatives with respect to the normal coordinates are identical with the symmetric part of the covariant derivative. A more complete set of formulas (62) is given by Veblen-Thomas (1923. 14, pp. 573–576).

As is easily found from (50) and (58) or from (59), $\Lambda^{\nu}_{\lambda\mu}$ vanishes for normal coordinates at P. This takes place for every transformation which has two terms in common with the right side of (59), in particular for

$$(63)\qquad x^{\nu} - x^{\nu}_0 = y^{\nu} - \frac{1}{2!}(\Lambda^{\nu}_{\lambda\mu})_0 y^{\lambda} y^{\mu}.$$

Such a system of coordinates is called *geodesic* (Weyl 1918. 2, p. 101 of 4th ed.) or *path coordinates* (Eisenhart, 1923. 11). The expansion for $\Lambda^{\nu}_{\lambda\mu}$ in normal coordinates is

$$(64)\qquad \Lambda^{\nu}_{\lambda\mu} = M^{\nu}_{\lambda\mu\alpha} y^{\alpha} + \frac{1}{2!} M^{\nu}_{\lambda\mu\alpha\beta} y^{\alpha} y^{\beta} + \cdots.$$

where

$$(65)\qquad M^{\nu}_{\lambda\mu\alpha_1\alpha_2\cdots\alpha_p} = \left(\frac{\partial^p \Lambda^{\nu}_{\lambda\mu}}{\partial y^{\alpha_1}\partial y^{\alpha_2}\cdots\partial y^{\alpha_p}}\right)_0.$$

The $\Lambda^{\nu}_{\lambda\mu}$ have no tensor character.

8. *Curvature.* If P and Q are points of a curve and v^{ν} is a field of vectors, the integral $\int_P^Q \delta v^{\nu}$ taken along s is the difference

of v^ν at Q and the vector obtained by parallel moving of the vector at P along s to Q. This integral depends not only on P and Q but also on the curve s. When it does not depend on the curve, but only on P and Q for every pair of points P and Q, the X_n is called *holonomous* (Cartan 1925.2, see 1923.21). If s is closed and $Q=P$, we find in general a difference after the return of the vector at P to P. This difference can be calculated easily if the closed curve s is infinitely small. We find for the difference Dv^ν at P

$$Dv^\nu = R_{\omega\mu\lambda}^{\cdots\nu} v^\lambda f^{\mu\omega} d\sigma, \tag{66}$$

where $f^{\omega\mu}$ is a unit bivector in the plane element which contains s (but for differential elements of higher than the first order), $d\sigma$ is the area of the element (both in such a measure that $f^{\omega\mu}\, d\sigma$ represents the bivector of the surface element determined by s), and

$$R_{\omega\mu\lambda}^{\cdots\nu} = \frac{\partial}{\partial x^\mu}\Gamma_{\omega\lambda}^{\nu} - \frac{\partial}{\partial x^\omega}\Gamma_{\mu\lambda}^{\nu} + \Gamma_{\kappa\mu}^{\nu}\Gamma_{\lambda\mu}^{\kappa} - \Gamma_{\kappa\omega}^{\mu}\Gamma_{\lambda\mu}^{\kappa}. \tag{67}$$

In the same way, we find

$$Dw_\lambda = - R_{\omega\mu\lambda}^{\cdots\nu} w_\nu f^{\omega\mu} d\sigma. \tag{68}$$

The symbol $R_{\omega\mu\lambda}^{\cdots\nu}$ is an affinor, the so called *curvature affinor* of the X_n. It always appears in investigations where the alternating part of the second derivative is considered, with $C_{\lambda\mu}^{\cdots\nu}=0$, according to the formulas

$$\begin{cases} 2\nabla_{[\omega}\nabla_{\mu]} v^\nu = - R_{\omega\mu\lambda}^{\cdots\nu} v^\lambda + 2S_{\omega\mu}^{\cdots\alpha}\Delta_\alpha v^\nu, \\ \\ 2\nabla_{[\omega}\nabla_{\mu]} w_\lambda = R_{\omega\mu\lambda}^{\cdots\nu} w_\nu + 2S_{\omega\mu}^{\cdots\alpha}\nabla_\alpha w_\lambda. \end{cases} \tag{69}$$

The interpretation (66), which for the case of a Riemannian geometry was given by Levi-Civita (1917.1), was given for Weyl's geometry by Weyl (1918.2). Compare for discussion and exact demonstration Synge (1923.12) and T. Y. Thomas (1926.3). See also Eddington (1923.4, p. 68, and p. 214 of second edition).

The effect of the operator $\nabla_{[\omega}\nabla_{\mu]}$ on quantities of higher degree is seen (see (14)) from the example

$$2\nabla_{[\omega}\nabla_{\mu]}v_{\alpha}^{\cdot\beta\gamma} = R_{\omega\mu\alpha}^{\cdots\rho}\, v_{\rho}^{\cdot\beta\gamma} - R_{\omega\mu\rho}^{\cdots\beta}\, v_{\alpha}^{\cdot\rho\gamma} - R_{\omega\mu\rho}^{\cdots\gamma}\, v_{\alpha}^{\cdot\beta\rho} + S_{\omega\mu}^{\cdot\cdot\delta}\,\nabla_{\delta}v_{\alpha}^{\cdot\beta\gamma}. \tag{70}$$

The affinor $R_{\omega\mu\lambda}^{\cdots\nu}$ is subject to the following identities. From the definition equation (66) follows the *first identity*:

$$R_{\omega\mu\lambda}^{\cdots\nu} = -R_{\mu\omega\lambda}^{\cdots\nu}. \tag{71}$$

Again, by calculating $\nabla_{[\omega}\nabla_{\mu}\nabla_{\lambda]}$, first by applying ∇_{ω} to $\nabla_{\mu}v_{\lambda}$ and then alternating, secondly by applying $\nabla_{[\omega}\nabla_{\mu]}$ to v_{λ} and then alternating, we find the *second identity*:

$$R_{[\omega\mu\lambda]}^{\cdots\nu} = 4S_{[\omega\mu}^{\cdot\cdot\alpha}S_{\lambda]\alpha}^{\cdot\cdot\nu} + 2\nabla_{[\omega}S_{\mu\lambda]}^{\cdot\cdot\nu}. \tag{72}$$

For a symmetric displacement this passes into the form

$$R_{[\omega\mu\lambda]}^{\cdots\nu} = 0, \tag{73}$$

or

$$R_{\omega\mu\lambda}^{\cdots\nu} + R_{\mu\lambda\omega}^{\cdots\nu} + R_{\lambda\omega\mu}^{\cdots\nu} = 0. \tag{74}$$

Moreover, by calculating $\nabla_{[\omega}\nabla_{\mu]}\, g_{\lambda\nu}$, first by applying ∇_{ω} to $\nabla_{\mu}\, g_{\lambda\nu}$, given as in (33), and then alternating, secondly by applying $\nabla_{[\omega}\nabla_{\mu]}$ to $g_{\lambda\nu}$ as in (70), we get the *third identity*:

$$R_{\omega\mu\lambda}^{\cdots\alpha}\, g_{\alpha\nu} + R_{\omega\mu\nu}^{\cdots\alpha}\, g_{\alpha\lambda} = 2\nabla_{[\omega}Q_{\mu]\lambda\nu}, \tag{75}$$

where

$$Q_{\mu\lambda\nu} = -\, g_{\lambda\alpha}g_{\nu\beta}Q_{\mu}^{\cdot\alpha\beta}.$$

For a Riemannian geometry, this passes into

$$R_{\omega\mu\lambda\nu} = -\, R_{\mu\omega\lambda\nu}. \tag{76}$$

In a Riemannian geometry, we can deduce from the three identities the fourth:

$$R_{\omega\mu\lambda\nu} = R_{\lambda\nu\omega\mu}. \tag{77}$$

The four identities of $R_{\omega\mu\lambda}^{\cdots\nu}$ for a Riemannian geometry have been given by Riemann (in his Paris memoir, 1861); for more general cases they appear in Weyl, Eddington, Veblen, Eisenhart and Schouten's papers, in Schouten's book also for the case of $C_{\lambda\mu}^{\cdots\nu} \neq 0$ (1924. 9, p. 87). The dependence of the fourth identity on the other three was first pointed out by Ricci* (see also Hessenberg 1916. 1). It also follows as a corollary from a general result of T. Y. Thomas (1926. 10).

For the derivative of $R_{\omega\mu\lambda}^{\cdots\nu}$ also an identity exists, which can be obtained, for example, by calculating $\nabla_{[\xi}\nabla_{\omega}\nabla_{\mu]}v_{\lambda}$ first by applying ∇_{ξ} on $\nabla_{[\omega}\nabla_{\mu]}v_{\lambda}$, secondly by applying $\nabla_{[\xi}\nabla_{\omega}$ on $\nabla_{\mu}q_{\lambda}$ and then alternating on ξ, ω, μ. For $C_{\lambda\mu}^{\cdots\nu} = 0$ it is

$$\nabla_{[\xi}R_{\omega\mu]\lambda}^{\cdots\nu} = -2S_{[\omega\mu}^{\cdots\alpha}R_{\xi]\alpha\lambda}^{\cdots\nu}. \tag{78}$$

This identity is called *Bianchi's identity*. For a symmetric displacement it becomes

$$\nabla_{[\xi}R_{\omega\mu]\lambda}^{\cdots\nu} = 0. \tag{79}$$

Bianchi published this identity in 1902 for a Riemannian geometry,† though it can be found in earlier papers of Padova (1889), who refers to a letter of Ricci.‡ Bach (1921. 3) proved it for a geometry of Weyl, Veblen (1922. 3) and Schouten (1923. 6) for symmetric displacements; Weitzenböck (1923. 25, p. 357) deduced (78). The general formulas for $C_{\lambda\mu}^{\cdots\nu} \neq 0$ can be found in Schouten (1924. 9, p. 91). (See also Schouten-Struik 1924. 10.) Cartan obtained (78) in another way (1923. 21, vol. 40, p. 373).

From $R_{\omega\lambda\mu}^{\cdots\nu}$, the following affinors can be deduced:

$$\begin{cases} R_{\mu\lambda} = R_{\omega\mu\lambda}^{\cdots\omega} \\ V_{\omega\mu} = R_{\omega\mu\lambda}^{\cdots\lambda} \\ F_{\mu\lambda} = R_{[\mu\lambda]} = R_{\omega[\mu\lambda]}^{\cdots\omega}. \end{cases} \tag{80}$$

* G. Ricci, *Sulla determinazione di varietà, dotate di proprietà intrinseche date a priori*, Rendiconti dei Lincei, (5), vol. 19^{II} (1910), pp. 85–90.

† L. Bianchi, *Sui simboli a quattro indice e sulla curvatura di Riemann*, Rendiconti dei Lincei, (5), vol. 11^{II} (1902), pp. 3–7.

‡ E. Padova, *Sulle deformazioni infinitesime*, Rendiconti dei Lincei, (4), vol. 5^{I} (1889), pp. 174–178.

In a Riemannian geometry $V_{\omega\mu}$ and $F_{\mu\lambda}$ vanish, and $R_{\mu\lambda}$ is symmetric. We then can build up a scalar

$$R = g^{\lambda\mu}R_{\mu\lambda} = g^{\lambda\mu}g^{\omega\nu}R_{\omega\mu\lambda\nu}. \tag{81}$$

From the differential equations for the quantities (80), obtained by contraction from the identity of Bianchi, we mention only

$$\nabla_\mu G^{\mu\lambda} = 0, \qquad G_{\mu\lambda} = R_{\mu\lambda} - \tfrac{1}{2}Rg_{\mu\lambda}, \tag{82}$$

which only holds for a Riemannian geometry. It was obtained by Ricci* and is one of the principal formulas of Einstein's theory of relativity. The equation

$$R_{\omega\mu\lambda}^{\cdots\lambda} = V_{\omega\mu} = 0 \tag{83}$$

has a simple geometric meaning. It means that the displacement conserves the volume of an n-dimensional element after a circuit along a closed curve. Hence we can introduce the conception of *volume* in the X_n, and there exists an integral invariant

$$\int U_{\lambda_1 \cdots \lambda_n} f^{\lambda_1 \cdots \lambda_n} d\tau_n, \tag{84}$$

where $U_{\lambda_1 \cdots \lambda_n}$ is an arbitrary n-vector, and $f^{\lambda_1 \cdots \lambda_n}$ a unit n-vector, $d\tau_n = dx^1 \cdots dx^n$ (Eddington 1921. 1, page 111; Eisenhart 1922.5; and Veblen 1923. 8). In normal coordinates, hence for a symmetric displacement, the curvature affinor arises from the expansion of $\Lambda_{\lambda\mu}^{\nu}$ in normal coordinates (64) in the form

$$(R_{\omega\mu\lambda}^{\cdots\nu})_0 = \left(\frac{\partial}{\partial y^\mu}\Lambda_{\lambda\omega}^{\nu}\right)_0 - \left(\frac{\partial}{\partial y^\omega}\Lambda_{\lambda\mu}^{\nu}\right)_0. \tag{85}$$

From this property the two first identities and the identity of Bianchi are easily proved (Veblen-Thomas 1923. 14, pp.

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* G. Ricci, *Sulle superficie geodetiche in una varietà qualunque e in particolare nella varietà a tre dimensioni*, Rendiconti dei Lincei, (5), vol. 12 (1903), pp. 409–420.

579–580). The holonomous case is characterized by $R_{\omega\mu\lambda}^{\cdots\nu}=0$. An example is the euclidean case.

9. *Projective Transformations of the Paths.* We consider a symmetric displacement with $C_{\lambda\mu}^{\cdots\nu}=0$ (affine displacement). We then can ask if there are transformations of the $\Gamma_{\lambda\mu}^{\nu}$, which leave the paths invariant. They are found to be

$$'\Gamma_{\lambda\mu}^{\nu} = \Gamma_{\lambda\mu}^{\nu} + A_{\lambda}^{\nu}p_{\mu} + A_{\mu}^{\nu}p_{\lambda}, \tag{86}$$

where p_λ is an arbitrary covariant vector. (Weyl 1921. 2; Veblen-Thomas 1926. 4, p. 281.) Such transformations are called *projective*. Under such a transformation the curvature affinor $R_{\omega\mu\lambda}^{\cdots\nu}$ passes into

$$\begin{aligned} '\!R_{\omega\mu\lambda}^{\cdots\nu} = R_{\omega\mu\lambda}^{\cdots\nu} &- 2A_{\lambda}^{\nu}\nabla_{[\omega}p_{\mu]} \\ &+ 2A_{[\omega}\nabla_{\mu]}p_{\lambda} + 2A_{[\omega}^{\nu}p_{\mu]}p_{\lambda}. \end{aligned} \tag{87}$$

From (87) follows

$$'\!R_{\omega\mu\lambda}^{\cdots\lambda} = 'V_{\omega\mu} = V_{\omega\mu} - 2(n+1)\nabla_{[\omega}p_{\mu]}. \tag{88}$$

This equation is completely integrable on account of Bianchi's identity; hence we find

$$\nabla_{[\xi}R_{\omega\mu]\lambda}^{\cdots}v^{\lambda} = \Delta_{[\xi}v_{\omega\mu]} = 0, \tag{89}$$

so that we see, that *an affine displacement can always be projectively transformed into a displacement which preserves volume* (Eisenhart, 1922. 5; Veblen, 1922. 6; 1922. 13).

The projective theory of affine displacements is the theory of those tensors which are unchanged by the transformation (86). See, however, a remark of Veblen-Thomas (1926. 4). One of these tensors, the *projective curvature tensor*, was found by Weyl (1921. 2); it is

$$P_{\omega\mu\lambda}^{\cdots\nu} = R_{\omega\mu\lambda}^{\cdots\nu} - 2P_{[\omega\mu]}A_{\lambda}^{\nu} + 2A_{[\omega}^{\nu}P_{\mu]\lambda}, \tag{90}$$

where

$$P_{\mu\lambda} = -\frac{1}{n^2-1}(nR_{\mu\lambda} + R_{\lambda\mu}). \tag{91}$$

For $n=1$ and $n=2$, $P^{\cdots\nu}_{\omega\mu\lambda}$ vanishes identically. It satisfies, for $n>2$, the identities

$$(92)\qquad \begin{cases} P^{\cdots\nu}_{\omega\mu\lambda} = -P^{\cdots\nu}_{\mu\omega\lambda}, \\ P^{\cdots\nu}_{[\omega\mu\lambda]} = 0, \\ \nabla_{[\xi}P^{\cdots\nu}_{\omega\mu]\lambda} = \dfrac{1}{n-2}\nabla_\alpha A^\nu{}_{[\omega}P^{\cdots\omega}_{\mu\xi]\lambda}, \end{cases}$$

which were given by Schouten (1924.9, pp. 131–132).

To obtain the projective tensors in a systematic way, we start with the observation of T. Y. Thomas (1925. 7) that the functions

$$(93)\qquad \Pi^\nu_{\lambda\mu} = \Gamma^\nu_{\lambda\mu} - \frac{1}{n+1}(A^\nu_\lambda\Gamma^\alpha_{\mu\alpha} + A^\nu_\mu\Gamma^\alpha_{\lambda\alpha})$$

are unaltered by transformations (86). The $\Pi^\nu_{\lambda\mu}$ build up a *projective displacement*, and are subject to the identity

$$(94)\qquad \Pi^\alpha_{\lambda\alpha} = 0.$$

The paths of the projective displacements can now be made solutions of the equations

$$(95)\qquad \frac{d^2x^\nu}{dp^2} + \Pi^\nu_{\lambda\mu}\frac{dx^\lambda}{dp}\frac{dx^\mu}{dp} = 0,$$

where p is the projective parameter, defined but for a transformation

$$(96)\qquad 'p = A\int\Delta^{2/(n+1)}dp + B,$$

where $\Delta = |\partial' x^\lambda/\partial x^\mu|$ (see (3)), and where A and B are constants. This transformation involves Δ, and the same holds for that of $\Pi^\nu_{\lambda\mu}$:

$$(97)\qquad \Pi^\nu_{\lambda\mu}\frac{\partial' x^\omega}{\partial x^\nu} = {}'\Pi^\omega_{\rho\sigma}\frac{\partial' x^\rho}{\partial x^\lambda}\frac{\partial' x^\sigma}{\partial x^\mu} + \frac{\partial^2 x^\omega}{\partial x^\lambda\partial x^\mu} + \frac{1}{n+1}\left(\frac{\partial' x^\omega}{\partial x^\lambda}\frac{\partial \log\Delta^{-1}}{\partial x^\mu} + \frac{\partial' x^\omega}{\partial x^\mu}\frac{\partial\log\Delta^{-1}}{\partial x^\lambda}\right).$$

The problem of finding covariants of linear displacements may be stated as the study of the integrability conditions of the equations (9) for $\Gamma^{\nu}_{\lambda\mu}$. In the same way the problem of projective equivalence of two affine displacements may be reduced to the study of the integrability conditions of the equations (97). This has been done by Veblen-Thomas (1925. 8; 1926. 4), anticipated by J. M. Thomas (1925. 9) and T. Y. Thomas (1925. 7). They introduce *projective normal coordinates*, and use them for the deduction of projective covariants. These, however, need not have tensor character. This is only the case if $P^{\cdots\nu}_{\omega\mu\lambda}=0$. As the main result of this study we may consider a set of *completeness theorems* and the theorem on *projective equivalence* of two affine displacements.

For transformations with $\Delta=1$ the $\Pi^{\nu}_{\lambda\mu}$ transform like the $\Gamma^{\nu}_{\lambda\mu}$ and the p is uniquely determined. This case is called *equi-projective* (Thomas 1925. 7).

If the affine displacement can be projectively transformed so that $R^{\cdots\nu}_{\omega\mu\lambda}=0$, we call this case *projective-euclidean*. We then obtain from (87)

$$R^{\cdots\nu}_{\xi\omega\mu\lambda} = 2P_{[\omega\mu]}A^{\nu}_{\lambda} - 2A^{\nu}_{[\omega}P_{\mu]\lambda}. \tag{98}$$

The integrability conditions, that (91) may exist under the auxiliary condition (98), reduce to

$$\nabla_{[\omega}P_{\mu]\lambda} = 0. \tag{99}$$

When however, we apply the identity of Bianchi to (98), we obtain

$$\nabla_{[\xi}P_{\lambda\mu]} = \tfrac{1}{3}(n-2)\nabla_{[\xi}P_{\mu]\lambda}, \tag{100}$$

which for $n>2$ is identical with (99). In this case the integrability conditions of (91) are a consequence of (98). Then, as $P_{\mu\lambda}$ is defined by (91), (98) passes (see (90)) into

$$P^{\cdots\nu}_{\omega\mu\lambda} = 0. \tag{101}$$

For $n=2$ condition (98) is always satisfied. Hence we have the theorem that *an affine displacement for $n>2$ is projective-*

euclidean when and only when $P_{\omega\mu\lambda}^{\cdots\nu}=0$, *and for* $n=2$ *when and only when* $P_{\omega\mu}$, *defined by* (99) *satisfies* (91) (Weyl 1921. 2).

10. *Conformal Transformation of a Riemannian Manifold.* A close analogy exists between the projective transformations of affine manifolds and the conformal transformations

$$'g_{\lambda\mu} = \sigma g_{\lambda\mu}, \tag{102}$$

where σ is an arbitrary function of the x^ν, in a Riemannian manifold (102) implies

$$'g^{\lambda\mu} = \sigma^{-1}g^{\lambda\mu}. \tag{103}$$

Then (see (86)) we have

$$'\left\{\begin{matrix}\lambda\mu\\ \nu\end{matrix}\right\} = \left\{\begin{matrix}\lambda\mu\\ \nu\end{matrix}\right\} + \frac{1}{2}(A_\lambda^{\ \nu}s_\mu + A_\mu^{\ \nu}s_\lambda - g^{\nu\alpha}g_{\mu\lambda}s_\alpha), \tag{104}$$

where

$$s_\mu = \frac{\partial \log\sigma}{\partial x^\mu}. \tag{105}$$

These conformal transformations leave the angle of two vectors invariant. Under them the curvature affinor $R_{\omega\mu\lambda}^{\cdots\nu}$ passes into the form

$$'R_{\omega\mu\lambda}^{\cdots\nu} = R_{\omega\mu\lambda}^{\cdots\nu} - g_{\omega\lambda}s_{\mu\alpha}g^{\alpha\nu} + g_{\lambda\mu}s_{\omega\alpha}g^{\alpha\nu}, \tag{106}$$

where (see (87))

$$s_{\mu\alpha} = 2\nabla_\mu s_\alpha - s_\mu s_\alpha + \tfrac{1}{2}s_\beta s^\beta g_{\mu\alpha}. \tag{107}$$

Equations (104) were obtained by Fubini*; equation (106) by Weyl (1918. 3).

We can ask if a conformal transformation can preserve the geodesic lines. Then, by (86) and (104) we have

$$\tfrac{1}{2}A_\lambda^{\ \nu}s_\mu + \tfrac{1}{2}A_\mu^{\ \nu}s_\lambda - \tfrac{1}{2}s^\nu g_{\lambda\mu} = A_\lambda^{\ \nu}p_\mu + A_\mu^{\ \nu}p_\lambda, \tag{108}$$

* G. Fubini, *Sulla teoria degli spazi che ammettono un gruppo conforme*, Atti di Torino, vol. 38, (1903), pp. 262–276.

which cannot be fulfilled, as is easily shown by contraction with respect to ν, λ; and by transvection with $g^{\lambda\mu}$; for both operations give

$$s_\mu = \frac{2(n+1)}{n} p_\mu = -\frac{4}{n-2} p_\mu, \tag{109}$$

which has no solutions for integer n. Hence we have the theorem *that a Riemannian displacement is fully determined when we know its geodesic lines and the fundamental tensor but for a factor* (Weyl 1921. 2).

Hence projective and conformal transformations exclude each other. Eyraud (1925. 4) has shown how a general deformation of the paths can be decomposed into two deformations, one a generalization of the projective, the other of the conformal transformation.

The conformal theory of Riemannian displacements is the theory of those tensors which are unchanged by the transformation (102) (or 104). (See, however, Veblen-Thomas' remark 1926. 4.) One of these, the conformal curvature tensor, was found by Weyl (1918. 2); it is

$$C_{\omega\mu\lambda\nu} = R_{\omega\mu\lambda\nu} - \frac{2}{n-2}(g_{\omega\lambda}L_{\mu\nu} - g_{\mu\lambda}L_{\omega\nu}), \tag{110}$$

when

$$L_{\mu\lambda} = -R_{\mu\lambda} + \frac{1}{2(n-1)} R g_{\mu\lambda}. \tag{111}$$

For $n=1$, $n=2$, and $n=3$, $C_{\omega\mu\lambda\nu}$ vanishes identically. It satisfies, for $n>3$, the four identities

$$\left\{\begin{aligned} C_{\omega\mu\lambda\nu} &= -C_{\mu\omega\lambda\nu}, \\ C_{[\omega\mu\lambda]\nu} &= 0, \\ C_{\omega\mu\lambda\nu} &= -C_{\omega\mu\nu\lambda}, \\ C_{\omega\mu\lambda\nu} &= C_{\lambda\nu\omega\mu}. \end{aligned}\right. \tag{112}$$

To obtain the conformal affinor in a systematic way, we start with the remark of J. M. Thomas (1925. 12) that the functions

$$(113)\quad Z_{\lambda\mu}^{\nu} = \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\} - \frac{1}{n}\left[A_{\lambda}^{\nu} \left\{ \begin{matrix} \mu\alpha \\ \alpha \end{matrix} \right\} + A_{\mu}^{\nu} \left\{ \begin{matrix} \lambda\alpha \\ \alpha \end{matrix} \right\} \right] + \frac{1}{n} g^{\nu\alpha} g_{\lambda\mu} \left\{ \begin{matrix} \alpha\beta \\ \beta \end{matrix} \right\}$$

are unaltered by transformations (102). The $Z_{\lambda\mu}^{\nu}$ build up a *conformal displacement*, and are subject to the identity

$$(114)\quad Z_{\lambda\alpha}^{\alpha} = 0.$$

The paths of the conformal displacement are given by

$$(115)\quad \frac{d^2x^{\nu}}{dc^2} + Z_{\lambda\mu}^{\nu} \frac{dx^{\lambda}}{dc} \frac{dx^{\mu}}{dc} = 0,$$

where c is the *conformal parameter*. For a discussion of this equation in the same way as (95) see J. M. Thomas (1925. 12). See also T. Y. Thomas (1925. 15; 1926. 14).

If the Riemannian displacement can be conformally transformed so that $R_{\omega\mu\lambda}^{\nu} = 0$, this case is called *conformal-euclidean*. We then obtain, from (104),

$$(116)\quad R_{\omega\mu\lambda\nu} = g_{\omega\lambda} s_{\mu\nu} - g_{\mu\lambda} s_{\omega\nu}.$$

The integrability conditions that (116) may exist under the auxiliary condition (107), reduce to

$$(117)\quad \nabla_{[\omega} s_{\mu]\lambda} = 0.$$

When, however, we apply the identity of Bianchi to (116), we obtain

$$(n-3)\nabla_{[\omega} s_{\mu]\lambda} = 0,$$

which, for $n > 3$, is identical with (117). Then, as $s_{\mu\lambda}$ is defined by (107), (116) passes into (see (110))

$$C_{\omega\mu\lambda\nu} = 0.$$

For $n=3$, condition (116) is always satisfied. We thus have (see the end of §9) *that a Riemannian displacement for* $n>3$ *is conformal-euclidean when and only when* $C_{\omega\mu\lambda\nu}=0$, *and for* $n=3$, *when and only when* $s_{\omega\mu}$, *defined by* (116), *satisfies* (117) (Schouten 1921. 7; for $n=3$ Cotton,* Finzi†).

11. *The Point of View of Point Displacement.* The ideas expressed above all start with the consideration of *vectors* in the E_n at one point P of the X_n that are compared to *vectors* in an E_n at another point P' at infinitesimal distance. Cartan has made the remark that this is not the only way to proceed. We can also start with the *points* in the E_n at a point P of X_n and define our displacement as a law that enables us to compare the *points* at P with those at P'. In this manner, Cartan first derived in a new way the asymmetric affine displacement (1922. 8), and later on could come to generalizations of projective and conformal geometry. We limit ourselves here to an exposition of Cartan's ideas for an affine displacement, and only give some outlines of his further generalizations. Consider, for this (Cartan 1923. 21, the notation is chiefly based on Schouten 1924. 8 and 1926. 12), the E_n of the dx^ν at a point P of the X_n and the E_n' at P'. Then the formulas give a law of comparison of the *free* vectors E_n and E_n'. We get a law of comparison of the *points* at E_n and E_n' if we also know how a definite point of E_n passes into a definite point of E_n'. Let us now first obtain in this new way the affine displacements. Introduce in the E_n *homogeneous* coordinates. Every set u^α, $\alpha=0, 1, \cdots, n+1$, abbreviated $\bar{u}$, determines a *point* by its proportion. Then a fundamental $(n+1)$-cell is defined by $n+1$ points $\bar{u}_0, \bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$. Let $\bar{u}_0$ be P itself, and let $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ be the points at infinity at n oblique axes issuing from P. If

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* E. Cotton, *Sur les variétés à trois dimensions*, Annales de Toulouse, (2), vol. 1 (1899), pp. 385–438; also Thèse, Paris, 1899, 61 pp.

† A. Finzi, *Le ipersuperficie a tre dimensioni che si possono rappresentare conformemente sullo spazio euclideo*, Atti, Istituto Veneto, (8), vol. 5(=62), (1902–03), pp. 1049–1062.

we now assume an affine correspondence between E_n at P and E_n' at P', we get our linear connection defined by

$$\delta \bar{u}_\lambda = \Gamma^{\kappa}_{\lambda\mu} \bar{u}_\kappa dx^\mu, \quad (\lambda, \mu, \kappa = 1, \cdots, n). \tag{118}$$

If we now fix the correspondence of P to its image in E' by the equation

$$\delta \bar{u}_0 = - \bar{u}_\mu dx^\mu, \tag{119}$$

a linear connection is determined:

$$\left\{ \begin{array}{l} \nabla_\mu v^\alpha = \dfrac{\partial v^\alpha}{\partial x^\mu} + \Lambda^{\alpha}_{\beta\mu} v^\beta, \\ \\ \nabla_\mu w_\alpha = \dfrac{\partial w_\alpha}{\partial x^\mu} - \Lambda^{\beta}_{\alpha\mu} w_\beta, \qquad (\alpha, \beta = 0, 1, \cdots, n), \end{array} \right. \tag{120}$$

where

$$\left\{ \begin{array}{l} \Lambda^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{\lambda\mu}, \\ \\ -\Lambda_{\lambda 0} = A^{\nu}_{\mu} \begin{cases} = 1, & \nu = \mu, \\ = 0, & \nu \neq \mu, \end{cases} \\ \\ \Lambda^{0}_{\alpha\mu} = 0. \qquad (\lambda, \mu, \nu = 1, \cdots, n). \end{array} \right. \tag{121}$$

A special case of (119) was treated by Hlavatý (1924. 21). These covariant differentials are mixed quantities, because the differentials dx^μ and the vectors $\bar{u}$ belong to different manifolds. In X_n, however, a displacement is not fixed. But if we assume

$$2\nabla_{[\mu} w_{\lambda]} = \frac{\partial w_\lambda}{\partial x^\mu} - \frac{\partial w_\mu}{\partial x^\lambda} \tag{122}$$

which means that this displacement is at least symmetric, we can define $\nabla_{[\omega}\nabla_{\mu]} v^\nu$ and $\nabla_{[\omega}\nabla_{\mu]} v_\lambda$ and obtain a curvature quantity

$$\mathfrak{A}^{\cdots\beta}_{\omega\mu\alpha} = \frac{\partial}{\partial x^\mu}\Lambda^{\beta}_{\alpha\omega} - \frac{\partial}{\partial x^\omega}\Lambda^{\beta}_{\alpha\mu} + \Lambda^{\beta}_{\delta\mu}\Lambda^{\delta}_{\alpha\mu} - \Lambda^{\beta}_{\delta\omega}\Lambda^{\delta}_{\alpha\mu}, \tag{123}$$
$$(\omega, \mu = 1, \cdots, n\ ;\ \alpha, \beta, \delta = 0, 1, \cdots, n).$$

For this tensor we have the first identity and Bianchi's identity

$$(124)\qquad \begin{cases} \mathfrak{A}_{\omega\mu\alpha}^{\cdots\beta} = -\mathfrak{A}_{\mu\omega\alpha}^{\cdots\beta} \\ \nabla_{[\xi}\mathfrak{A}_{\omega\mu]\alpha}^{\cdots\beta} = 0. \end{cases}$$

$\mathfrak{A}_{\omega\mu\alpha}^{\cdots\beta}$ can be derived from $R_{\omega\mu\lambda}^{\cdots\nu}$ and $S_{\mu\lambda}^{\cdots\nu}$. We have

$$(125)\qquad \begin{cases} \mathfrak{A}_{\omega\mu\lambda}^{\cdots\nu} = R_{\omega\mu\lambda}^{\cdots\nu}\ ; \quad \mathfrak{A}_{\omega\mu\lambda}^{\cdots 0} = 0 \\ \mathfrak{A}_{\omega\mu 0}^{\cdots\nu} = -2S_{\omega\mu}^{\cdots\nu}\ ; \quad \mathfrak{A}_{\omega\mu 0}^{\cdots 0} = 0. \end{cases}$$

By means of the formula $Dv^\beta = f^{\omega\mu}\, d\sigma\ \mathfrak{A}_{\omega\mu\alpha}^{\cdots\beta} v^\alpha$, we see that a symmetric displacement is characterized by the fact that $\tilde{u}_0$ returns in its own place after a transport along a closed circuit. This is another statement of the fact expressed by formula (24), that in the non-symmetric case the manifold is "infinitely crinkled."

If $R_{\omega\mu\alpha}^{\cdots\gamma} = 0$, we get an E_n. In this case the displacement passes *into the correspondence of all points of the E_n into themselves*.

This last remark leads toward a generalization for the projective and conformal cases, which can now be treated in a way analogous to the preceding paragraph (Cartan 1923. 20; 1924. 4; 1924. 3). The difference is that the E_n at a point P of an X_n must be considered as a projective or conformal space. By the introduction of homogeneous coordinates, projective space is equivalent to affine space of $n+1$ dimensions; by the introduction of tetracyclical, pentaspherical, etc., coordinates, conformal space is equivalent to affine space of $n+2$ dimensions. We thus come to the following generalization: Associate to every point P of an X_n an E_{n+k}, $k = 0, 1, 2, \cdots$ and generalize the method of the preceding paragraph to this case. We then have three kinds of quantities:

(a) Quantities in X_n, defined for the E_n of the dx^ν.

(b) Quantities in the E_{n+k} defined in every E_{n+k} with respect to a system of $n+k$ covariant and $n+k$ contravariant fundamental vectors.

(c) Quantities obtained by summing and multiplying the quantities of first and second kind.

We obtain a displacement if we determine how the quantities of the second kind of the E_{n+k} at P correspond to those of the E_{n+k} at a neighboring point P'. We thus obtain a method of treating the results of point displacements by the methods of vector displacement, and find a kind of geometry previously introduced, at least in general, by König (1919. 1; 1920. 1).

Cartan, using his point displacement, now proceeds in an analogous way as in the affine case. He deduces a curvature tensor for the projective and the conformal displacement

$$\mathfrak{P}_{\omega\mu a}^{\cdots c}; \qquad \begin{matrix} \omega,\mu = 1, \cdots, n \text{ in } X_n; \\ a,c = 0,1, \cdots, n \text{ in } E_{n+1}, \end{matrix} \tag{126}$$

and

$$\mathfrak{C}_{\omega\mu ac}; \qquad \begin{matrix} \omega,\mu = 1, \cdots, n \text{ in } X_n; \\ a,c = 0,1, \cdots, n+1 \text{ in } E_{n+2}. \end{matrix} \tag{127}$$

The relation between point displacements of Cartan and the vector displacements are given by the theorems:

If in an X_n an affine displacement is fixed but for transformations that leave the paths invariant, there is one and only one symmetric projective point connection with the same geodesic lines, for which

$$\sum_k \mathfrak{P}_{kli}^{\cdots k} = 0; \qquad k,l = 1, \cdots, n.$$

Moreover, we have

$$\mathfrak{P}_{kli}^{\cdots m} = P_{kli}^{\cdots m}.$$

The special coordinate system in X_n with respect to which their components are taken is one for which the given affine displacement preserves volume (Cartan 1924. 4, Schouten 1924. 8).

If in an X_n, $n \geqq 3$, *Riemann displacement is fixed but for conformal displacements, there is one and only one symmetric conformal point connection with the orthogonality fixed by* $g_{\mu\nu}$, *for which* $\sum_k \mathfrak{C}_{klik} = 0$; $k, l = 1, \cdots, n$. Moreover, $\mathfrak{C}_{klim} = C_{klim}$.

The coordinate system in X_n with respect to which their components are taken is an orthogonal system (Cartan 1923. 20). Cartan's results are discussed from the point of view of vector displacement by Schouten (1926. 12), where also some general remarks on their relation toward the group program of Felix Klein's Erlanger address can be found. A good introduction to Cartan's conceptions is given by his papers of a more general character (1924. 5; 1925. 1).

12. *Manifolds Immersed in Manifolds with a Linear Displacement.* An important reason for the study of X_m, $m < n$, in an X_n with a linear displacement was the desire to have a treatment of affine differential geometry, developed by Blaschke and others,* on the basis of linear displacement. König (1919. 1) pointed out the way, and showed that this affine geometry was the geometry of a linear displacement with

$$(128) \qquad C_{\lambda\mu}^{\cdots\nu} = 0, \quad \Gamma_{\lambda\mu}^{\nu} = \Gamma_{\mu\lambda}^{\nu}, \quad R_{\alpha\mu\lambda}^{\cdots\nu} = 0.$$

Schouten (1923. 7; 1924. 9) took up König's suggestion and gave an elaborate study of this subject. Cartan (1924. 22; 1924. 1; 1923. 21) studied the same subject independently of König. Schouten starts with an X_n and a displacement for which $C_{\lambda\mu}^{\cdots\nu} = 0$, $\Gamma_{\lambda\mu}^{\nu} = \Gamma_{\mu\lambda}^{\nu}$. Consider now an X_{n-1} immersed in the X_n, and a point P of X_{n-1}. Let v^ν be a vector field in X_n; if a vector v^ν is not situated in the tangential E_{n-1} at P, we cannot speak without further assumption of the v^ν-*component* of v^ν in E_{n-1}. This is only possible after the

* See W. Blaschke, *Vorlesungen über Differentialgeometrie*, II, *Affine Differentialgeometrie*, Berlin, Springer, 1923.

introduction of a special direction at P not situated in E_{n-1}, the *pseudonormal* direction. As a covariant vector can be represented by a set of two parallel E_{n-1}, we see that a covariant vector determines, by its intersection with the tangential E_{n-1}, a covariant vector in this E_{n-1} (representable by two parallel E_{n-2}), without any further assumption. After the introduction of a pseudonormal direction, however, it is possible to associate to a covariant vector field in X_{n-1} also a covariant vector field in X_n.

In the same way, we can consider tensors, especially the covariant derivatives of scalars and vectors. Let B_β^α be the unit tensor of the X_{n-1}, t_λ the tangential covariant vector of the X_{n-1}, and n^ν a vector in the pseudonormal direction. If we now assume $t_\lambda n^\lambda = 1$, which does not fix the scale of measure of t_λ, we have between B^ν and the unit tensor A_λ^ν of X_n the relation

$$B_\lambda^\nu = A_\lambda^\nu \, t_\lambda n^\nu. \tag{129}$$

We now define, ∇_μ being the differential operator with respect to the X_n, a linear displacement in the X_{n-1} by means of the equations

$$\begin{cases} {}'\nabla_\mu v^\nu = B_\mu^\alpha B_\beta^\nu \nabla_\alpha v^\beta \\ {}'\nabla_\mu w_\lambda = B_\mu^\alpha B_\lambda^\beta \nabla_\alpha w_\beta. \end{cases} \tag{130}$$

Even before introducing a normal vector, we may define the derivative of a scalar field

$${}'\nabla_\mu p = B_\mu^\alpha \nabla_\alpha p. \tag{131}$$

The displacement thus defined is also affine (see (17) and (27), since we have

$$\begin{cases} {}'\nabla_\mu B_\lambda^\nu = B_\mu^\alpha B_\lambda^\beta B_\delta \nabla_\alpha^\nu B_\beta^\delta = - B_\mu^\nu B_\lambda^\beta B_\lambda^\nu \nabla_\alpha t_\beta n^\delta = 0, \\ {}'\nabla_{[\omega} {}'\nabla_{\mu]} p = {}'\nabla_{[\omega}(B_{\mu]}^\alpha \nabla_\alpha p) = B_\mu^\alpha \nabla'_{[\omega} \nabla_{\alpha]} p \\ \qquad\qquad = B_\mu^\alpha B_\omega^\beta \nabla_{[\beta} \nabla_{\alpha]} p = 0, \end{cases} \tag{132}$$

We thus have the fundamental theorem :

If we define at every point of the X_{n-1} a pseudonormal direction, we can define an affine displacement in the X_{n-1} in

such a way that the covariant derivative of a quantity in X_{n-1} is the X_{n-1}-component of the covariant derivative with respect to the X_n.

The displacement in X_{n-1} is called *induced*. We now can induce the following tensor :

$$h_{\mu\lambda} = B_\mu^\alpha B_\lambda^\beta \nabla_\alpha t_\beta, \tag{133}$$

which, by choice of the part of t_β outside of the X_{n-1}, can be made symmetric. Change of the scale of measure for t_λ of the form $'t_\lambda = \sigma t_\lambda$, changes the tensor $h_{\mu\lambda}$ conformally:

$$'h_{\mu\lambda} = \sigma B_\mu^\alpha B_\lambda^\beta \nabla_\alpha t_\beta + B_\mu^\alpha B_\lambda^\beta t_\beta \nabla_\alpha \sigma = \sigma h_{\mu\lambda}. \tag{134}$$

This tensor is perfectly determined after choice of the scale of measure t_λ. In this case a Riemannian displacement is fixed in the X_{n-1}, if $h_{\lambda\mu}$ has the rank $n-1$, and then we take $h_{\lambda\mu}$ as fundamental tensor.

If $P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = 0$ in the X_n we can show that

$$T_{\lambda\mu\nu} = T_{\lambda\mu}^{\cdot\cdot\alpha} h_{\alpha\nu} \tag{135}$$

(see (38)) is symmetric, and also that (see (32))

$$T_{\lambda\mu\nu} = \tfrac{1}{2} Q_{\lambda\mu\nu}, \quad h^{\mu\nu} Q_{\lambda\mu\nu} = 0.$$

This condition of apolarity (135) is one of the signs that for the case of ordinary affine differential geometry we have in $h_{\mu\lambda}$ and $Q_{\lambda\mu\nu}$, the so-called first and second fundamental tensors. This geometry is obtained, when the X_n becomes an E_n, through the condition $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = 0$. The theory of X_{n-1} in X_n and also X_k in X_n $(k < n-1)$ can be developed on this basis. We can obtain, for example, the generalizations of the formulas of Gauss and Codazzi for these cases.

The case of $\Gamma_{\lambda\mu}^\nu \neq \Gamma_{\mu\lambda}^\nu$ for X_k in X_n was treated by Hlavatý (1926. 13).

Another treatment of these geometries was given by Cartan (1923. 21, vol. 40, p. 403; Cartan has many results not mentioned here). Moreover, Schouten (1924. 9) developed the theory of X_k in X_n for a Weyl geometry.

Frenet formulas for curves in a general linear displacement and a Weyl displacement, but with fixed $g_{\lambda\mu}$, were found by Juvet (1921. 5 ; 1921. 6 ; 1924. 13 ; 1925. 20). Schouten gave the general formulas (1924. 9); see also Juvet (1924. 14).

Eisenhart discussed parallel vector fields in an affine displacement (1922. 4). He calls the vectors of a field *parallel* to one another, if a scalar α exists, so that we have, independent of the direction of displacement dx^ν,

$$dx^\lambda \nabla_\lambda \alpha v^\nu = 0.$$

13. *Linear Displacements and Continuous Groups.* If we consider the ∞^r transformations of an r-parametric continuous group in an X_n as points of an X_r, we obtain the *group-manifold*, in which the parameters are coordinates. Let the transformations be

$$(136)\quad {}'x^\nu = {}'x^\nu(x^1, \cdots, x^n, a^1, \cdots, a^r), \quad (\nu = 1, \cdots, n);$$

then the a^λ, $\lambda = 1, \cdots, r$, are the coordinates of X_r. The transformation (136) may be denoted by T_a.

If T_a corresponds with the point a^ν and T_{a+da} with the point $a^\nu + da^\nu$, then the infinitesimal transformation $T_{a+da}T_a^{-1}$ corresponds to the line element da^ν. The line element db^ν at point b^ν is characterized by $T_{b+db}T_b^{-1}$. This correspondence of both transformations can be expressed by

$$T_{b+db}T_b^{-1} = T_{a+da}T_a^{-1}.$$

This defines a linear displacement, through which a vector at a point corresponds to one and only one vector at another point. Hence $R_{\omega\mu\lambda}{}^\nu = 0$. It can be shown, however, that $\Gamma^\nu_{\lambda\mu} \neq \Gamma_{\mu\lambda}$. Another displacement with these two properties can be defined by

$$T_b^{-1}T_{b+db} = T_a^{-1}T_{a+da}.$$

The study of simple and semi-simple groups* under this point of view was started by Cartan-Schouten (1925. 3). Related to these investigations is a paper of Eisenhart (1925. 10).

* First introduced by E. Cartan, *La structure des groupes de transformations continus et la théorie du trièdre mobile*, Bulletin des Sciences Mathématiques, (2), vol. 34 (1910), pp. 250–283.

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CONCERNING THE BOUNDARIES OF DOMAINS OF A CONTINUOUS CURVE*

BY W. L. AYRES

We shall consider a space M consisting of all the points of a plane continuous curve M†, and all point sets mentioned are assumed to be subsets of M. A connected set of points D of M is said to be an *M-domain* if $M-D$ is closed. The set of all limit points of D which do not belong to D is called the *M-boundary* of D. If D is an M-domain, D' denotes the M-boundary of D, and $\overline{D}$ denotes the set $D+D'$. The M-boundary of an M-domain D is closed but not necessarily connected, even if D is simply connected, as we may easily show by examples. If N is a continuum, a maximal connected subset of $M-N$ is called a *complementary M-domain* of N.

THEOREM I.‡ *Every closed and connected subset of the M-boundary of a complementary M-domain of a continuous curve N is a continuous curve.*

PROOF. Let K denote a closed and connected subset of the M-boundary of a complementary M-domain D of N. Suppose K is not connected im kleinen. Then there exist§ two concentric circles C_1 and C_2 (let r_i denote the radius of C_i and let $r=r_1-r_2>0$) and an infinite sequence of subcon-

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† A point set which is closed, connected and connected im kleinen is called a *continuous curve*. In general it may be either bounded or unbounded.

‡ For the case where M is the entire plane and N is bounded, see R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 361.

§ See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, this Bulletin, vol. 29 (1923), p. 296. This theorem is stated by Moore for a bounded continuum, but the theorem remains true without the condition of boundedness.

tinua of K, K_∞, K_1, K_2, K_3, $\cdots$, such that (1) each of these continua $K_\alpha(\alpha = \infty, 1, 2, \cdots)$ contains a point a_α on C_1 and a point b_α on C_2, but no point exterior to C_1 or interior to C_2, (2) no two of these continua have a point in common and no one of them, except possibly K_∞, is a proper subset of any connected subset of K which contains no point without C_1 or within C_2, (3) the set K_∞ is the sequential limiting set of the sequence of continua K_1, K_2, K_3, $\cdots$. For any i, let $K_i^* = K_\infty + K_{i+1} + K_{i+2} + \cdots$. Let d_1 be the smaller of the numbers $\frac{1}{4}r$ and $\frac{1}{2}d(K_1, K_1^*)$†.

Let R_1 denote the set of all points $[P]$ of N such that the distance from P to some point of K_1 is less than d_1. The set R_1 is an open subset of N and hence R_1 contains an arc from a_1 to b_1.‡ This arc contains a subarc x_1y_1 such that x_1 is on C_2 and y_1 is on C_1 and every other point of x_1y_1 is between C_1 and C_2.

For each $n > 1$, let d_n be the smallest of the numbers d_{n-1}, $r2^{-n-1}$, and $\frac{1}{2}d(K_n, K_n^* + x_1y_1 + x_2y_2 + \cdots + x_{n-1}y_{n-1})$. Let R_n denote the set of all points $[P]$ of N such that there is some point of K_n whose distance from P is less than d_n. The set R_n contains an arc from a_n to b_n and this arc contains a subarc x_ny_n such that x_n is on C_2, y_n is on C_1, and every other point of x_ny_n is between C_1 and C_2.

There exists an increasing sequence of positive integers n_1, n_2, n_3, $\cdots$, such that (1) C_2 contains three points X_1, X_2, X_3 such that every point x_{n_i} lies on the arc $X_1X_2X_3$ of C_2 and in the order $X_1X_2x_{n_1}x_{n_2}\cdots X_3$, (2) C_1 contains three points Y_1, Y_2, Y_3 such that every point y_{n_i} lies on the arc $Y_1Y_2Y_3$ of C_1 and in the order $Y_1Y_2y_{n_1}y_{n_2}\cdots Y_3$, (3) X_3 is the sequential limit point of $[x_{n_i}]$ and Y_3 is the sequential limit point of $[y_{n_i}]$. Clearly K_∞ contains X_3 and Y_3.

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† If A and B are sets of points, the symbol $d(A,B)$ denotes the lower bound of all numbers $d(x,y)$, where x is a point of A, y is a point of B, and $d(x,y)$ is the distance from x to y.

‡ R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 255.

The set K_∞ contains points X and Y on the circles which are concentric with C_1 and with radii $r_2+r/10$ and $r_1-r/10$ respectively. Let η be the smaller of the two numbers $r/10$ and r_2. Since N is connected im kleinen, there exists a positive number δ_η such that any point of N within a distance δ_η of X or Y may be joined to X or Y, as the case may be, by an arc of N every point of which is within a distance η of X or Y. Let n_s be the smallest integer such that $x_{n_s}y_{n_s}$ contains two points P and Q such that $d(P, X)<\delta_\eta$ and $d(Q, Y)<\delta_\eta$. Then N contains arcs PX and QY, every point of which is within a distance η of X and Y respectively. Let order be defined on these arcs as *from P to X* and *from Q to Y*. The arcs PX and QY have points in common with every arc $x_{n_i}y_{n_i}$ for $i\geqq s$.

Let V_{10} and V_{20} be the last points the arcs PX and QY have in common with $x_{n_s}y_{n_s}$. Let U_{11} and U_{21} be the first points and V_{11} and V_{21} be the last points the subarcs $V_{10}X$ and $V_{20}Y$ have in common with $x_{n_{s+1}}y_{n_{s+1}}$. Let U_{12} and U_{22} be the first points the subarcs $V_{11}X$ and $V_{21}Y$ have in common with $x_{n_{s+2}}y_{n_{s+2}}$. The set J, composed of the arcs $V_{10}V_{20}$ of $x_{n_s}y_{n_s}$, $U_{11}V_{11}$ and $U_{21}V_{21}$ of $x_{n_{s+1}}y_{n_{s+1}}$, $U_{12}U_{22}$ of $x_{n_{s+2}}y_{n_{s+2}}$, $V_{10}U_{11}$ and $V_{11}U_{12}$ of PX, $V_{20}U_{21}$ and $V_{21}U_{22}$ of QY, is a simple closed curve. The subarc of $x_{n_{s+1}}y_{n_{s+1}}$ lying within J and the arc $x_{n_{s+3}}y_{n_{s+3}}$ have points p_1 and p_3, respectively, in common with the circle concentric with C_1 and with radius $r_2+\frac{1}{2}r$. The point p_1 is interior to J and within a distance $d_{n_{s+1}}$ of some point of K and this point is a limit point of D. Thus D contains a point in the interior of J. Similarly D contains a point within a distance $d_{n_{s+3}}$ of p_3 and thus in the exterior of J. Since D is connected D must contain a point of J. But as J belongs to N and D to $M-N$, D cannot contain a point of J. Thus the assumption that K is not connected im kleinen has led to a contradiction.

THEOREM II. *If a maximal connected subset K of the boundary of an M-domain is a continuous curve, every closed and connected subset of K is a continuous curve.*

DEFINITION. If D is an M-domain and P is a point of $M-\overline{D}$, the M-boundary of the maximal connected subset of $M-\overline{D}$ containing P will be called the *M-boundary of D with respect to P*. This is a generalization of the notion of outer boundary as defined by R. L. Moore.* If M is the entire plane, D is bounded, and P is a point of the maximal connected subset of $M-\overline{D}$ which is unbounded, then the M-boundary of D with respect to P is exactly the outer boundary of D as defined by Moore.*

THEOREM III.† *If D is an M-domain, P is a point of $M-\overline{D}$, and B is the M-boundary of D with respect to P, then B is the entire M-boundary of some M-domain which contains D.*

PROOF. The entire set D lies in the same maximal connected subset of $M-B$ and let R denote this maximal connected subset. Evidently R is an M-domain containing D. Suppose Q is a point of the M-boundary of R. If Q does not belong to B, Q belongs to a maximal connected subset of $M-B$ which is different from R. Then Q is not a limit point of R.‡ Therefore every point of the M-boundary of R is a point of B. Conversely every point of B is an M-boundary point of D and thus of R. Hence B is identical with the M-boundary of R, an M-domain containing D.

THEOREM IV. *If (1) D is an M-domain and P is a point of $M-\overline{D}$, (2) every maximal connected subset of D' is a continuous curve, (3) the M-boundary of D with respect to P, which we denote by B, is bounded, then every maximal connected subset of B is either a point, a simple continuous arc or a simple closed curve.*

PROOF. Let R be the maximal connected subset of $M-\overline{D}$ containing P, and let B_1 be a maximal connected subset of B.

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* *Concerning continuous curves in the plane*, loc. cit., p. 256.

† Compare R. L. Moore, *Concerning continuous curves in the plane*, loc. cit., Theorem 3, p. 258.

‡ See R. L. Moore, *A characterization of a continuous curve*, Fundamenta Mathematicae, vol. 7 (1925), Lemma 1, p. 302.

By Theorem II, B_1 is a continuous curve and B_1 is bounded by hypothesis. If B_1 consists of a single point, our theorem is proved. If B_1 consists of more than a single point then by a theorem due to Mazurkiewicz,* B_1 contains two points x and y which do not cut B_1. The continuous curve B_1 contains an arc xzy from x to y. If $B_1 \equiv xzy$, our theorem is proved. If not, let p be a point of B_1 which does not lie on xzy. By Theorem III, B is the entire M-boundary of some M-domain H which contains D. Clearly R and H are mutually exclusive and B is the entire M-boundary of each. By a theorem due to Wilder,† if p_1 and p_2 are points of R and H respectively there exist arcs p_1x and p_1y which lie except for x and y wholly in R and arcs p_2x and p_2y which lie except for x and y wholly in H. The sets p_1x+p_1y and p_2x+p_2y contain arcs xuy and xvy which lie wholly in R and H respectively except for the points x and y.

Let J_1, J_2, J_3 be the simple closed curves formed of $xuy+xzy$, $xvy+xzy$, $xuy+xvy$ respectively and let I_i and E_i denote the interior and exterior of $J_i (i=1, 2, 3)$. We have three cases to consider:

Case (1). Suppose $I_3 = I_1 + I_2 + <xzy>$.‡ Any point q of $B - xzy$ lies either in I_1, I_2 or E_3. If q lies in I_1, I_1 contains a point of H since q is a limit point of H. The exterior E_1 contains $<xvy>$ of H. But H is connected and contains no point of J_1. Hence I_1 contains no point of $B - xzy$. Similarly I_2 contains no point of $B - xzy$. Then every point of $B - xzy$ lies in E_3. Since x and y are not cut-points of B_1, the continuous curve B_1 contains an arc px which does not contain y and an arc py which does not contain x. The set $px+py$ contains an arc xwy from x to y. Since p is in E_3 and no point of J_3 except x and y is a point of B_1, the set $<xwy>$ lies

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* *Un théorème sur les lignes de Jordan*, Fundamenta Mathematicae vol. 2 (1921), pp. 119–130.

† Loc. cit., Theorem 1, p. 342.

‡ If xzy denotes a simple continuous arc with end-points x and y $<xzy>$ denotes $xzy - (x+y)$.

entirely in E_3, and thus the arcs xwy and xzy have only x and y in common. Let J_4 be the simple closed curve $xzy+xwy$. We will show that $B-J_4$ is vacuous and thus prove $B\equiv B_1\equiv J_4$. Suppose $B-J_4$ contains a point q_1. If q_1 lies in the interior of J_4 both R and H have points in the interior of J_4 since q_1 is a limit point of both domains. One of the two sets $<xuy>$ or $<xvy>$, say $<xuy>$, lies entirely in the exterior of J_4. Then R contains points interior and exterior to J_4 but contains no point of J_4, which is impossible. If q_1 lies in the exterior of J_4, both R and H contain points in the exterior of J_4, and one of them contains points in the interior, which is impossible. Therefore $B-J_4$ is vacuous, which proves the theorem for this case.

Case (2). Suppose $I_2=I_1+I_3+<xuy>$.

Case (3). Suppose $I_1=I_2+I_3+<xvy>$.

In Cases (2) and (3), it may be proved by methods similar to those of Case (1) that $B\equiv B_1$ and B_1 is a simple closed curve. Therefore B_1 is either a point, a simple continuous arc, or a simple closed curve.

In proving Theorem IV we have obtained this result:

THEOREM V. *Under the hypothesis of Theorem* IV, *if any maximal connected subset J of the M-boundary of D with respect to P is a simple closed curve, then J is the entire M-boundary of D with respect to P.*

THEOREM VI. *If D is an M-domain, P is a point of $M-\overline{D}$, R is the maximal connected subset of $M-\overline{D}$ containing P, and Q is a point of the maximal connected subset of $M-\overline{R}$ which contains D, then R' is the M-boundary of R with respect to Q.*

PROOF. Let H denote the maximal connected subset of $M-\overline{R}$ which contans D. By definition H' is the M-boundary of R with respect to Q. The set H' is a subset of R' since the M-boundary of a domain with respect to a point is always a subset of the M-boundary of the domain. By definition, R' is the M-boundary of D with respect to P. Then every point of R' is a limit point of D and thus of H. As H is a subset of

$M-\overline{R}$, no point of R' is a point of H. Therefore every point of R' is an M-boundary point of H, that is, R' is a subset of H'. Hence $R' \equiv H'$.

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A THEOREM ON CONNECTED POINT SETS

BY C. KURATOWSKI AND C. ZARANKIEWICZ

1. *Introduction.* The purpose of this paper is to prove the following theorem.

THEOREM. *If S is a connected point set and Z is the set of all points such that $S-p$ is neither connected nor the sum of two connected sets, then Z is finite or countable.*

2. LEMMA. *If S, P, and Q are three non-vacuous connected sets (or points), and if*

(1) $P+Q \subset S$, (2) $P \cdot Q = 0$,

(3) $A \subset S-P$, (4) $B \subset S-Q$,

(5) $A \cdot Q = 0$, (6) $B \cdot P = 0$,

(7) *A and $S-P-A$ are mutually separated,*

(8) *B and $S-Q-B$ are mutually separated,*

then $A \cdot B = 0$.

PROOF. By (1) and (3), $A+P \subset S$. Hence, by (2) and (5), $A+P=(A+P)\cdot(S-Q)$. By (4), $S-Q=B+(S-Q-B)$. Therefore

$$A + P = (A + P)\cdot B + (A + P)\cdot(S - Q - B). \tag{9}$$

It follows from (2) and (6) that $P=P-Q-B \subset (A+P)\cdot(S-Q-B)$. Since $P \neq 0$, we have

$$(A + P)\cdot(S - Q - B) \neq 0. \tag{10}$$

Now, by (8), the sets $(A+P)\cdot B$ and $(A+P)\cdot(S-Q-B)$ are mutually separated. On the other hand, by virtue of a

theorem of Knaster and Kuratowski,* and by (3) and (7), the set $A+P$ is connected. Hence, by (9) and (10), $(A+P)\cdot B=0$, that is, $A\cdot B=0$.

3. *Proof of the Theorem.* Let

$$R_1, R_2, \cdots, R_n, \cdots \tag{11}$$

be the sequence of all circles (spheres) which are such that their radii and the coordinates of the center are rational; and let

$$(1,2,3),(1,2,4), \cdots ,(i,j,k), \cdots \tag{12}$$

be the sequence of all systems composed of three different positive integers.

We shall first show that, for every point p of Z, there exist three sets A, C, and D and a system of indices (i, j, k) such that

(13) *A, C, and D are mutually separated,*

$$S - p = A + C + D, \tag{14}$$

$$S\cdot R_i = A\cdot R_i \neq 0, \tag{15}$$

$$S\cdot R_j = C\cdot R_j \neq 0 \neq D\cdot R_k = S\cdot R_k. \tag{16}$$

By hypothesis, $S-p$ is not connected. Hence $S-p$ may be decomposed into two mutually separated sets; one of these two sets can itself be decomposed into two mutually separated sets, since by hypothesis $S-p$ is not the sum of two connected sets. It follows that there exists at least one decomposition of $S-p$ into three non-vacuous mutually separated sets A, C, and D. Let a, c, and d be three points in A, C, and D, respectively. It is obvious that the point a may be surrounded by a circle R_i which contains no point of $C+D+p$. Similarly c and d belong to two circles R_j and R_k that contain no point of $A+D+p$ and of $A+C+p$, respectively. Therefore the conditions (15) and (16) are fulfilled.

* *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), p. 210, Theorem VI.

Thus we have shown the existence of a system (i, j, k) having the property that there exists at least one system of three sets (A, C, D) satisfying the formulas (13)–(16). We denote by $\sigma(p)$ the *first* system in the sequence (12) having that property.

We shall prove that, if

$$p \neq q, \tag{17}$$

then $\sigma(p) \neq \sigma(q)$.

Suppose, on the contrary, that $\sigma(p) = \sigma(q) = (i, j, k)$. Hence there exist three sets A, C, D satisfying the conditions (13)–(16) and three sets B, E, F such that

(18) *B, E, and F are mutually separated,*

$$S - q = B + E + F, \tag{19}$$

$$S \cdot R_i = B \cdot R_i \neq 0, \tag{20}$$

$$S \cdot R_j = E \cdot R_j \neq 0 \neq F \cdot R_k = S \cdot R_k. \tag{21}$$

From (15) and (20) it follows that $A \cdot R_i = B \cdot R_i \neq 0$. Hence

$$A \cdot B \neq 0 \tag{22}$$

and similarly, $C \cdot E \neq 0 \neq D \cdot F$.

By (17), (14) and (13) q belongs to one and only one of the sets A, C, and D. Similarly, p belongs to one and only one of the sets B, E, and F. Clearly we may suppose that neither q belongs to A nor p to B, so that

$$A \cdot q = 0 = B \cdot p. \tag{23}$$

Now, by (13) and (14), $C + D = S - p - A$. It follows by (13) that A and $S - p - A$ are mutually separated. Similarly, B and $S - q - B$ are mutually separated. Therefore, by virtue of the Lemma and by (17), (14), (19), and (23) we have $A \cdot B = 0$. But this is a contradiction of (22).

Thus we have shown that to different points of the set Z correspond different systems composed of three positive integers. Hence the set Z is countable.

4. *Remarks.* A. It is to be noted that the above proof was carried out *without the aid of the Zermelo Axiom** and that it gives a mode of enumerating the points of Z. That is, Z is *effectively countable* (in the sense of Borel-Sierpinski).

B. By means of a remark of B. Knaster, the argument used above may be applied to prove the following more, general theorem. *If S is a connected set and Z is a class of mutually exclusive, connected, and relatively (to S) closed sets X, such that $S-X$ is neither connected nor the sum of two connected sets, then the class Z is countable.*

C. In case S is an acyclic continuous curve (i. e., a continuous curve that contains no simple closed curves), then the points of the set Z are identical with *points of ramification* of S.† Thus, our Theorem has as a consequence the theorem of Wazewski-Menger‡ to the effect that *the set of points of ramification of a tree is countable.*

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* The Zermelo Axiom was used by Wazewski (Annales de la Société Polonaise de Mathématique, vol. 2 (1923), p. 169) to prove a particular case of our Theorem, namely the case S is a continuous curve.

By a method analogous to that used above, the following theorem that was proved with the aid of the Zermelo Axiom by R. L. Moore in a less general form (see Proceedings of the National Academy of Sciences, vol. 9 (1923), p. 102), and by C. Zarankiewicz (Fundamenta Mathematicae, vol. 9 (1927), p. 140, Theorem 9) may be proved without having reference to that axiom:

If T is a connected subset of a connected set S and W is the set of points each of which belongs to T and disconnects S but not T, then W is countable.

For, suppose p is a point of W. Then there exists a circle R having the property that there exists a set A such that (i) A and $A-p-S$ are mutually separated, (ii) $A \subset S-T$, (iii) $S \cdot R = A \cdot R \neq 0$.

Let $n(p)$ denote the lowest index such that the circle $R_{n(p)}$ of the sequence (11) has that property. Now, if $p \neq q$ then $n(p) \neq n(q)$. Suppose, on the contrary, that $n(p) = n(q)$. Hence there exists a set B such that (iv) B and $S-q-B$ are mutually separated, (v) $B \subset S-T$, (vi) $S \cdot R = B \cdot R \neq 0$. It follows, by (ii), that $A \cdot q = 0$ and, by (v), that $B \cdot p = 0$. Therefore, by (i), (iv), and the Lemma, we have $A \cdot B = 0$. But this is a contradiction of (iii) and (vi). Thus the set W is effectively countable.

† See K. Menger, *Ueber reguläre Baumkurven*, Mathematische Annalen, vol. 96 (1926), p. 574. A point p is said to be a *point of ramification* of a (regular) curve if the curve contains three arcs L_1, L_2, L_3 such that $p = L_1 \cdot L_2 = L_1 \cdot L_3 = L_2 \cdot L_3$.

‡ Wazewski, loc. cit., p. 169, Menger, loc. cit., p. 576.

This theorem may be generalized as follows. Suppose S is a acyclic continuous curve *locally* at each of its points (i. e., for each point p of S there exists an acyclic continuous curve T_p contained in S and such that p is not a limit point of $S-T_p$). It follows, by the Borel covering theorem, that the set of all points of ramification of S is still countable.

A direct consequence of this statement is the following result* obtained by Alexandroff: *If S is a continuous curve of finite connectivity† then the set of all points of ramification of S is countable.* For,‡ a continuous curve of finite connectivity is a tree locally at each of its points.

Now, if S is a plane non-dense continuous curve that disconnects the plane in a finite number of domains, then§ S has a finite connectivity. Thus we may state the following result. *The set of points of ramification of a plane non-dense continuous curve that disconnects the plane in a finite number of domains is countable.*‖

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* P. Alexandroff, *Ueber kombinatorische Eigenschaften allgemeiner Kurven*, Mathematische Annalen, vol. 96 (1926), p. 552, Corollary 4.

† A (one-dimensional bounded) continuous curve is said to be of *finite connectivity* if it contains at most a finite number of simple closed curves. This definition is equivalent to that given by Alexandroff, loc. cit., p. 541, by virtue of the Lemma 15 of the mentioned paper of Zarankiewicz.

‡ See Menger, loc. cit., p. 574.

§ See Alexandroff, loc. cit., p. 527.

‖ This statement does not remain true if the hypothesis that the number of complementary domains is *finite* be omitted, as is seen on an example of a continuous (regular) curve composed exclusively of points of ramification, given by W. Sierpinski, Comptes Rendus, vol. 160 (1915), p. 305; see also this Bulletin, vol. 33 (1927), p. 106.

ON CERTAIN SUFFICIENT CONDITIONS FOR THE CONVERGENCE AND CESÀRO SUMMABILITY OF THE ALLIED SERIES OF A DOUBLE FOURIER SERIES*

BY G. M. MERRIMAN†

1. *Introduction.* The double Fourier series of a summable function

$$f(\alpha,\beta) \sim \sum_1^\infty \sum_1^\infty \left\{ \begin{matrix} a_{i,j}\cos i\alpha\cos j\beta + b_{i,j}\cos i\alpha\sin j\beta \\ + c_{i,j}\sin i\alpha\cos j\beta + d_{i,j}\sin i\alpha\sin j\beta \end{matrix} \right\}$$

$$= \sum_1^\infty \sum_1^\infty (a,b,c,d,\alpha,\beta)_{i,j} \tag{1}$$

has three series said to be allied to it :

$$\text{(A)} \qquad \sum_1^\infty \sum_1^\infty (d,-c,-b,a,\alpha,\beta)_{i,j},$$

$$\text{(B)} \qquad \sum_1^\infty \sum_1^\infty (c,d,-a,-b,\alpha,\beta)_{i,j},$$

$$\text{(C)} \qquad \sum_1^\infty \sum_1^\infty (b,-a,d,-c,\alpha,\beta)_{i,j}.$$

These will be designated hereafter by the terms first, second, and third allied series, respectively.

Although the theorems concerning these series to be stated in this paper follow quite readily, in fact, so readily that most of the details of proof will be omitted, from the use of certain auxiliary functions and identities, yet they are useful and do not appear elsewhere in the literature. They were suggested by three papers of W. H. Young.‡

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† National Research Fellow.

‡ (1) *Multiple Fourier series*, Proceedings of the London Society, (2), vol. 11 (1912–1913); (2) *On the convergence of a Fourier series and its allied series*, Proceedings of the London Society, (2), vol. 10 (1911–1912); (3)

2. *Convergence Theorems.* If we write

$$\phi_1(x,y) \equiv \frac{1}{4}\{f(\alpha+x,\beta+y) - f(\alpha+x,\beta-y) - f(\alpha-x,\beta+y) + f(\alpha-x,\beta-y)\}$$

$$\sim \sum_1^\infty \sum_1^\infty (d,-c,-b,a,\alpha,\beta)_{i,j} \sin ix \sin jy,$$

$$\phi_2(x,y) \equiv \frac{1}{4}\{f(\alpha+x,\beta+y) + f(\alpha+x,\beta-y) - f(\alpha-x,\beta+y) - f(\alpha-x,\beta-y)\}$$

$$\sim \sum_1^\infty \sum_1^\infty (c,d,-a,-b,\alpha,\beta)_{i,j} \sin ix \cos jy,$$

$$\phi_3(x,y) \equiv \frac{1}{4}\{f(\alpha+x,\beta+y) - f(\alpha+x,\beta-y) + f(\alpha-x,\beta+y) - f(\alpha-x,\beta-y)\}$$

$$\sim \sum_1^\infty \sum_1^\infty (b,-a,d,-c,\alpha,\beta)_{i,j} \cos ix \sin jy,$$

we may state the following theorem.

THEOREM I. *If $f(\alpha, \beta)$ is a function of bounded variation*, and if each of the integrals*

$$(2) \qquad \lim_{\epsilon_1,\epsilon_2\to 0} \frac{1}{\pi^2} \int_{\epsilon_1}^{\pi} \int_{\epsilon_2}^{\pi} \phi_1(x,y) \operatorname{ctn}(x/2) \operatorname{ctn}(y/2)\, dx\, dy,$$

$$(3) \qquad \lim_{\epsilon_1\to 0} \frac{1}{\pi^2} \int_{\epsilon_1}^{\pi} \int_{0}^{\pi} \phi_2(x,y) \operatorname{ctn}(x/2)\, dx\, dy,$$

$$(4) \qquad \lim_{\epsilon_2\to 0} \frac{1}{\pi^2} \int_{0}^{\pi} \int_{\epsilon_2}^{\pi} \phi_3(x,y) \operatorname{ctn}(y/2)\, dx\, dy,$$

Konvergenzbedingungen für die verwandte Reihen einer Fouriersche Reihe, Akademie der Wissenschaften München, 1911. (1) will hereafter be referred to as M.F.S., (2) as C.A.S., (3) as K.V.R.

* See M.F.S., §3; a function of two variables is said to have bounded variation if it is of bounded variation not only with respect to both variables, but also with respect to each variable separately.

has a finite limit, then the three allied series of $f(\alpha, \beta)$ *converge,* (A) *to* (2), (B) *and* (C) *to the negatives of* (3) *and* (4), *respectively.*

Theorem I is a generalization of the result in K.V.R., §2. By slight changes in the details of proof, we may obtain also two more general theorems, which will be needed later.

THEOREM II. *If* $\psi(u, v)$ *is an odd-odd function of* u *and* v, *and has the property that* $uv\psi(u, v)$ *is the double integral of some function* $\phi(u, v)$ *such that*

$$\int_0^u \int_0^v |\phi(u,v)| \, du \, dv = o(uv),$$

then the series of double Fourier constants of $\psi(u, v)$ (*i.e., the first allied series of* $\psi(u, v)$ *at* $u=0, v=0$) *converges to the value*

$$\lim_{\epsilon_1,\epsilon_2 \to 0} \frac{1}{\pi^2} \int_{\epsilon_1}^{\pi} \int_{\epsilon_2}^{\pi} \psi(u,v) \operatorname{ctn}(u/2) \operatorname{ctn}(v/2) \, du \, dv,$$

provided that this limit exists.

THEOREM III. *If* $\psi(u, v)$ *is an odd-even function of* u *and* v, *such that* $u\psi(u, v)$ *is the integral with respect to* u *of a function* $\phi(u, v)$ *such that*

$$\int_0^u |\phi(u,v)| \, du = o(u),$$

and $\phi(u, v)$ *is, near* $u=0, v=0$, *a continuous function of* v, *and a bounded function of* u *and* v, *then the series of double Fourier constants of* $\psi(u, v)$ *converges to the limit*

$$\lim_{\epsilon_1 \to 0} \frac{-1}{\pi^2} \int_{\epsilon_1}^{\pi} \int_0^{\pi} \psi(u,v) \operatorname{ctn}(u/2) \, du \, dv,$$

*if it exists.**

A theorem similar to Theorem III may be stated for an even-odd function.

* The continuity of $\psi(u,v)$ at the origin, essential to the proof of this theorem, follows from a result of H. J. Ettlinger: *On continuity in several variables*, this Bulletin, vol. 33, No. 1 (Jan., 1927).

3. *Auxiliary Identities and Theorems.* We write

$$g_1(u,v) = \csc u \csc v \int_0^u \int_0^v \phi_1(2u,2v)\, du\, dv,$$

$$g_2(u,v) = \csc u \int_0^u \phi_2(2u,2v)\, du,$$

$$g_3(u,v) = \csc v \int_0^v \phi_3(2u,2v)\, dv.$$

If we carry out reductions akin to those in M.F.S., §§13, 14, and 21, we derive the following set of identities :

(I) $$s_{m,n} = S_{m,n} - p_{m,n} - q_{m,n} + mnA_{m,n},$$

(II) $$[s_{C,C}]_{m,n} = 4[S_{C,C}]_{m,n} - 2[S_{C,O}]_{m,n} - 2[S_{O,C}]_{m,n} + S_{m,n},$$

(III) $$[s_{C,C}]_{m,n} = 2[S_{C,C}]_{m,n} - [S_{O,C}]_{m,n},$$

(IV) $$[s_{C,C}]_{m,n} = 2[S_{C,C}]_{m,n} - [S_{C,O}]_{m,n}.$$

In (I), $s_{m,n}$ stands for the partial sum of (A), $S_{m,n}$ for the partial sum of the double Fourier constants of $g_1(u, v)$, $p_{m,n}$ for m times the sum of the first n terms of the mth column of the double Fourier constants of $g_1(u, v)$, $q_{m,n}$ for n times the sum of the first m terms of the nth row, and $A_{m,n}$ is the general term of the double Fourier constants of $g_1(u, v)$. In (II), the $s_{m,n}$ and $S_{m,n}$ stand for the same partial sums as in (I) and the subscripts C and O stand for "Cesàro" and "ordinary" summations respectively. In (III) and (IV), the $s_{m,n}$ is the partial sum of (B) and (C), the $S_{m,n}$ the partial sum of the double Fourier constants of $g_2(u, v)$ and $g_3(u, v)$, respectively. The last three identities can be extended to suitably defined Cesàro sums of higher indices. With these identities at hand, we may state the following results.

From repeated applications of (I) and (II), we obtain

THEOREM IV. *If $g_1(u, v)$ be a summable function of u and v such that the functions $p_{m,n}$, $q_{m,n}$ and $mnA_{m,n}$ of its double Fourier constants have the unique double limit zero, then, if it be known that either the first allied series of $f(\alpha, \beta)$ or that of*

$g_1(u, v)$, at $u=0, v=0$, converges doubly in any Cesàro manner, indices (r, r), it follows that both series converge doubly in the ordinary manner and have the same sum.*

COROLLARY. *If the functions $p_{m,n}$, $q_{m,n}$ and $mnA_{m,n}$ converge to limits other than zero, the theorem is still true, except that the series then have not the same sum ; the difference in their sums is the sum of the limits of the functions.*

If we use (II) alone, and its extensions noted above, we have the following theorem.

THEOREM V. *If the first allied series of the double Fourier series of $g_1(u, v)$ converges doubly in the ordinary way at the point $u=0, v=0$, then the first allied series of $f(\alpha, \beta)$ converges doubly in the Cesàro manner, indices* (1, 1), *and the sums of the series are the same.*

This theorem can be extended to general Cesàro indices, and the footnote appended to Theorem IV shows that they need not be the same.

If we use (III) in the same manner, we obtain the theorem

THEOREM VI. *If the second allied series of $g_2(u, v)$ converges doubly in the ordinary way at the point $u=0$, $v=0$, then the second allied series of $f(\alpha, \beta)$ converges $(C, 1, 1)$, and the sums of the series are the same.*

By use of (IV), a theorem similar to Theorem VI can be stated relative to the third allied series. Also, the theorems can be extended to higher Cesàro indices. Of course, we have not used the narrowest possible hypothesis by requiring the double convergence of the allied series of $g_2(u, v)$; we might have arrived at the same conclusions by requiring only the ordinary-Cesàro convergence. However, we shall need the result only as here stated.

We must note finally that we can replace the functions $g_1(u, v)$, $g_2(u, v)$, and $g_3(u, v)$ by other auxiliary functions

* That these indices of summation may be different, say (r, s), follows as in M.F.S., §§17, 18.

$h_1(u, v)$, $h_2(u, v)$, and $h_3(u, v)$, in which the factors csc u and csc v have been replaced by $1/u$ and $1/v$ respectively.*

4. *Summability* $(C, 1, 1)$. We proceed now to the main theorems of the paper.

THEOREM VII. *If*

$$\int_0^u \int_0^v |\phi_1(2u, 2v)| \, du \, dv = o(uv),$$

and the limit

$$\lim_{\epsilon_1, \epsilon_2 \to 0} \frac{1}{\pi^2} \int_{\epsilon_1}^{\pi} \int_{\epsilon_2}^{\pi} \phi_1(u, v) \operatorname{ctn}(u/2) \operatorname{ctn}(v/2) \, du \, dv \tag{5}$$

exists, then the first allied series of $f(\alpha, \beta)$ *sums* $(C, 1, 1)$ *to* (5).

The function $\phi_1(2u, 2v)$ fulfils the rôle of $\phi(u, v)$ in Theorem II, and therefore the first allied series of $h_1(u, v)$, and hence that of $g_1(u, v)$, converges at $u = 0$, $v = 0$, to the limit (if it exists)

$$\lim_{\epsilon_1, \epsilon_2 \to 0} \frac{1}{\pi^2} \int_{\epsilon_1}^{\pi} \int_{\epsilon_2}^{\pi} g_1(u, v) \operatorname{ctn}(u/2) \operatorname{ctn}(v/2) \, du \, dv.$$

Therefore, by Theorem V, the first allied series of $f(\alpha, \beta)$ sums $(C, 1, 1)$ to the value, if it exists,

$$\lim_{\epsilon_1, \epsilon_2 \to 0} \frac{1}{4\pi^2} \int_{\epsilon_1}^{\pi} \int_{\epsilon_2}^{\pi} \left\{ \int_0^u \int_0^v \phi_1(2u, 2v) du \, dv \right\} \cdot \csc^2(u/2) \csc^2(v/2) du \, dv. \tag{6}$$

Integrating this result doubly by parts and using hypothesis, we complete Theorem VII.

COROLLARY. *The limit* (5) *and the limit*

$$\lim_{\epsilon_1, \epsilon_2 \to 0} \frac{1}{\pi^2} \int_{\epsilon_1}^{\infty} \int_{\epsilon_2}^{\infty} \frac{\phi_1(u, v)}{uv} du \, dv \tag{7}$$

are interchangeable, provided the periodicity of $f(\alpha, \beta)$ *is used to extend its definition outside the given range.*

* See C.A.S., §§5, and 12.

The expression (6) can be written in the form

$$\lim_{\epsilon_1,\epsilon_2\to 0} \frac{4}{\pi^2} \int_{\epsilon_1}^{\infty} \int_{\epsilon_2}^{\infty} u^{-2}v^{-2} \left\{ \int_0^u \int_0^v \phi_1(2u, 2v)\, du\, dv \right\} du\, dv,$$

whence, integrating doubly by parts and using the hypothesis, the corollary is proved.

THEOREM VIII. *If*

$$\int_0^u \int_0^v |\phi_2(2u, 2v)|\, du\, dv = o(uv),$$

and the limit

$$\lim_{\epsilon_1\to 0} \frac{-1}{\pi^2} \int_{\epsilon_1}^{\pi} \int_0^{\pi} \phi_2(u, v) \operatorname{ctn} (u/2)\, du\, dv \tag{8}$$

exists, then the second allied series of $f(\alpha, \beta)$ *sums* $(C, 1, 1)$ *to* (8).

The function $\phi_2(2u, 2v)$ is easily seen to play the part of $\phi(u, v)$ in Theorem III, and hence the second allied series of $h_2(u, v)$ and therefore that of $g_2(u, v)$ converge to the limit (if it exists)

$$\lim_{\epsilon_1\to 0} \frac{-1}{\pi^2} \int_{\epsilon_1}^{\pi} \int_0^{\pi} g_2(u, v) \operatorname{ctn} (u/2)\, du\, dv.$$

Hence, by Theorem VI, the second allied series of $f(\alpha, \beta)$ sums $(C, 1, 1)$ to

$$\lim_{\epsilon_1\to 0} \frac{-1}{2\pi^2} \int_{\epsilon_1}^{\pi} \int_0^{\pi} \left\{ \int_0^u \phi_2(2u, 2v)\, du \right\} \csc^2(u/2)\, du\, dv. \tag{9}$$

An integration by parts with respect to u, and the use of hypothesis, complete Theorem VIII.

A theorem similar to Theorem VIII can be deduced concerning the third allied series of $f(\alpha, \beta)$.

In conclusion, mention should, be made perhaps, of theorems analogous to those of §§3 and 4, dealing with summability $(C, r, 0)$ and $(C, 0, s)$ of the allied series, which can be developed along the lines of this paper from functions

of the type of $g_2(u, v)$ and $g_3(u, v)$. It would be such theorems which would allow us to arrive at the statements contained in the footnote appended to Theorem IV and elsewhere.

In the proofs of Theorems V and VI, use was made of a "consistency theorem" to the effect that if a double series is summable (C, r, s), it is also summable (C, r', s'), $r' \geqq r$, $s' \geqq s$, and to the same sum as formerly. This result is contained in Theorem I′ of a paper by the author concerning the summability of double series of a certain type, which will appear in an early issue of the Annals of Mathematics.

HARVARD UNIVERSITY

A GENERAL THEOREM ON QUANTIC DETERMINANTS*

BY T. R. ROSEBRUGH

Let there be a homogeneous linear substitution

$$x_{rj}' = \sum_{i=1}^{n_r} a_{rji} x_{ri}, \qquad (j = 1, 2, \cdots, n_r),$$

having M_r for the determinant of its coefficients, and let there be derived therefrom quantics of degree p_r giving values of all possible monomials in these n_r new variables of that degree. There will be

$$f_r = \binom{p_r + n_r - 1}{p_r}$$

such new monomials, and an equal number of the old will be involved in these quantics.

Now consider this to be but one, namely the rth of a set of s such substitutions, each in its own set of variables of arbitrary (generally different) number n_r, similarly treated. Each new monomial of each of the s sets is now to be used,

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* Presented to the Society, February 26, 1927.

one only from each set, as a factor in a continued product which contains such factors.

It is now proposed to examine a direct method for finding the algebraic composition of the determinant M of the coefficients of the quantics so constructed, when placed in an array similarly ordered in the composition of the monomials of the new and old quantities. It is evident that this condition of similar ordering is sufficient to ensure the same value of M for any change in the order, since it effects the same permutation of the rows as of the columns.

The result of this examination will be found to be that

$$\log M = \sum_{r=1}^{s} u_r \log M_r$$

where

$$u_r = \frac{p_r}{n_r} \prod_{r=1}^{s} f_r.$$

It is desirable to agree first for descriptive purposes on some rule of order more definite than the mere essential condition of similar order. Let the factor-groups of new variables from all other substitutions than the rth, combined, appearing associated with factor-groups from the rth on the left side of the equations be assigned an arbitrary order which is likewise applied to the corresponding factor-groups of old variables on the right.

There will then be

$$q_r = \frac{1}{f_r} \prod_{r=1}^{s} f_r$$

such factor-groups. Now let the value be supposed written out of the product of each of the f_r primed monomials of the rth group (taken in an assigned order) with the first of the other factor-groups; then the products of each of them in the same order with the second of the q_r other factor-groups, and so on. For convenience each set of f_r rows of the *coefficients* so obtained may be called a block. There will thus be q_r blocks.

The order of the columns of coefficients is fixed by that of the associated monomials of the old variables and is determined by application of the rule of similar order. Thus there will be blocks running horizontally as well as vertically in each of which the same order for rth substitution variables will apply. For convenience, let anything referring to variables or coefficients connected with the rth substitution be called "rth stage." Considering now details of the rth stage coefficients within individual blocks horizontally and vertically, and the associated variables of that stage, and ignoring the factors from the other stages which are present, we proceed as follows:

There is a row in each block giving the value of a new term in which the factor

$$\prod_{j=1}^{n_r} (x'_{rj})^{t'_{rj}}$$

occurs. The ordinal number of this row in the block has been fixed in an assigned manner by the exponents of the system $(t'_{r1}, t'_{r2}, \cdots, t'_{rn_r})$, admitting zero values. The column in which the coefficient of the old term in which the factor

$$\prod_{j=1}^{n_r} (x_{rj})^{t_{rj}}$$

occurs has accordingly been fixed by the same rule of order by the exponents of the system $(t_{r1}, t_{r2}, \cdots, t_{rn_r})$ also admitting zero values.

Suppose now we examine the effect of the operator

$$\sum_{i=1}^{n_r} a_{rli} \frac{\partial}{\partial a_{rji}}$$

on M, the determinant under study, in case (1), $l=j$, and in case (2), $l \neq j$. If a determinant some of whose elements are functions of a variable be differentiated with regard to that variable the result is the sum of the products of the derivative of each element into the corresponding unchanged co-factor. Consequently an operator of the above form acting on a

determinant gives the sum of the products of the result of its application to each element into the corresponding unchanged co-factor. Now these products may be grouped by rows, each row of the results of operation being associated with all the other rows unchanged, and thus a total of as many determinants as the order of the determinant may be so formed.

First consider, however,

$$\sum_{i=1}^{n_r} a_{rli} \frac{\partial}{\partial a_{rji}}$$

applied not to an individual element of the determinant M but to any one of the quantics in all the variables, containing, together with factors from all the other stages, the rth stage factor

$$\prod_{j=1}^{n_r} (x'_{rj})^{t'_{rj}}.$$

This quantic may be written, for a chosen value of j,

$$C(x'_{rj})^{t'_{rj}}$$

where C contains all factors independent of $a_{rji} (i = 1, 2, \cdots, n_r)$ partly from its own stage and partly from the aggregate of the others. This is therefore

$$C\left\{ \sum_{i=1}^{n_r} a_{rji} x_{ri} \right\}^{t'_{rj}}.$$

Now when $t'_{rj} \neq 0$, we have

$$\sum_{i=1}^{n_r} a_{rli} \frac{\partial}{\partial a_{rji}} \cdot C \left\{ \sum_{i=1}^{n_r} a_{rji} x_{ri} \right\}^{t'_{rj}}$$

$$= t'_{rj} C \left\{ \sum_{i=1}^{n_r} a_{rli} x_{ri} \right\} \left\{ \sum_{i=1}^{n_r} a_{rji} x_{ri} \right\}^{t'_{rj}-1}.$$

Thus in case (1), $(l = j)$, this quantic is converted into t'_{rj} times itself by the operation (as in Euler's Theorem), while in case (2), $(l \neq j)$, it is converted into t'_{rj} times the

quantic of another row, identifiable by the system of exponents in which t'_{rl} and t'_{rj} are respectively replaced by $t'_{rl}+1$ and $t'_{rj}-1$. The case $t'_{rj}=0$ requires special consideration. In this case the a_{rji} not being present, the result of the operation both for $l=j$ and $l\neq j$ is zero. Hence *every coefficient* in a given row is, for $t'_{rj}=0$, in both cases converted into zero, and for $t'_{rj}\neq 0$, is converted either into t'_{rj} times itself, case (1): or, case (2), into t'_{rj} times the coefficient in the same column in another row identifiable as described above.

In case (2), since the operation yields either zero for each coefficient or else a constant multiple of the corresponding coefficient of another row, the value of each determinant so produced will be zero. That is

$$(1) \qquad \sum_{i=1}^{n_r} a_{rli}\frac{\partial M}{\partial a_{rji}} = 0, \quad (l,j = 1,2,\cdots,n_r),(l\neq j).$$

This denotes $n_r^2-n_r$ equations. But in case (1),

$$\sum_{i=1}^{n_r} a_{rli}\frac{\partial M}{\partial a_{rji}}$$

includes for every row a determinant having the value $t'_{rj}\ M$, the case $t'_{rj}=0$ being now included. Now the set of values of t'_{rj} for the individual rows in a block are the same for each block. The total contribution from any one block of rows is $M\sum t'_{rj}$ where it is easily seen that $n_r\sum t'_{rj}=f_r p_r$.

The contribution from one block of rows is thus $f_r p_r M/n_r$ and from all the blocks, which are

$$\frac{1}{f_r}\prod_{r=1}^{s} f_r$$

in number, it is

$$\frac{p_r}{n_r}M\prod_{r=1}^{s} f_r = u_r M,$$

say. Hence

$$(2) \qquad \sum_{i=1}^{n_r} a_{rji}\frac{\partial M}{\partial a_{rji}} = u_r M, \quad (j = 1,2,\cdots,n_r);$$

that is, n_r equations. Now let (1) and (2) be combined in a single statement thus:

$$(3) \quad \sum_{i=1}^{n_r} a_{rli} \frac{\partial(u_r^{-1} \log M)}{\partial a_{rji}} = \begin{cases} 1, l = j, \\ 0, l \neq j, \end{cases} (l,j = 1,2, \cdots, n_r),$$

and compared with

$$(4) \quad \sum_{i=1}^{n_r} a_{rli} \frac{\partial \log M_r}{\partial a_{rji}} = \begin{cases} 1, l = j, \\ 0, l \neq j, \end{cases} \quad (l,j = 1,2, \cdots, n_r),$$

which describes known properties of a simple determinant.

It is seen at once that the unique solution for the n_r^2 unknown partial derivatives in (3) is

$$\frac{\partial \log M}{\partial a_{rji}} = u_r \frac{\partial \log M_r}{\partial a_{rji}}, \quad (i,j = 1,2, \cdots, n_r),$$

that is

$$(5) \quad \frac{\partial (\log M - u_r \log M_r)}{\partial a_{rji}} = 0.$$

That is, the n_r^2 quantities a_{rji} occur in $\log M$ in such a manner that when $u_r \log M_r$ involving these n_r^2 quantities only has been subtracted from it they have all disappeared from it, and the remainder consists therefore only of an unaltered function of the coefficients of the other stages.

Taking account also of the fact that the simplest possible substitutions corresponding to unity along all principal diagonals and zero elsewhere would give a similar form to the determinant yielding M, and thus give $M_r = 1$, $(r = 1, 2, \cdots, s)$ and $M = 1$, we conclude that

$$\log M = \sum_{r=1}^{s} u_r \log M_r,$$

where

$$(6) \quad u_r = \frac{p_r}{n_r} \prod_{r=1}^{s} \binom{n_r + p_r - 1}{p_r}.$$

From this general result others less general may of course be obtained by successive specialization. For this purpose the use of a notation will be convenient. Let

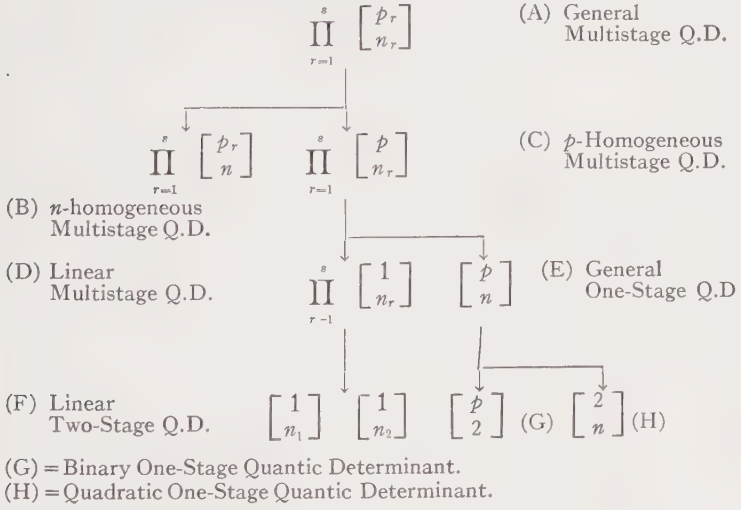
$$\begin{bmatrix} p \\ n \end{bmatrix}$$

denote the theorem relating to the determinant of the coefficients of the quantics representing all monomials of degree p in the n variables given by n equations linearly in terms of others. Then

$$\Pi \begin{bmatrix} p_r \\ n_r \end{bmatrix}$$

may be understood to denote that for the case where all the monomials formed as products from different substitutions are expressed as quantics in variables from the other sets, namely the general theorem proved above.

The relationship between the general theorem and others derivable as special cases may be expressed in the following diagram, in which the abbreviation Q.D. is used for Quantic Determinant.



(G) = Binary One-Stage Quantic Determinant.
(H) = Quadratic One-Stage Quantic Determinant.

Of the above, the only cases known to the writer as having been previously studied are those designated (F), (G) and (H) as follows:

(F): E. Pascal, *I Determinanti*, Milan, 1897, page 138, names this theorem "Teorema di Kronecker," but the reference which he gives is incorrect. The exponents are $u_1 = n_2$ and $u_2 = n_1$.

(G): A theorem of Faà di Bruno deals with this case. The demonstration is given on pages 228–29 of his *Théorie des Formes Binaires*, Turin, 1876. The exponent u of the original determinant here becomes $p(p+1)/2$.

(H): These determinants are discussed by Hunyady, Journal für Mathematik (vol. 89 (1879), p. 58). H. Leitzmann, in his translation of Pascal's *I Determinanti*, credits the theorem to Scholtz in place of Hunyady, as in the original. The exponent u proves to be $n+1$.

The University of Toronto

ON THE DEGREE OF APPROXIMATION TO A HARMONIC FUNCTION*

BY J. L. WALSH

It is well known that if a function $u(x, y)$ is harmonic in a closed Jordan region of the (x, y)-plane, then in that closed region, $u(x, y)$ can be expanded in a uniformly convergent series of harmonic polynomials in (x, y). A more explicit result can be proved, however, and it is the object of the present note to establish such a more general theorem.

THEOREM I. *Let C be an arbitrary closed Jordan region of the (x, y)-plane, and let $w=\phi(z)$, $z=x+iy$, be a function which maps conformally the exterior of C onto the exterior of the unit circle in the w-plane so that the points at infinity correspond to each other. Let C_R denote the curve $|\phi(z)|=R$, $R>1$, that is, the transform onto the z-plane of the circle $|w|=R$.*

A necessary and sufficient condition that an arbitrary function $u(x, y)$, defined in C, be harmonic in (the closed region) C is that there should exist harmonic polynomials $p_n(x, y)$ of degree n,† $n=0, 1, 2, \cdots$, and numbers M, $R>1$, such that the inequalities

$$|u(x,y) - p_n(x,y)| \leqq \frac{M}{R^n}, \tag{1}$$

where M and R are constants, that is, independent of n and of x, y, and where $R>1$, should be valid for every point (x, y) in C.

If the polynomials $p_n(x, y)$ are given so that (1) is satisfied for every (x, y) in C, the sequence $\{p_n(x, y)\}$ converges everywhere interior to C_R and uniformly on any closed point set

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* Presented to the Society, September 9, 1926.

† A harmonic polynomial of degree n is a harmonic polynomial of the form $\sum_{p+q\leqq n} a_{pq}x^p y^q$. No assumption of the non-vanishing of the a_{pq} is made. Compare, however, the discussion given in the paper referred to in the second following footnote. A similar remark applies to the degree of $V_n(z)$.

*interior to C_R, so the function $u(x, y)$ is harmonic throughout the interior of C_R.**

If $u(x, y)$ is given harmonic in the closed region interior to C_ρ, the polynomials $p_n(x, y)$ can be chosen to satisfy (1) with $R=\rho$, for (x, y) in C.

Theorem I is analogous to, and will be proved by means of, the following theorem.†

THEOREM II. *Let C be an arbitrary closed Jordan region of the z-plane, and let $w=\phi(z)$ be a function which maps conformally the exterior of C onto the exterior of the unit circle in the w-plane so that the points at infinity correspond to each other. Let C denote the curve $|\phi(z)|=R$, $R>1$, that is, the transform onto the z-plane of the circle $|w|=R$.*

A necessary and sufficient condition that an arbitrary function $F(z)$ defined in C be analytic in the closed region C is that there should exist polynomials $V_n(z)$ of degree n, $n=0, 1, 2, \cdots$, and numbers M, $R>1$, such that the inequalities

$$|F(z) - V_n(z)| \leqq \frac{M}{R^n}, \tag{2}$$

where M and R are constants, that is, independent of n and of z, and where $R>1$, are valid for every z in C.

If the polynomials $V_n(z)$ are given so that (2) is satisfied, the sequence $\{V_n(z)\}$ converges everywhere interior to C_R and uniformly on any closed point set interior to C_R, and thus $F(z)$ is analytic throughout the interior of C_R.

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* Here and below we tacitly assume that if $u(x, y)$ is not originally supposed to be defined on the entire point set considered, then the definition in the new points is to be made by harmonic extension—or what amounts to the same thing, by means of the convergent series of harmonic polynomials. A similar remark applies in Theorem II.

† The names of S. Bernstein, M. Riesz, Faber, Fejér, Szegö should be particularly mentioned in connection with Theorem II. Proof of that theorem and detailed references to the literature are to be found in a paper by the present writer, Sitzungsberichte der Bayerischen Akademie der Wissenschaften, 1926, pp. 223–229. Compare also a forthcoming paper by Szegö in the same journal.

If $F(z)$ is given analytic in the closed region interior to C_ρ, the polynomials $V_n(z)$ can be chosen so that (1) *is satisfied for $R=\rho$ and z in C.*

If the function $u(x, y)$ of Theorem I is given harmonic in the closed region interior to C_ρ, the function

$$F(z) = u(x,y) + iv(x,y), \quad z = x + iy,$$

where $v(x, y)$ is a function conjugate to $u(x, y)$, is analytic likewise in the closed region interior to C_ρ, and hence the polynomials $V_n(z)$ exist so that inequalities (2) are satisfied for z in C and for $R=\rho$. The real part of $F(z)-V_n(z)$ is $u(x, y)-p_n(x, y)$, where $p_n(x, y)$ is a harmonic polynomial of degree n, so inequalities (1) are satisfied, $R=\rho$, (x, y) in C.

It remains merely to prove that part of Theorem I concerning itself with given polynomials $p_n(x, y)$. Here we find it convenient to prove two lemmas.

LEMMA I. *Let Γ be an arbitrary analytic Jordan curve in the (x, y)-plane and Γ' an analytic Jordan curve interior to Γ. Then there exists a constant γ depending only on Γ and Γ' such that the inequality*

$$|u'(x,y)| \leqq \mu, \tag{3}$$

where (x, y) is in or on Γ, and where $u'(x, y)$ is an arbitrary function harmonic in the closed interior of Γ, implies the inequalities

$$\left|\frac{\partial u'(x,y)}{\partial x}\right| \leqq \gamma\mu, \quad \left|\frac{\partial u'(x,y)}{\partial y}\right| \leqq \gamma\mu, \tag{4}$$

where (x, y) is in or on Γ'.

The proof of Lemma I is easy, by Green's formula:

$$u'(x,y) = \frac{1}{2\pi}\int_\Gamma u'(\xi,\eta)\frac{\partial g(\xi,\eta\,;\,x,y)}{\partial n}ds, \tag{5}$$

where g denotes the Green function for the region interior to Γ, and n the inner normal. Partial differentiation of (5)

with respect to x and with respect to y, if in the right-hand member we differentiate under the integral sign, yields inequalities (4), where γ is the larger of the upper bounds of the two quantities

$$\frac{1}{2\pi}\int_\Gamma \left|\frac{\partial^2 g(\xi,\eta\,;\,x,y)}{\partial x \partial n}\right| ds, \qquad \frac{1}{2\pi}\int_\Gamma \left|\frac{\partial^2 g(\xi,\eta\,;\,x,y)}{\partial y \partial n}\right| ds.$$

These two quantities are of course bounded when (x, y) lies (as we here suppose) on or within Γ'.

Lemma I will be used in proving the following lemma.

LEMMA II. *Let Γ be an analytic Jordan curve in the (x, y)-plane and Γ' an analytic Jordan curve interior to Γ. Then there exists a constant γ' depending only on Γ and Γ' such that the inequality (3), $|u'(x, y)| \leqq \mu$, where (x, y) is in or on Γ and where $u'(x, y)$ is an arbitrary function harmonic in the closed interior of Γ, implies the existence of a function $v'(x, y)$ conjugate to $u'(x, y)$ on and interior to Γ such that*

$$|v'(x,y)| \leqq \gamma'\mu, \tag{6}$$

where (x, y) is in or on Γ'.

Denote by (a, b) any fixed point interior to Γ'; then the lengths of the shortest curves in the closed interior of Γ' from (a, b) to the various points in and on Γ' have a finite upper limit L.* If we introduce the definition

$$v'(x,y) = \int_{(a,b)}^{(x,y)} \left(-\frac{\partial u'}{\partial y}dx + \frac{\partial u'}{\partial x}dy\right) = \int_{(a,b)}^{(x,y)} \frac{\partial u'}{\partial n}ds,$$

$$(x, y \text{ in or on } \Gamma),$$

where the normal n has the same relation to the direction s as the positive axis of y to the positive axis of x, the function $v'(x, y)$ is conjugate to $u'(x, y)$ in and on Γ. This integral is independent of the path of integration in Γ; we take, then, the integral over the shortest path in Γ' from (a, b) to (x, y), when (x, y) lies in or on Γ'. Inequalities (4) give us

* The reader should have no difficulty in proving this statement.

$$\left|\frac{\partial u'}{\partial n}\right| \leqq 2^{1/2}\gamma\mu,$$

so we have, if (x, y) is in or on Γ', $|v'(x, y)| \leqq 2^{1/2}\mu\gamma L$, which is of the form (6).

We return now to the proof of Theorem I. Let Γ be an analytic Jordan curve interior to C, and Γ' an analytic Jordan curve interior to Γ. If the harmonic polynomials $p_n(x, y)$ satisfying (1) are given, we have

$$|u(x,y) - p_n(x,y)| \leqq \frac{M}{R^n}, \quad |u(x,y) - p_{n-1}(x,y| \leqq \frac{M}{R^{n-1}},$$

$$|p_n(x,y) - p_{n-1}(x,y)| \leqq M\,\frac{1+R}{R^n},$$

where (x, y) is in or on Γ. There exists therefore a function $q_n(x, y) - q_{n-1}(x, y)$ defined in and on Γ and conjugate to $p_n(x, y) - p_{n-1}(x, y)$ such that we have by (6)

$$|q_n(x,y) - q_{n-1}(x,y)| \leqq \gamma' M\,\frac{1+R}{R^n},$$

where (x, y) is in or on Γ'. If we set $V_n(z) = p_n(x, y) + iq_n(x, y)$, then $V_n(z)$ is a polynomial in z of degree n and we have

$$|V_n(z) - V_{n-1}(z)| \leqq M\,\frac{(1+\gamma')(1+R)}{R^n},$$

where z is in or on Γ'. It follows that the sequence $\{V_n(z)\}$ converges in and on Γ' to a function $F(z)$, and in such a way that we have

$$\begin{aligned}|F(z) - V_n(z)| &= |[V_{n+1}(z) - V_n(z)] \\ &\quad + [V_{n+2}(z) - V_{n+1}(z)] + \cdots| \\ &\leqq M(1+\gamma')(1+R)\left[\frac{1}{R^{n+1}} + \frac{1}{R^{n+2}} + \cdots\right] \\ &= \frac{M(1+\gamma')(1+R)}{(1-R)R^{n+1}}, \qquad (z \text{ in or on } \Gamma'),\end{aligned}$$

which is of form (2). Hence by Theorem II, the sequences $\{V_n(z)\}$ and $\{p_n(x, y)\}$ converge interior to C_R', and uniformly on any closed point set interior to C_R', where C_R' is the transform onto the z-plane of the circle $|w| = R$, when the exterior of the unit circle in the w-plane is mapped onto the exterior of Γ' in the z-plane so that the points at infinity correspond to each other.

But Γ' is really an *arbitrary* analytic Jordan curve interior to C, and can be chosen as near to the boundary of C as we please. When Γ' approaches the boundary of C uniformly, it follows from the classical results of Carathéodory* on conformal mapping that the curve C_R' approaches the curve C_R uniformly, so Theorem I is established.

Theorem II is valid under a more general assumption than that stated, namely when instead of the closed interior of a Jordan curve we consider C to be an arbitrary closed limited point set not a single point whose complementary set, with respect to the entire plane, is simply connected. We cannot extend Theorem I to this most general case, but can nevertheless prove a more general result than Theorem I.

If C is an arbitrary limited region, simply or multiply connected, there may be points P, not belonging to the closed region C, which cannot be joined to the point at infinity by means of a simple arc containing no point of the closed region C. These points P make up a finite or infinite number of regions G whose boundary points are all points of the boundary of C. Thus if a sequence of harmonic polynomials or

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* Mathematische Annalen, vol. 27 (1912), pp. 107–144, especially pp. 126–127. It is convenient to think of the curve Γ' as approaching the boundary of C monotonically—that is, each successive curve Γ' lies exterior to the preceding, so that the curves C_R' likewise vary monotonically, and hence C_R' approaches C_R uniformly. For the details connected with non-monotonic approach, see Carathéodory, loc. cit.

Here we may also apply the results of Courant, Göttinger Nachrichten, 1914, pp. 101–109; 1922, pp. 69–70, or of Lebesgue, Rendiconti di Palermo, vol. 24 (1907), pp. 371–402, especially pp. 398–399. But the results of Carathéodory (his Satz V, together with the remark in §§12a and 19 on uniform convergence) are in precisely the proper form for our application.

polynomials in z converges uniformly in the closed region C, that sequence converges uniformly likewise in each of the closed regions G, for it converges uniformly on the boundary of every such region.

Such a sequence of harmonic polynomials may, however, represent different monogenic harmonic functions in C and in a region G. We give here an illustration without proof. Let C be a strip, closed at one end, that winds infinitely many times about the unit circle and approaches that circle. The circumference of the unit circle then belongs to the closed region C and the interior of that circle is a region G. The function $u(x, y) = \log (x^2+y^2)^{1/2}$ is harmonic in the closed region C and in that closed region may be represented by a uniformly convergent series of harmonic polynomials.* This series of polynomials converges uniformly on the unit circle, but interior to that circle represents not $\log (x^2+y^2)^{1/2}$, but the function which is harmonic interior to that circle and takes on the values $\log (x^2+y^2)^{1/2}$ on the circumference, namely zero.

Let us in the general case call the *extended interior of* C the closed point set $\overline{C}$ which is the totality of all points P which cannot be joined to the point at infinity by a simple arc containing no point of the closed region C. We shall then prove a theorem which is particularly interesting in view of the illustration just given.

THEOREM III. *Let C be an arbitrary limited region of the (x, y)-plane, and let B be the complementary set (with respect to the entire plane) of $\overline{C}$, the extended interior of C. Denote by $w=\phi(z)$, $z=x+iy$, a function which maps B conformally onto the exterior of the unit circle in the w-plane so that the points at infinity correspond to each other, and denote by C_R the curve $|\phi(z)|=R$, $R>1$, that is, the transform onto the z-plane of the circle $|w|=R$.*

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* See a forthcoming paper by the present writer in Journal für Mathematik.

A necessary and sufficient condition that a function $u(x, y)$ defined on $\overline{C}$ (or in the closed region C) be harmonic on $\overline{C}$ is that there shall exist harmonic polynomials $p_n(x, y)$ of degree n, $n=0, 1, 2, \cdots$, and numbers M and $R>1$, such that inequalities (1) *are valid for all points (x, y) of $\overline{C}$ (or of the closed region C).*

If the polynomials $p_n(x, y)$ are given so that (1) *is satisfied for all points (x, y) on $\overline{C}$ (or all points (x, y) in the closed region C), the sequence $\{p_n(x, y)\}$ converges everywhere interior to C_R and uniformly on any closed point set interior to C_R, so the function $u(x, y)$ is harmonic everywhere interior to C_R.*

If $u(x, y)$ is given harmonic in the closed region interior to C_ρ, the polynomials $p_n(x, y)$ can be chosen to satisfy (1) *with $R=\rho$, for (x, y) on $\overline{C}$.*

If the function $u(x, y)$ is given harmonic in the closed interior of C_ρ, a conjugate function $v(x, y)$ exists which is likewise harmonic in that closed region. By the extension of Theorem II, the polynomials $V_n(z)$ can be found so that (2) is satisfied for z in $\overline{C}$ and for $R=\rho$, where $F(z)=u(x, y)+iv(x,y)$. The real part $p_n(x, y)$ of the polynomial $V_n(z)$ is such that (1) is satisfied for $R=\rho$ and for (x, y) in $\overline{C}$.

Suppose finally that the polynomials $p_n(x, y)$ are given so that (1) is satisfied for (x, y) in the closed region C. The reasoning already used is then valid, even if C is multiply connected, if a sequence Γ_n' of curves Γ' is chosen, the curve Γ_n' lying interior to C, every curve Γ_n' containing all points on and interior to Γ_{n-1}', every point interior to C likewise interior to some Γ_n'. For application of the results of Carathéodory it is sufficient to notice that no region B' containing B but different from B lies exterior to all the curves Γ_n'. Any such region B' would contain at least one interior point of either C or a region G. Such a point cannot, however, be joined to the point at infinity by a curve containing no point of the boundary of C, hence cannot be joined to the point at infinity by a curve lying wholly in the open region B'.

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ON INDUCTIVE RELATIONS*

BY C. H. LANGFORD

1. *Introduction.* Difficulties, connected with the occurrence of reflexive fallacies, appear when we attempt to give straightforward definitions of inductive series. These difficulties arise inevitably from the fact that, if a series is to be inductive, in the ordinary sense, properties of all orders must be transmitted in the series, and it would seem that no proposition, nor any finite set of propositions, can assert that properties of all orders are transmitted.† In the discussion which follows, we shall confine attention to the definition of a particular type of inductive series, namely the order-type ω. This restriction is made for the sake of definiteness and simplicity, and it does not entail any loss of generality in the points we shall wish to illustrate.

In a note in Mind for 1923, Dr. J. E. McTaggart holds that there is a sense in which the dictum, *No proposition can be about itself*, is false;‡ and he maintains, on the contrary, that propositions in intension can be about themselves in the sense that they are applicable to themselves; and that distinctions of type are transcended in the case of these propositions. I think it just possible that this distinction is an important one, and wish to inquire what can be made of it by way of overcoming certain difficulties in the theory of types. McTaggart's view requires that we make a sharp distinction between modal and non-modal propositions, and that non-modal or material propositions be strictly subject to differences of type, in contrast to modal propositions.

* Presented to the Society, May 7, 1927.

† Compare, however, *Principia Mathematica*, 2d ed., vol. 1, p. xliii ff. and Appendix B.

‡ *Propositions applicable to themselves*, Mind, vol. 32, p. 462. We may, for the moment, overlook the fact that this dictum itself seems to be a proposition about itself; it is a proposition in intension. Compare L. Wittgenstein, *Tractatus Logico-Philosophicus*, p. 57.

We may begin by distinguishing, roughly, three senses in which a proposition may be said to be about its subject-matter. (1) Primary propositions, propositions like "*a* gives *b* to *c*," are, in a well-understood sense, directly about the terms involved in them, and express a relation among these terms, or ascribe a characteristic to a single term. (2a) Elementary functions of primary propositions, propositions of the form $p \vee q$ for example, are not directly about the terms involved in p, q; but they are about these terms indirectly, since they restrict the truth-values of p, q,—"*at least one of the propositions* p, q *is true*." Tertiary propositions, like $(p \vee q) . (r \vee s)$, are about the terms in the primary propositions still less directly; and so on. (2b) General propositions, propositions of the form $(x) . \phi x$ for example, differ from primary and elementary propositions in that proper names are replaced by apparent variables. Propositions of the form $(x).\phi x$ are not directly about the values of x; such a proposition may be rendered, Every value of ϕx is true. Propositions involving a double quantification, as for example $(\exists y):(x).\phi(x,y)$, are about the values of y indirectly, and about the values of x doubly indirectly,—some value of $(x).\phi(x,y)$, say $(x).\phi(x,b)$, is such that $(x).\phi(x,b)$ is true; and so on. Again, $\phi(x,y)$ may have elementary functions of primary propositions for values; in which case we get a combination of (2a) and (2b). Or, we may have elementary functions of functions which are not elementary, as for example $(x).\phi x. \vee .(\exists y).\psi y$. These functions are always reducible to functions which do not involve elementary functions of functions that are not elementary—the function just cited is equivalent to $(x):(\exists y).\phi x \vee \psi y$—but even when such reduction is not effected, the principle of interpretation is the same: we have a definite hierarchy of values and values of values. (3) This is the sense in which a modal proposition is about the propositions to which it applies; the sense in which, for example, "*being a proposition entails being true or false*" is applicable to a given proposition. I shall wish to express these propositions in a

somewhat different form, and will defer discussion of (3) until some other distinctions have been drawn. The contrast to be emphasized here falls between (2b) and (3), between general propositions in extension and modal propositions.

We have now to consider two senses in which an expression may be said to be a value of another expression, senses in which one expression may be validly obtained from another by substitution. (i) In a strict sense, when a proposition is a value of a function, the function characterizes the proposition; the proposition is an instance of the function. Thus, $(x).\phi x$, Every value of ϕx is true, involves the totality of instances of ϕx; and it is in connection with this sense of value that illegitimate totalities arise, when we attempt to include as values propositions which are not instances of ϕx. The same remarks apply to multiply-quantified propositions—as for example, $(x):(\exists y).\phi(x,y)$, Every value of $(\exists y).\phi(x,y)$ is true—which involve a hierarchy of values. In this sense of value, ϕa is inferred from $(x).\phi x$ by substituting a for x in ϕx; and it is essential, for substitution of this sort to be valid, that a replace x in all of the occurrences of x. This point is of some importance in connection with the occurrence of certain pseudo-propositions. For example, if we should attempt to substitute ϕx for x in ϕx, we should get $\phi(\phi x)$; but this substitution is not complete, for ϕx must be substituted for the occurrence of x in this latter expression, and so on. The regression is infinite and vicious. (ii) We have, in this case, to consider a function f' as a value of a function f in the sense that the intension of f' involves that of f, and is, in general, a further determination of the intension of f. This relation holds when f' entails f. We may begin with an illustration. If we turn to the theory of elementary functions of propositions, as given in *Principia Mathematica*, vol. 1, *1–*5, and consider any particular proposition, as for example

$$p.p \supset q. \supset .q, \tag{1}$$

there appears to be some doubt as to the precise force of

this proposition; and there are, in fact, two alternative readings, either of which might be meant. It might be meant that

(2) $$(p,q):p.p \supset q. \supset .q,$$

every value of $p.p \supset q. \supset .q$ is true. On the other hand, it might be held that (1) is a modal proposition, involving the function $p.p \supset q. \supset .q$, which could be written

(3) "$p.p \supset q. \supset .q$" cannot fail.

It is to be noted that the terms *hold* and *fail* apply to properties or functions, while *true* and *false* apply to propositions; so that (2) is about the totality of values of the function involved, while (3) asserts a modal characteristic of the function itself. Now any function which is a further determination of the function in (3), as for example

(4) $$\sim p.\sim p \supset q. \supset .q,$$

is, in a sense, a value of the function in (3). But (4) is not a value in a sense (i); for substitution of $\sim p$ for p, in sense (i), would lead to a vicious regression, since $\sim p$ would have to replace p in every occurrence of p; and in any case, the expression obtained by substitution is not a proposition, as it always is in sense (i). In the language of Mr. W. E. Johnson, (4) is a relative determinate under the function in (3) as determinable;* the function in (3) is related to (4) as "being colored" is related to "being red," and not as "x is colored" is related to "*this* is colored"—this latter being the relation of function to value in sense (i).

In the second edition of *Principia Mathematica*, the ambiguity of (1), as between (2) and (3), is decided in favor of (2).† This is natural in view of the abandonment of the real variable, as distinguished from a universal apparent variable having the whole of the asserted proposition or scope, and in view of the fact that the reference of the

* *Logic*, Part 1, p. 173.

† Introduction, vol. 1, p. xiii.

variables in (1) was confined to propositions of one order. We shall, on the other hand, adopt (3) as the interpretation of (1); and this is why numbers *1–*5 were referred to as the theory of elementary functions of propositions, rather than as the theory of elementary propositions. If we denote by p_1 a proposition of the first order, and by p_2 a proposition of the second order, then it is to be held that

$$(2_1) \qquad (p_1, q_1) : p_1 . p_1 \supset q_1 . \supset q_1$$

follows from (3), but not conversely; and that

$$(2_2) \qquad (p_2, q_2) : p_2 . p_2 \supset q_2 . \supset . q_2$$

follows from (3), but not conversely. Again, the proposition

$$(5) \qquad (\phi) : . (\exists y) : (x) . \phi(x, y) : \supset : (x) : (\exists y) . \phi(x, y)$$

follows from the modal proposition

(6) "$(\exists y) : (x) . \phi(x, y) : \supset : (x) : (\exists y) . \phi(x, y)$" cannot fail ;

and this holds no matter what the order of ϕ, in 5, may be. A modal proposition entails a corresponding factual universal, and it may entail many such propositions of different orders, but no factual universal entails a modal proposition.*

2. *Modal, Material, and Apparent Variables.* In distinguishing modal and material propositions, we may recognize three kinds of variables. The variables p, q in (3) and the variable ϕ in (6) are modal variables; p, q in (2) and x, y in (5) and (6) are, of course, apparent variables. Material variables occur only in properties or functions. Thus, in the function $(x) : (\exists y) . \phi(x, y)$, ϕ is a material variable, and must be of a definite order. In a function like "x is colored," x is a material variable; and we may, when necessary, indicate material variables, such as ϕ and x, by writing $\ddot{\phi}$ and $\ddot{x}$. Similarly, modal variables may be written $\bar{\phi}$, $\bar{x}$; ϕ, $\bar{x}$; so that (3) becomes $\bar{p} . \bar{p} \supset \bar{q} . \supset . \bar{q}$. If this analysis should

* See C. I. Lewis, *A Survey of Symbolic Logic*, Chap. V, on strict relations.

be right, it would seem that what originally gave rise to the notion of a real variable was the recognition of propositions in which what we have called modal variables occur; but that the real variable was confused, on the one hand, with the universal apparent variable, and on the other with modal variables.

3. *Modal and Material Properties.* A property or function is derived from a proposition by replacing constant constituents of the proposition by appropriate material variables; and this holds for both material and modal propositions, for the proposition is, in either case, a value in sense (i) of the derived function. Consider, for example, the material function

(7) $$\phi x \supset \phi y.$$

This function will be satisfied by any value of x which lacks ϕ, for all values of y, and by any value of y which has ϕ, for all values of x. But the modal property,

(8) "$\phi x \supset \phi y$" cannot fail,

will hold only for $x=y$, or when ϕ is some special function and x, y are so chosen that x having ϕ necessitates y having ϕ. The relation of (7) to (8) is simply that the values for which (8) holds are a proper part of the values for which (7) holds. Again, the relation of identity is not properly analyzed by the material function $(\phi).\phi x \supset \phi y$, but rather by the modal function $\phi x \supset \phi y$, that is, being a function of x necessitates being a function of y. Modal variables occur in modal propositions and in modal properties, and the occurrence of at least one modal variable is necessary for these propositions and properties; apparent variables occur in modal and material propositions and in modal and material properties; material variables occur in modal and material properties, and their presence is necessary and sufficient for a property.

4. *Definition of the Order-Type* ω. If we consider a series of the form $a_0, \cdots, a_n, \cdots$, which is inductive, what we mean by saying that the series is inductive is always strictly equivalent to what we mean by saying that, beginning with a_0, any term, a_n, can be reached step by step. We require an analysis of what we mean when we say that a_n is an inductive distance from a_0, or that a_n can be reached step by step from a_0; and it is clear that, if a proposition p_1 is to be a right analysis of a proposition p_2, then p_1 must entail and follow from p_2, that is, must be strictly equivalent to p_2. If now we consider some property f, which belongs to a_0, and is transmitted in the series, f will also belong to a_n; and we may say that it is not the case that $f(a_0)$ is true and $f(a_n)$ false. But, of course, if b_n be some term not belonging to the series, it may also be true that $f(b_n)$; so that we may say that it is not the case that $f(a_0)$ is true and $f(b_n)$ false. There is, however, this important difference: f belongs to a_n for the reason that is belongs to a_0 and is transmitted; whereas f does not belong to b_n for this reason.* Expressed otherwise, being an hereditary property belonging to a_0 entails being a property belonging to a_n; it follows from $f(a_0)$ and f is hereditary, alone, that $f(a_n)$, but not that $f(b_n)$. It might be supposed that we could express the fact that a_n is an inductive distance from a_0 by saying that a_n has every hereditary property which belongs to a_0; but this is not the case. For example, in a series of the form $a_0, \cdots, a_n, \cdots, a_\omega$, where a_ω is the limit of the progression, it might be true that a_ω has every hereditary property which belongs to a_0; whereas it is of course false that a_ω can be reached step by step from a_0, and false that $f(a_0)$ and f is hereditary entails $f(a_\omega)$.

Properties of the order-type ω will be formulated, in the usual way, on $\phi x, \psi(x, y)$, that is, on a class K and a dyadic relation R_2. There will be the usual properties for a discrete series having a first but no last element, which are material properties; and a single modal property. This modal prop-

* Compare Lewis, loc. cit., on strict implication.

erty distinguishes ω from such order-types as $\omega+^*\omega+\omega$, which also satisfy all of the material properties. A property is hereditary in $a_0, \cdots, a_n, \cdots$, if there is no term having this property whose immediate successor lacks it. Now t_2 immediately succeeds t_1 if, and only if,

$$\phi(t_1).\phi(t_2).\psi(t_1,t_2):.(z):\phi(z).z\neq t_2.\psi(t_1,z).\supset.\psi(t_2,z).$$

This is a function of t_1, t_2, and we may denote it by $S(t_1, t_2)$. A property f is hereditary in the series if, and only if,

$$(t_1,t_2):S(t_1,t_2).f(t_1).\supset.f(t_2)\ ;$$

and this function of f may be denoted by $T(f)$. If a_i, a_j are such that $\psi(a_i, a_j)$, then $T(f).f(a_i).\supset.f(a_j)$. But this relation is stronger, for $T(f).f(a_i)$ entails $f(a_j)$, that is,

$$T(\bar{f}).\bar{f}(a_i).\supset.\bar{f}(a_j)\ ;$$

so that the inductive property may be formulated,

p_1 $\quad (a_i,a_j):.\phi(a_i).\phi(a_j).\psi(a_i,a_j).\supset:T(\bar{f}).\bar{f}(a_i).\supset.\bar{f}(a_j).$

The following material properties are familiar:

p_2 $\quad (\exists x).\phi x;$ $\qquad$ p_3 $\quad (x).\phi x\supset\sim\psi(x,x);$

p_4 $\quad (x_1,x_2):\phi(x_1).\phi(x_2).\sim x_1=x_2.\supset.\psi(x_1,x_2)\vee\psi(x_2,x_1);$

p_5 $\quad (x_1,x_2):\phi(x_1).\phi(x_2).\sim x_1=x_2.\supset.\sim\psi(x_1,x_2)\vee\sim\psi(x_2,x_1);$

p_6 $\quad (x_1,x_2,x_3):\phi(x_1).\phi(x_2).\phi(x_3).\sim x_1=x_2.\sim x_2=x_3.$
$\qquad\qquad \sim x_1=x_3.\psi(x_1,x_2).\psi(x_2,x_3).\supset.\psi(x_1,x_3);$

p_7 $\quad (\exists x):.(y):\phi x:\phi y.\sim x=y.\supset.\psi(x,y);$

p_8 $\quad (x):::(\exists y)::.(z)::\phi x.\supset:.\phi y.\psi(x,y):.\phi z.\supset:$
$\qquad \psi(x,z).\psi(y,z).\vee.\psi(z,x).\psi(z,y).\vee.z=x\vee z=(y).$

5. *Note on Classes.* It may be suggested that modal propositions have some bearing on the customary treatment of classes through their determining functions. There are two related questions which arise in this connection; namely, whether there exist determining functions of the proper order, and whether there exist determining functions at all. We have suggested how, in certain cases, difficulties

as to order might be overcome. With regard to the second question, we wish to know that every class has a determining function; but this is not sufficient: we wish to know that every class must have a determining function. An illustration will make this clear. Suppose we have two classes α_1, α_2 and respective determining functions f_1, f_2, such that $(x).f_1(x) \supset f_2(x)$. Now we have occasion very often to make such assertions as,

(i) Every subclass of α_1 is a subclass of α_2;

and wish to express this by

(ii) $$(\phi):(x).\phi x \supset f_1(x).\supset .(x).\phi x \supset f_2(x).$$

Aside from difficulties of type, (ii) is not logically equivalent to (i) even if every subclass of α_1 has a determining function, unless it is also true that every subclass must have. But neglecting this point, let us suppose that there is some subclass β, of α_1, which has no determining function. Then, clearly, (ii) is not a right analysis of (i). But if we use a modal proposition in the analysis of (i), it seems to be unimportant whether every class has a determining function, or whether it must have; and we seem to be able to treat classes through functions whether classes have determining functions or not. For (i) may be expressed

(iii) $$(x).\bar{\phi}x \supset f_1(x).\supset .(x).\bar{\phi}x \supset f_2(x)\ ;$$

that is, being a function which determines a subclass of the class determined by f_1 entails being a function which determines a subclass of the class determined by f_2. Now, by hypothesis, β is a subclass of f_1 having no determining function; but (iii) is relevant to β, for it is nevertheless true that, if β had a determining function, that function would determine a subclass of f_2.

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THREE-PARAMETER AND FOUR-PARAMETER LINEAR FAMILIES OF CONICS IN THE GALOIS FIELDS OF ORDER 2^n*

BY A. D. CAMPBELL

1. *Three-Parameter Families.* We shall denote a Galois field of order 2^n by the symbol $GF(2^n)$. We define a three-parameter linear family of conics in such a $GF(2^n)$ as the locus of all points whose coordinates x, y, z satisfy an equation of the form

$$\lambda C_1 + \mu C_2 + \nu C_3 + \rho C_4 = 0, \tag{1}$$

where

$$C_i = a_i x^2 + b_i y^2 + c_i z^2 + f_i yz + g_i zx + h_i xy = 0, \quad (i = 1,2,3,4),$$

and where the variables, coefficients, and parameters represent numbers in this domain. The conics $C_1 = 0, \cdots, C_4 = 0$ are linearly independent, and are called fundamental conics of this family. In this paper we derive the classes of these families, and we give a typical family for each class. We note that in any $GF(2^n)$ every number is a perfect square with just one square root, and $(\alpha x + \beta y + \gamma z)^2 = \alpha^2 x^2 + \beta^2 y^2 + \gamma^2 z^2$. We note that every such family has at least one double line.

We first divide these families into the following distinct sets.

Set I. Each family contains a net of conics reducible to the form

$$\lambda x^2 + \mu y^2 + \nu z^2 = 0. \tag{2}$$

Set II. Each family contains no net reducible to (2), but does contain a net reducible to the form

$$\lambda x^2 + \mu y^2 + 2\nu xy = 0. \tag{3}$$

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Set III. Each family contains at least two double lines, but no net reducible to (2) or (3).

Set IV. Each family contains only one double line, but has also a net reducible to the form

$$\lambda x^2 + \mu xy + \nu xz = 0. \tag{4}$$

Set V. Each family has none of the preceding properties, but has a pencil reducible to the form

$$\lambda x^2 + \mu xy = 0. \tag{5}$$

Set VI. Each family has just one double line, but none of the preceding degenerate pencils and nets.

Set I. Such a family can be put in the form

$$\lambda x^2 + \mu y^2 + \nu z^2 + \rho(fyz + gzx + hxy) = 0.$$

We easily get

$$\text{Class (1)}: \quad \lambda x^2 + \mu y^2 + \nu z^2 + \rho xy = 0. \tag{6}$$

Set II. We obtain

$$\text{Class (2)}: \quad \lambda x^2 + \mu y^2 + \nu xy + \rho(z^2 + zx) = 0. \tag{7}$$

$$\text{Class (3)}: \quad \lambda x^2 + \mu y^2 + \nu xy + \rho zx = 0. \tag{8}$$

Any transformation

$$x = \alpha_1 x' + \beta_1 y' + \gamma_1 z', \quad y = \alpha_2 x' + \beta_2 y' + \gamma_2 z', \tag{9}$$
$$z = \alpha_3 x' + \beta_3 y' + \gamma_3 z',$$

where $|\alpha_1, \beta_2, \gamma_3| \neq 0$, that is to send (7) into (8) must have $\gamma_1 = \gamma_2 = 0$, $\gamma_3 \neq 0$, since the net $\rho = 0$ of one family must go into that of the other. But (9) then cannot send (7) into a family lacking the term in z^2.

Set III. We can reduce such a family to the form

$$\lambda x^2 + \mu y^2 + \nu xz + \rho(cz^2 + fyz + hxy) = 0.$$

We get the following three classes, according as $cf \neq 0$; or $c = 0$, $f \neq 0$; or $c \neq 0$, $f = 0$.

(10) Class (4) : $\lambda x^2 + \mu y^2 + \nu zx + \rho(z^2 + yz) = 0.$

(11) Class (5) : $\lambda x^2 + \mu y^2 + \nu zx + \rho yz = 0.$

An argument similar to that used in the study of (7) and (8) shows that (10) and (11) are non-equivalent. Here the pencil $\nu = \rho = 0$ of (10) must go into the similar pencil of (11) by any projectivity (9).

(12) Class (6) : $\lambda x^2 + \mu y^2 + \nu zx + \rho(z^2 + xy) = 0.$

It is easy to prove (12) non-equivalent to (10) or (11).

Set IV. We put (1) in the form

$$\lambda x^2 + \mu xy + \nu xz + \rho(by^2 + cz^2 + fyz) = 0.$$

We get two classes, according as $C_4 = 0$ in the above family is a pair of real or conjugate imaginary lines.

(13) Class (7) : $\lambda x^2 + \mu xy + \nu xz + \rho yz = 0.$

(14) Class (8) : $\lambda x^2 + \mu xy + \nu xz + \rho(y^2 + z^2 + \alpha yz) = 0,$

where $C_4 = 0$ is irreducible. Any transformation (9) that is to send (14) into (13) must have $\beta_1 = \gamma_1 = 0$. But then the conic $C_4 = 0$ of (14) goes into a conic $y'^2(\beta_2^2 + \beta_3^2 + \alpha\beta_2\beta_3) + z'^2(\gamma_2^2 + \gamma_3^2 + \alpha\gamma_2\gamma_3) + \cdots = 0$. So we must have $\beta_2^2 + \beta_3^2 + \alpha\beta_2\beta_3 = 0$, $\gamma_2^2 + \gamma_3^2 + \alpha\gamma_2\gamma_3 = 0$ which give us $\beta_2 = \beta_3 = \gamma_2 = \gamma_3 = 0$ and (9) is singular.

Set V. We can put the family in one of the three following forms:

(A) $\lambda x^2 + \mu xy + \nu(z^2 + zx) + \rho(by^2 + fyz + gzx) = 0,$

(B) $\lambda x^2 + \mu xy + \nu yz + \rho(by^2 + cz^2 + gzx) = 0,$

(C) $\lambda x^2 + \mu xy + \nu(z^2 + yz) + \rho(by^2 + cz^2 + gzx) = 0.$

Case (A) gives us two classes.

(15) Class (9) : $\lambda x^2 + \mu xy + \nu(z^2 + zx) + \rho(y^2 + yz) = 0.$

(16) Class (10) : $\lambda x^2 + \mu xy + \nu(z^2 + zx) + \rho yz = 0.$

Any projectivity (9) that is to send (15) into (16) must have $\beta_1 = \gamma_1 = \gamma_2 = 0$, $\beta_2\gamma_3 \neq 0$. But then $C_3 = 0$ and $C_4 = 0$ of (10)

must go separately into conics lacking the term in y^2. This gives us $\beta_3^2 = 0$, $\beta_2^2 + \beta_2\beta_3 = 0$; hence $\beta_2 = 0$.

Case (B) gives us two classes.

(17) Class (11) : $\lambda x^2 + \mu xy + \nu yz + \rho(y^2 + z^2 + zx) = 0.$

(18) Class (12) : $\lambda x^2 + \mu xy + \nu yz + \rho(y^2 + zx) = 0.$

The families (17) and (18) are easily proved to be non-equivalent. They are non-equivalent to families (15) and (16) because of the presence in these two latter families of the degenerate pencil $\lambda x^2 + \nu(z^2 + zx) = 0$.

Case (C) gives us the following class.

(19) Class (13) : $\lambda x^2 + \mu xy + \nu(z^2 + yz) + \rho(y^2 + \alpha z^2 + zx) = 0.$

Set VI. We can put the family in one of two forms

(A) $\lambda x^2 + \mu(y^2 + xy) + \nu yz + \rho(by^2 + cz^2 + gzx) = 0,$

(B) $\lambda x^2 + \mu yz + \nu(by^2 + cz^2 + gzx) + \rho(b'y^2 + c'z^2 + h'xy) = 0.$

Case (A) gives us

(20) Class (14) :

$$\lambda x^2 + \mu(y^2 + xy) + \nu yz + \rho(y^2 + \alpha z^2 + zx) = 0, \quad \alpha \neq 0.$$

Case (B) gives us

(21) Class (15) :

$$\lambda x^2 + \mu yz + \nu(y^2 + \alpha z^2 + zx) + \rho(\beta y^2 + \gamma z^2 + xy) = 0,$$

where $\gamma = 1$ or $\gamma \neq$ cube, $\gamma/\beta \neq \alpha$. The family (20) has a degenerate pencil $\lambda x^2 + \mu(y^2 + xy) = 0$, which does not occur in (21).

2. *Four-Parameter Families.* We define a four-parameter linear family of conics in a $GF(2^n)$ by the equation

(22) $$\lambda C_1 + \mu C_2 + \nu C_3 + \rho C_4 + \sigma C_5 = 0,$$

where the details of notation are as indicated in the description of equation (1). It is easy to show that every such family has at least two double lines. First we assume that

(22) has a degenerate net reducible to (2). Then we assume that (22) has no net (2), but a net (3). Then we assume that (22) has neither (2) nor (3).

If (22) has a net (2) we can put the family in the form

$$\lambda x^2 + \mu y^2 + \nu z^2 + \rho xy + \sigma(fyz + gzx) = 0,$$

which gives us

(23) Class (1) : $\lambda x^2 + \mu y^2 + \nu z^2 + \rho xy + \sigma xz = 0.$

If (22) has a net (3), but not a net (2), we get

$$\lambda x^2 + \mu y^2 + \nu xy + \rho xz + \sigma(cz^2 + fyz) = 0,$$

which gives us, according as $c \neq 0$, or $c = 0$,

(24) Class (2) : $\lambda x^2 + \mu y^2 + \nu xy + \rho xz + \sigma(z^2 + yz) = 0,$

(25) Class (3) : $\lambda x^2 + \mu y^2 + \nu xy + \rho xz + \sigma yz = 0.$

If (9) is to send (24) into (25) we must have $\gamma_1 = \gamma_2 = 0$, $\gamma_3 \neq 0$. But such a transformation cannot send (24) into a family lacking the term in z^2.

If (22) has no net (2) and no net (3) we easily reduce the family to the form

(26) Class (4) : $\lambda x^2 + \mu y^2 + \nu xz + \rho yz + \sigma(z^2 + xy) = 0.$

Syracuse University

CLARENCE ABIATHAR WALDO

BY W. H. ROEVER

It has been said that next to those who are directly laboring to extend and advance science, they contribute to its progress who prepare fitting aids to the young beginner and remove the difficulties in his way. The late Clarence Abiathar Waldo was thus outstanding as a contributor to scientific progress in this country. He was essentially a teacher, having once said of himself: "My greatest contribution to education has been my intimate contact with thousands of young men during their college period. . . . . These are my intellectual children, whose ideals I have helped to form. As a teacher of *men*, I will be best known and remembered."

He was among the pioneers in mathematical organization in this country. In 1891 he became member of the New York Mathematical Society, which was reorganized in 1894 as the American Mathematical Society. He was one of the small group of mathematicians present at the Mathematical Congress at Chicago and the Colloquium at Evanston in 1893. He was active on several committees of the Society, serving with Professors Bolza and Townsend in framing a report on the requirements for the Master's Degree (this Bulletin, vol. 10, p. 380). He was also a charter member of the Mathematical Association of America, being particularly active in the Missouri Section.

Professor Waldo's mathematical interests were chiefly in applications to engineering. He was author of a *Manual of Descriptive Geometry* (D. C. Heath, 1888), to which the remark above concerning "fitting aids to the young beginner" was particularly applicable. He followed the practice, now too much neglected, of having the student make models of the space objects under consideration in the course. His address as retiring chairman of Section D (now M) of the American Association for the Advancement of Science on the subject *The relation of mathematics to engineering* (Proceedings, vol. 53 (1904)) is interesting and instructive. In the Association for the Advancement of Science he was also secretary of the council, 1903–04, and general secretary and editor of the Proceedings, 1904–05. He was very active in the Society for the Promotion of Engineering Education, serving as treasurer, 1900, and editor of its Proceedings, 1901–03. The Indiana Academy of Science was also fortunate in having Professor Waldo as editor of its Proceedings, 1896–98, and as president, 1898. In connection with his work in the Academy, Professor Waldo told with amusement of a bill which had been introduced in the Indiana Legislature fixing a "new and correct value of π" as a definite rational number. The bill was being favorably considered when he heard of it, and quick action was necessary to prevent its passage. It was due to his efforts that the bill was rejected. He was a member of Sigma Xi and Phi Beta Kappa.

Other activities than those strictly scientific also claimed his interest, and attention. He was a member and president, 1890, of the Indiana

College Association. His interest in athletics was very keen. He was arbitrator of the Chicago Conference of Colleges, 1901–12, and of the Ohio Conference of Colleges, 1903–05.

Professor Waldo received the degrees A.B. and A.M. from Wesleyan University in 1875 and 1878, respectively, and the degree Ph.D. from Syracuse University in 1894. He spent the academic year 1882–83 in study at the universities of Leipzig and Munich.

He held the following academic positions: Instructor in Mathematics and Registrar, Wesleyan University, Conn., 1877–81; Professor of Mathematics, 1883–91, and Acting President, 1885–86, 1888–89, Rose Polytechnic Institute; Professor of Mathematics, De Pauw University, 1891–95; Head Professor of Mathematics, 1895–98, Purdue University; Professor of Mathematics, 1908–10, Thayer Professor of Mathematics and Applied Mechanics and Head of the Department of Mathematics, 1910–17, Professor Emeritus after 1917, Washington University. After leaving Washington University, he worked actively on exemption boards in New York City in 1917–18. In 1919 he was College Visitor for the Carnegie Foundation.

Professor Waldo was born in Hammond, New York, January 21, 1852. He died in Bridgeport, Conn., October 1, 1926. He married Miss Abby Wright Allen, August 2, 1881, in Stamford, Conn. He had two children, Alice Goddard and Clarence A. (deceased).

His pleasing personality and his kindly nature endeared him to all who knew him. His high ideals were built on deep religious foundations and were reflected in his daily life.

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EDDINGTON ON CONSTITUTION OF STARS

The Internal Constitution of the Stars. By A. S. Eddington. Cambridge University Press, 1926. viii+407 pp.

The author says that this book was written between May 1924 and November 1925 and that during this period theoretical papers in stellar constitution in the Monthly Notices alone amounted to more than 400 pages. He has tried to cover everything up to November 1925 and, in proof, some further material up to March 1926. Composition under such circumstances is difficult even for one who has contributed very actively to the subject for the previous decade, and particularly when he has been currently engaged in a running fight with his most eminent compatriot specializing in the same field. A popular account of the subject may be found in Science (May 6, 1927, pp. 431–438) by Menzel, and there is no need here to repeat it. Suffice it to remark that this book reads like a romance to those who are not deterred by the mathematics whether because they can follow it or because they ignore it; the difficulties of composition do not show in the style.

One may raise the question: Is it romance or science? We all recall the "Cosmic Crucibles" of G. E. Hale. We see only the thinnest fumes hovering over the crucibles; how much do we know about the conditions within them? Certainly very little. The highest steady temperatures with which we can work in the laboratory are only a few thousand degrees; by spectral observations on the exterior of stars we record spectra which by the law of Wien (or Planck) we estimate at most as toward a very few tens of thousands; just what the temperatures are, or even whether there is anything truly corresponding to temperature in the explosive discharges of Anderson we perhaps do not know. When therefore through a course of reasoning we set 40,000,000 degrees as the normal temperature of the center of a sun we are stretching our laboratory laws a great deal on the side of extrapolation toward infinity, we are dealing certainly not with scientific facts, in the sense of verifiable facts of observation, we are wholly in the realm of theory, in that of faith which has ever had a higher psychic appeal than fact.

True, we work in our vacuum tubes at high potentials with high speed electrons and observe their effects upon matter; true, too, that to produce free electrons of such speeds under conditions of thermal equilibrium would require intense temperatures; but how safe is the passage from an observed high tension electric dis-equilibrium with its multifarious manifestations to an inferential intense thermal equilibrium? And then there are the tremendous pressures; what are their effects on the imagined and inaccessible phenomena we are studying? Bridgman's highest laboratory pressures do not take us much if any below our lithosphere and yet they at least approach those that justify our speaking of the compressibility of the molecules and atoms. May it be that in the upper stellar atmospheres

the luminous effects are chiefly electrical, as in our vacuum tubes or possibly in our aurora, rather than thermal as such, and that in the stellar interiors the physical state is one governed more by pressure than by temperature?

The author does not minimize the physical difficulties. What he does is to take our finite physical laws of observation as they are, add to them some inferential generalizations, assume both to hold everywhere even under conditions far more extreme than any realized in the laboratory, and upon this to build a mathematical physical theory of the internal constitution of the stars, which in turn he tries to justify, and with a large measure of success, upon the record of astronomical, chiefly astrophysical, classification of the stars. This is a perfectly sound procedure. The business of a scientific theory is to bind together the scientific observations. We have certain variables or parameters $a, b, c, \cdots$ which represent the facts observed; we desire to find rational relations between them; we introduce certain auxiliary unknown variables $x, y, z, \cdots$ with laws connecting them to $a, b, c, \cdots$ and among themselves; we ultimately eliminate the unknowns and come out with the desired relations between the knowns and verify those relations on the facts. In a field so actively worked as stellar physics is today, both observationally and theoretically, in a field in part so much in dispute as is this between Eddington and Jeans, changes may well come; such is science in the romantic, perhaps better in the heroic or epic phase when an Achilles fights his Hector for some lady who, forsooth, belongs to a third (or fourth) party.

There is something more involved than the rational correlation of the observed quantities $a, b, c, \cdots$, some constraints on the manner of that correlation; the real riddle is the permanence of this planet. Geology and paleontology need half a billion years, radioactive studies set the age of the earth's crust as around a billion years, we apparently must think of the luminous life of a star as measured in centuries of billions of years. Where does the energy come from? Evolution goes too slowly for the physics we know and can only be expounded on extensive extrapolations. Laplace, Helmholtz, Kelvin, all who have tried to set a time scale by old fashioned physics have fallen far short of the mark; our present hope seems to reside in the commutation of material mass into radiant energy through Einstein's equation $E=mc^2$, each disappearing gram of matter giving rise to 9×10^{20} ergs of radiation. The synthesis of hydrogen into helium, when, if, and as it takes place, would by this equation release a store of energy vast in comparison with ordinary chemical reactions, but apparently a bit too small in amount to maintain for its life the radiation of a star; we apparently have to fall back on some sort of super-radioactivity (Jeans) or on the coalescence of electron and proton into nothingness (Eddington).

After the immediate satisfaction of our needs for individual sustenance and for the continuation of the race, our most imperious demand seems to be to solve the infinite problems of our finite minds—a truly imperial ambition. It is right and proper that we should offer a crown to such a Caesar as our author. At the risk of marking ourselves as some miserable

Cinna we may make a few additional comments quoting from p. 163: "At the beginning of 1924 the giant and dwarf theory of Hertzsprung and Russell was almost universally accepted. . . . . I do not think it is too blunt an expression to say that this is now overthrown; at least it has been gutted, and it remains to be seen whether the empty shell is still standing." Note the great change in two years. Things are moving fast, not faster perhaps than a genius may develop new theory, but surely far faster than new facts of observation may be revealed and established. And herein lies a source of possible future overthrow for any present theory; we need more and better facts and they are slow to get. Moreover we need facts that are not colored or selected by our preconceived but not yet established theories. There is a marked tendency for an accepted, albeit perhaps not established, theory to bring to light those facts which may perchance support it and to obscure those which might contravert it. This is particularly true when the argument is statistical as is now common in astronomy. I do not say that the author is unfair; the question is one of judgment and his judgment should be as good as that of anybody and certainly incomparably better than mine; I merely cite his inevitable liabilities. In speaking of the synthesis of hydrogen into helium he says (p. 301): "We do not argue with the critic who urges that the stars are not hot enough for this process; we tell him to go and find *a hotter place*." I am not finding fault with the facts that we have but merely urging that we need better ones and more of them.

Although the chief interest of the author is in the deep interiors of the stars and in astronomical observations his text is full of developments of and suggestions for physics and of astronomical conditions on the surface of or exterior to the stars. A list of the chapter headings gives the best short summary of the content of the book. I. Survey of the Problem. II. Thermodynamics of Radiation. III. Quantum Theory. IV. Polytropic Gas Spheres. V. Radiative Equilibrium. VI. Solution of the Equations. VII. The Mass-Luminosity Relation. VIII. Variable Stars. IX. The Coefficient of Opacity. X. Ionization, Diffusion, Rotation. XI. The Source of Stellar Energy. XII. The Outside of a Star. XIII. Diffuse Matter in Space. There is a table of physical and astronomical constants, a long list of references with some brief summaries of their content, and a good index. If any reader thinks the author at times is too zealous, too dogmatic, let him persevere until he can distinguish brilliancy of style from fixation of mind, even unto these words from the final paragraph: "The history of scientific progress teaches us to keep an open mind. I do not think we need to feel greatly concerned as to whether these rude attempts to explore the interior of a star have brought us to anything like the final truth. We have learned something of the varied interests involved."

EDWIN B. WILSON

CULLIS ON MATRICES

Matrices and Determinoids. By C. E. Cullis. University of Calcutta Readership Lectures. Cambridge University Press, Vol. III, Part 1, 1925. xviii+681 pp.

The first two volumes of this treatise were reviewed in this Bulletin (vol. 26 (1920), pp. 224–233). The present volume continues the development, which has become more extensive than at first contemplated, so that the third volume has had to appear in two parts. The second part, it is stated, will deal chiefly with structural matrices. The author's avowed aim is to carry forward the program of Sylvester, who conceived a magnificent treatise on universal algebra, which he never carried out. The present treatise to a large extent is realizing this program. To Professor Cullis, matrices means practically all of algebra, and it is easy to see from his standpoint that every expression in algebra, whether it depends upon one or upon more variables, is in reality a function of some sort of matrix. To the reviewer this is tantamount to saying that every function of one or more variables is ipso facto a function of a vector of a space of the same number of dimensions, or of various vector products in such space. The difference is simply one of point of view or mode of statement. It is simpler, in the reviewer's opinion, to make the whole subject a branch of the theory of hypernumbers, than to build up a theory of "sets," although this is really following the tradition of Sir W. R. Hamilton, the creator of quaternions. The author insists rightly upon the matrix as a single entity, and considers that a *pure* calculus will be determined by a system of scalar numbers, so that he would introduce hypernumbers later, the ordinary quaternion, for instance, being a product of a scalar matrix and a matrix whose elements are the usual $1, i, j, k$. While this is logical, and is a justifiable line of development, it seems to the reviewer to complicate the subject unnecessarily. The author further considers that on the applied side, every physical entity can be represented by a matrix, and he certainly would find plenty of justification for this view in modern physics. His assertion (p. vi) that the expressions would be two-dimensional tables, hardly seems tenable, however. For witness the modern tensor theory.

The scope of this volume is best indicated by a brief indication of the contents of the various chapters. The first three (XX, XXI, XXII) deal with rational integral functions of any number of variables, irresoluble and irreducible functions and factors, common factors, resultants, eliminants, discriminants, and common roots. Symmetric functions are considered specially in Chapter XXII. In Chapter XXIII begins the consideration of rational integral functional matrices, that is, matrices whose elements are rational integral functions of scalar variables, $x, y, z, \cdots$. The definition is given and properties proved of irreducible and irresoluble divisors of such matrices. These divisors are rational integral functions of the same variables as the matrix, and are of order i when they will divide every minor determinant of order i which arises

from the matrix. A *regular* minor determinant of order i is one which, with respect to an irreducible or irresoluble divisor t, does not vanish identically, and is not divisible by a higher power of t than that power which is the highest that will divide all minors of the order i. The terms invariant factors, and elementary divisors, of the classic literature, are replaced in this chapter by *potent divisor*. The greatest common divisors D are called maximum factors, the invariant factors E are called maximum divisors, the divisors of these are called potent factors and potent divisors. These are all rational in the domain considered. The former elementary divisors might have been irrational. The potent factors and divisors of the product of two matrices are considered. Following this the same divisors are considered for *compartite* matrices, that is, matrices which may be considered to be the *direct sum* (in the hypernumber sense) of simpler matrices. The next chapter, XXIV, takes up the treatment of *equipotent* matrices, that is, matrices which are *equivalent* in the sense of having the same rank, and the same potent divisors. The equipotent transformations are discussed in detail. With regard to these topics the reviewer will remark only that the fundamental elements of a matrix are what he has previously called its *shear regions*. When these are given everything is virtually given and it would be simpler to treat the subject on such a basis. The two chapters just discussed introduce the idea of rationality for a given domain, which is after all an *arithmetic* notion as contrasted with an algebraic notion. The reviewer has in mind that algebra must deal with what he calls *hypernumbers*, which are frequently called in France *relative numbers*, that is to say, numbers with qualitative signs. Even the introduction of $+$ and $-$ was necessary to produce *algebra*. Mere literal notations do not take one out of the region of *arithmetic*. In Arithmetic we find the root of the notion of *domain of rationality*. Its extension to algebraic fields followed. Thus in all discussions of such topics as we have before us we find these two aspects, the hypernumber structure, and the domain of rationality, or sometimes of integrity, as for instance in Dickson's Arithmetic of Algebras.

Chapter XXV is concerned with rational integral functions of a square matrix. It introduces the notions of *latent roots*, and the *characteristic matrix*, which is the matrix $\phi - \lambda I$, where I is the identity matrix, and the elements of the matrix ϕ are constant. It includes also the determination of all *rational integral equations* satisfied by ϕ. At the end of the chapter is Frobenius' solution of the matrix equation $\psi^2 = \phi$. The Hamilton-Cayley equation is developed.

Chapter XXVI develops the subject of *equimutant transformations*, or transformations of the form $A(\ )A^{-1}$ where A is undegenerate. This transformation corresponds directly to what is usually called a transformation in the theory of groups, or in the theory of linear homogeneous groups to an orthogonal transformation. It might be added it is also in the general theory of vectors in N dimensions, a rotation. It enables us to bring a matrix with constant elements to the canonical form. Also the idea of *transmute* is introduced. By this is meant that if $\phi = \alpha\beta$ where α and β are of rank r, the first with m rows and r columns, the second with

r rows and m columns, then $\phi_1 = \beta\alpha$ is the transmute of ϕ, a notion connecting with previous results (vol. I, page 154) and useful in stating the theorem that two square matrices with constant coefficients are connected by an equimutant transformation when and only when they have a common first transmute. Transmutes are not unique, there being an infinity of first transmutes for the matrix ϕ, which, however, are all transformable into one another by equimutant transformations. In fact, the best way to arrive at a first transmute is to consider the matrix as a linear vector operator which annuls a region of dimensions equal to the nullity, and if for the region left κ is the identity matrix, then a first transmute of ϕ is $\kappa\phi\kappa$. Or in the array form, if we transform the matrix so that it has zero columns equal in number to the nullity, then by erasing the same number of rows at the bottom, producing a square matrix, we will have a first transmute. There is much material in this chapter which is related to the characteristic equation of a matrix, including the standard theorems.

Chapter XXVII deals with *commutants*. A commutant X is a matrix solution of the equation $AX = XB$ where A and B are square matrices. It is either a zero matrix or has a determinate structure. If A and B have only constant elements X is non-zero only if A and B have a latent root in common. When A and B are canonical square matrices with the same latent root, X is a general "ruled compound slope." The ideas advanced in connection with this notion have a direct connection with what the reviewer has previously called "*associative units*" (Transactions of this Society, vol. 4 (1903), 251–287). These play an important part in the linear associative algebras. There is a great deal of new material in the chapter which can scarcely be detailed here. It evidently is in preparation for the developments of the second part of volume three. Chapter XXVIII deals with commutants of commutants, which are commutants of every commutant of a given matrix A. They are proved to be the rational integral functions of A. Chapter XXIX deals with invariant transformands which are solutions of the equation $AXB = X$. When there are solutions which are non-zero they have a definite structure. It is evident that if B is invertible the invariant transformands go back into the commutant list.

The theory of matrices, in whatever notation it may be expressed, has a growing importance. From the study of finite matrices have followed many recent developments that involve the properties of infinite matrices. The study of quadratic forms on an infinity of variables, either denumerable or non-denumerable, the various developments of integral equations and of linear operators in general, and other related subjects, all lead back to the fundamental structure of matrices. These in turn are special cases of hypernumbers, and consequently we find that in hypernumbers we have as universal an algebra as we may at present study. The matrices are practically the associative hypernumbers, and special structures in these forms give the special algebras. The non-associative systems however are very much more numerous and very much more complicated. There has been up to the present time little use for such systems, perhaps due to the fact that their algebra has not been worked out, but this will be done in

the future. It is evident that from the study of such systems associativity, important as it is, is of trifling importance in the general problems of structure. It may be then that the day will come when the matrix will be of only very limited importance in the study of structural physics, and the non-associative hypernumbers will give us the keys to the universe.

It is nevertheless a matter for congratulation to Professor Cullis that he is carrying out such a heavy piece of work as the present development of matrices, and it is to be hoped that he may bring it to a successful conclusion.

J. B. Shaw

STENSTRÖM ON THE 27 LINES

Synthetische Untersuchungen des Systems von 27 Geraden einer Fläche dritter Ordnung. By Olof Stenström. Upsala, Appelbergs Boktryckeri Aktiebolag, 1925. 128 pp.

It was a matter of course, in the history of geometry, that the cubic surface should be much studied. The zest with which many have pursued this study was not a matter of course but was largely due to the elegant configuration of the 27 lines on the surface. Since their fortunate discovery by Cayley and Salmon in 1849, they have been essential to nearly all investigations concerning it—investigations in which conspicuous names are Schläfli, Clebsch, Sylvester, Cremona, Sturm, Klein, Reye, Zeuthen, Schur, Henderson, and Baker. No decade has, indeed, been without notable work on the surface of third order. Since 1920 the contributions of two men deserve especial mention. The first was E. Stenfors,* who worked on the projective transformations of a Schläfli double-six into itself, as well as on like transformations of the whole system of lines, and their groups. Olof Stenström followed in 1925 with the booklet which is the subject of this review.

If, contrary to custom, a formal Table of Contents is included in a review, it may perhaps be justified by the fact that the book itself contains none, although such a table would aid both those who wish to read it and those who wish to learn more quickly of its content.

Chapter		Page
	Introduction—Historical	1
I.	Types of involutions on the lines	13

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* *Die Schläflische Konfiguration von zwölf Geraden einer Fläche dritter Ordnung*, 1921. *Über die Geradenkonfiguration einer Fläche dritter Ordnung, bzw. Klasse*, 1922. Both memoirs are contained in Series A, vol. 18, of that valuable periodical which the Union List of Serials names Suomalaisen Tiedeakatemian Toimituksia, but which the Revue Semestrielle designates—equally correctly—as Annales Academiae Scientiarum Fennicae. One who searches for a volume of this rather scarce journal wishes that there were agreement on nomenclature.

Chapter		Page
II.	Types of double-six, and of corresponding Schur quadric	20
III.	Partial systems of lines (The heading is omitted, but should follow line 2.)	23
IV.	Correspondence of the configuration to the division of the 6 indices in two halves	26
V.	Determination of cubic surface by its line system	27
VI.	Separation of surface into regions by lines; mapping on a plane	29
VII.	Singular forms of line system, arising from coincidence of lines	37
VIII.	Systems in which, once or oftener, three lines are concurrent	56
IX.	Singularity of trilinearity determining the surface; regions of surfaces with singular systems of lines	58
X.	Partly imaginary line systems	63
XI.	Other lines through the intersections of the 27, their planes, etc.	83
XII.	Special forms of systems in which three lines are concurrent	90
XIII.	Cross ratios of quadruples of planes through a line	95

From the title two things are clear: that Stenström follows the synthetic method which had its first great impetus in the study of the cubic surface with Cremona's memoir of 1868*; and that his researches concern the lines primarily, not the surface. Nevertheless, the Table of Contents indicates, among its wealth of subjects, three chapters (V, VI, and IX) in which the surface is treated also.

Any one of the 27 lines meets ten others in pairs of an involution. In the case of twelve lines, which happen to constitute a double-six, the involution is elliptic; on the other lines it is hyperbolic. Here we are considering, of course, the surface whose lines are real and distinct. This fact, early recognized by Klein and Zeuthen, is the basis for Stenström's detailed analysis of the various figures formed by the lines; as a result the geometry, always alluringly intricate, becomes truly elaborate. Thus our author distinguishes three types of planes determined by lines of the system, four types of double-sixes, twelve of skew triples of lines, five of Steinerian triedral pairs, and so on. The dissection of the surface by means of each kind of double-six is displayed; in particular, it appears that all parabolic points lie within those ten triangles whose boundaries carry hyperbolic involutions.

The last chapter, devoted to the cross ratios of quadruples of planes through lines of the system, continues work begun by Kasner,† but brings

* *Mémoire de géométrie pure sur les surfaces du troisième ordre*, Journal für Mathematik, vol. 68.

† *The double-six configuration connected with the cubic surface and a related group of Cremona transformations*, American Journal of Mathematics, vol. 25 (1903).

results in a new and decidedly elegant form. Let us, for instance, consider the five planes of Type II—the type in which each of the three lines has a hyperbolic involution, with the intersecting lines outside the plane giving two pairs of points on each segment bounded by lines in the plane. Let the other four planes through each of the three lines have cross ratio λ_n; this will be the same for the three lines, but will differ from plane to plane. Then these five cross ratios satisfy the symmetric equation

$$(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)(1-\frac{1}{\lambda_1}-\frac{1}{\lambda_2}-\frac{1}{\lambda_3}-\frac{1}{\lambda_4}-\frac{1}{\lambda_5})+1$$
$$-\lambda_1-\lambda_2-\lambda_3-\lambda_4-\lambda_5=0.$$

A notable characteristic of Stenström's work is his ingenuity in notation, and in the use of diagrams. The capitalization of the usual a_n and b_n, in order to set off that double-six whose involutions are elliptic, was fairly obvious. Decidedly clever is the indication of degenerate double-sixes of Type IV, by means of the vertices and centers of two regular pentagons—in each pentagon six points for the six indices. This type of double-six

$$\begin{array}{cccccc} A_i & C_{np} & A_k & C_{mp} & A_l & C_{mn} \\ C_{kl} & B_m & C_{il} & B_n & C_{ik} & B_p \end{array}$$

obviously individuated by the indices i, k, l, "is represented by (one of) the ten triangles with an obtuse angle which can be formed from these points." The other three indices give an acute triangle in the other pentagon. 26 cases of degeneracy, caused by coincidence of conjugate lines in such double-sixes, are enumerated; and the triangles drawn in the pentagonal skeleton indicate at once the collapsed double-sixes, the coincident lines, the number and multiplicity of the lines and planes.

The pentagonal representation does duty only so long as the lines are real; admission of imaginary lines raises the number of types to 75. These 75 cubic surfaces are spread out on a family tree. From the common ancestor —the system of 27 real and distinct lines—one passes downward and to the right whenever lines come to coincidence, then upward, but still to the right, when they separate again and become imaginary. Thus the level on which the symbol for a particular system (or surface) is found shows at once the number of collapsed double-sixes.

All diagrams are decidedly neat, even if they lack the perfection of Henderson's plates* and the gaiety of Stenfors' black, red, and green pictures.

Proof-reading on Stenström's book was faulty. However, most of the errors were corrected in the copy supplied the reviewer, and it is to be hoped that these corrections will become part of the standard equipment of the book. One of them, indeed, is distressing (page 97, line 23), annulling a predicate and allowing the subject to survive as best it may.

In the excellent historical introduction we are puzzled by a reference to a "Probeartikel der Encyclopädie der Mathematischen Wissenschaften

* *The Twenty-Seven Lines upon the Cubic Surface*, Cambridge Tracts, No. 13.

von 1896"; for the publication of the encyclopedia began in 1898, the article on cubic surfaces has not yet appeared, and Stenström's "Literaturverzeichnis" gives no item dated 1896. There is confusion (page 22) in the discussion of Schur quadrics associated with various types of double-sixes. "Eine *nicht nullteilige* Fläche zweiter Ordnung teilt den Raum in zwei Bereiche. . . . . Liegen (die Geraden des Paares) auf derselben Seite (der Fläche), so ist sie *nullteilig*."

The configuration which Stenström (page 26) names "Sturmsche Doppelfünfseite" is not that which Sturm himself describes.* A double-five satisfying Sturm's specifications, but not Stenström's, is

$$\begin{matrix} C_{23} & C_{45} & C_{26} & C_{34} & C_{56} \\ C_{46} & C_{36} & C_{35} & C_{25} & C_{24}. \end{matrix}$$

The reverse is true of

$$\begin{matrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56}. \end{matrix}$$

On page 37 it is stated that "einander nicht schneidende Gerade bleiben einander nicht schneidende oder fallen ganz und gar zusammen." Yet when, in

$$\begin{matrix} A_1 & A_2 & A_3 & C_{56} & C_{64} & C_{45} \\ C_{23} & C_{31} & C_{12} & B_4 & B_5 & B_6, \end{matrix}$$

corresponding pairs come to coincidence, all lines become concurrent, and A_1 and A_2 intersect without coinciding. This section could be stated more carefully. And what is the meaning of the assertion (page 63) that a double-six whose lines have become imaginary is "im ganzen betrachtet immer noch reell"?

Stenström devotes some attention to the Sylvester pentahedron, whose vertices are the double points of the cubic surface's Hessian; and he hints that in this matter he plans further investigation, which is to develop the relation between the system of lines and the pentahedron. It is to be hoped that we shall learn of the success of this attempt; for the present book establishes its author's ability very substantially to extend our knowledge in this field of geometry.

E. S. ALLEN

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* *Die 27 Geraden der cubischen Fläche*, Mathematische Annalen, vol. 23 (1884), p. 296.

SHORTER NOTICES

A Course of Differential Geometry. By J. E. Campbell. Oxford, Oxford University Press, 1926. xv+261 pp.

The manuscript for this book was completed by Professor Campbell shortly before his death in 1924. It was prepared for the press by Professor E. B. Elliott. The book was evidently planned to prepare the reader to understand differential geometry as applied in modern relativity theory. The methods of tensor analysis have been employed throughout.

The first chapter gives a brief but quite satisfactory introduction to the tensor theory associated with a symmetric quadratic differential form with n independent variables. The Christoffel symbols of three and four indices are defined and discussed and the operation of covariant differentiation is introduced without being so called. At the end of this chapter an Einstein space is defined as one in which certain tensor components vanish but no explanation of the definition is given at this point.

The next nine chapters contain a treatment of differential geometry of ordinary space, tensor methods being used almost exclusively. Particular attention is given to the following subjects: equivalence of forms, geodesics, curvature deformation of surfaces, congruences, curves in space and on a surface, ruled surfaces, minimal surfaces, conformal representation, orthogonal surfaces.

In order to understand these nine chapters the reader must have previously acquired the main facts of differential geometry. Little explanation outside of the analysis is offered and the geometrical concepts are usually introduced without definition. The purposes of this part of the book undoubtedly are to show the power of vector methods and to prepare the way for extensions to hyperspace.

In the last four chapters we return to the consideration of the quadratic form with n independent variables and are thus led to geometry in n-way space. As a generalization of the straight line in the plane we have the geodesic. Gauss's measure of curvature has as a natural extension Riemann's measure of curvature, in which a tangential n-fold takes the place of the tangent plane.

The fundamental form for an $(n+1)$-way space may be taken as

$$\phi du^2 + b_{ik}\, dx_i\, dx_k \qquad (i, k = 1, \cdots, n),$$

where ϕ, b_{ik} are functions of $x_1, \cdots, x_n$ and u. The surface $u=0$ is an arbitrary surface in the space. This $(n+1)$-way space is an Einstein space if for all such surfaces lying in it

$$\sum \frac{1}{R_i R_k} + \frac{1}{2} A = 0,$$

where the first term represents the sum of the products, taken two at a time, of the reciprocals of the principal radii of curvature of the surface, and where

A is a certain invariant dependent on the Christoffel symbols of four indices for the n-way space of the surface. For $n=2$ the above formula reduces immediately to Gauss's well known formula for the product of the reciprocals of the principal radii of curvature of a surface. The above definition is easily seen to be equivalent to that stated in the first chapter.

A given n-way space is always surrounded by an Einstein $(n+1)$-way space whose generation can be accomplished by infinitesimal methods. The Einstein space which surrounds the n-way space $u=0$ is stationary if ϕ and b_{ik} are independent of u.

The last two chapters of the book are given to the discussion of n-way space of constant Riemannian curvature and of n-way space as a locus in $(n+1)$-way space.

Professor Campbell planned to add an appendix treating of the relation of the contents of the book to the physics of Einstein. Such an appendix would undoubtedly have added greatly to the understanding of the physical meanings of some of the mathematical results.

There is a complete absence of references in the book. Important results are not stated in the form of theorems or made to stand out on the page in any way.

E. B. Stouffer

Les Fondements des Mathématiques. By F. Gonseth. Paris, Blanchard, 1926.

This book, which commences with a preface by J. Hadamard, deals with the foundations of geometry and kinematics and the relation between mathematics and logic.

In the first few chapters a discussion is given of various sets of axioms for euclidean space, the axioms of Geiger being compared with those of Hilbert.

Following a chapter on the continuum there is next a short chapter on the compatibility and independence of the axioms of a system. The author then passes on to the construction of continua in which he makes use of Jordan curves, topological transformations and groups of movements. His program is expressed as follows: Topology should be constructed with the aid only of the axioms of order and continuity.

Chapter VI is devoted to non-euclidean geometry and the next chapter to the relation between theory and experience. The discussion of time and relativity is enlivened by a short dialogue in which Bergson, Metz, LeRoux, Fabre, and the author are supposed to take part.

There is next a chapter on the idea of motion and general relativity and the book closes with a chapter on mathematics and logic in which special attention is devoted to the vicious circle of Weyl, Brouwer's remarks on the principle of the excluded third, Weyl's formulation of the continuum and Hilbert's logical foundations of mathematics. There is also a short article on formal logic.

H. Bateman

Oeuvres complètes de Christiaan Huygens publiées par la Société Hollandaise des Sciences. Tome Quinzième. Observations Astronomiques. Système de Saturne. Travaux Astronomiques, 1658–1666. La Haye, Martinus Nijhoff, 1925. Pp. 622.

This fifteenth volume of the works of Huygens is a model of what can be done for the convenience of readers. A general table of contents (sommaire) containing twelve entries is supplemented by a fuller one (pièces et mémoires) covering eight pages. There are alphabetical indexes to persons and to subject matter. Numerous footnotes contain explanations and cross references. The longer documents appear in the original Latin and in French translation. Facsimile reproductions of parts of manuscripts and original drawings add to the interest of the book. Introductions to the various parts contain information in great detail. Among the mathematical notations of interest are the Cartesian sign of equality (the sign for taurus turned so that the opening is on the left) and the "scratch method" used in extracting square roots.

Huygens' early astronomical observations relate mainly to Saturn, Jupiter and their companions. Huygens and his brother Constantine constructed their own telescopes. The first one, 12 feet in length, promptly revealed a new satellite of Saturn. This instrument and a 23 foot telescope furnished data to solve the enigma relating to the varying appearance of Saturn. Galileo had seen it threefold. Hevelius saw it in three parts. Fontana made drawings widely distorted. Huygens in 1655 drew Saturn as a sphere with two handles or arms on opposite sides and extending in opposite directions. When Saturn was near the sun, in January, 1656, the arms disappeared. Nevertheless, Huygens had a hypothesis which he published in 1656, disguised as an anagram: aaaaaaaccccccdeeeeeghiii

50 pages each and the last two 80 pages each. To this is added an appendix and index of 40 pages.

The first five chapters include the fundamental elementary topics which are studied in a first course in analytic geometry. These are developed in the familiar non-homogeneous analytical language but from a broader and more general point of view than the method of the ordinary text book. After explaining various coordinate systems the equations of the conics are derived in both the cartesian and polar systems together with the parametric equations of the circle and ellipse. The distance of a point from a line is found by means of projections. This relation and the transformation of coordinates precede the chapter on the straight line in which the importance of the Hesse normal form is shown and which includes a good treatment of the theory of two lines and of three lines.

In Chapters VI–VIII inclusive the fundamental framework of modern analytic geometry is constructed. The projective method predominates as this gives the most natural approach to homogeneous coordinates. The author develops this part in logical order and with great clearness. It includes line coordinates, duality. the complete 4-point and 4-line, the theorems of Desargues (perspective triangles) and Pascal, double ratio, and the theory of harmonic pairs. Linear substitution is shown to be a projective relation; quadratic involutions are discussed; and the conic is obtained as a locus or envelope of corresponding projective forms. Homogeneous coordinates are developed first from cartesian point and line coordinates, then linear projective coordinates are introduced. Finally the general plane homogeneous coordinates are defined by means of a triangle of reference, and the ratios of these coordinates are shown to be given by means of a double ratio as soon as the unit point is fixed.

Chapters IX–XII inclusive deal principally with a more extensive treatment of the conics, the general equation of the second degree, and collineations in the plane. Orthogonal pencils of circles, circular points and minimal lines, and transformation by reciprocal radii are included in the chapter on the circle. In the following chapters are discussed confocal conics, conjugate diameters, invariants, polar systems and involutions, the pencil of conics, collinear and reciprocal transformations, and the theorems of Pascal and Brianchon.

The last five chapters are devoted to the geometry of three dimensions following a procedure analogous to that used in the plane. Coordinate systems, dual elements and loci, collineations in space, a discussion of the quadric surfaces both with non-homogeneous and with the general homogeneous coordinates, and the polar system are the principal subjects considered. Useful related algebraic topics are given a supplementary treatment in the appendix, including determinants, linear equations, and transformations. The book ends with a selected list of 115 exercises and problems.

After introducing the homogeneous coordinate systems it seems unfortunate that the author did not make more extensive use of them. Had the equation of the circle, for instance, been given with an inscribed triangle of reference it would have opened to the student a most attractive

method for investigating the geometry of the circle and triangle. However, the arrangement of the material is logical and successive topics are developed in a masterly fashion. It should be kept in mind that the author has attempted to fulfill not only the formal requirements of the Wissenschaften but also the demands of the student in geometry both as to the selection and arrangement of material. In this sense the book is a compromise; to quote from the preface "ein einführendes Lehrbuch soll in erster Linie ein Lernbuch sein." A few minor errors in type were observed but these do not detract from the merits of the book. It is a welcome addition to our libraries and will be a valuable reference for students in their first course in modern analytic geometry.

J. I. Tracey

La Logique des Mathématiques. By Stanislas Zaremba. Mémorial des Sciences Mathématiques, No. 15. Paris, Gauthier-Villars, 1926. 52 pp.

In this number of the Mémorial des Sciences Mathématiques, Professor Zaremba, of the University of Cracow, views logic as a "theory of deductive demonstration" and sketches a "logistic" by means of which a "complete" mathematical demonstration can be effected. In the course of his exposition, in which he "carefully refrains from engaging in psychologic and philosophic speculations," the author formulates fundamental problems in the logic of mathematics still awaiting solution, and gives an explanation of the paradoxes in aggregate theory based on a distinction between the notions *class* and *category*. The text is followed by a good-sized bibliography.

The author excludes from his logic the logic of classes and the logic of relations—theories that "really should not enter into the domain of a general theory of demonstration." His logic is a logic of propositions consisting (1) of a theory of "propositional identities" and (2) of a theory of propositions of logic of the "second kind." The first of these seems to be the theory of "elementary" propositions of Whitehead and Russell's *Principia Mathematica*, whose "theory of deduction" is "the most complete exposition of the theory of propositional identities." The second theory is presumably that of "apparent variables" of the *Principia*. The author, however, does not accept Russel's "theory of types," which has brought into logic "numerous obscurities and extreme complications."

B. A. Bernstein

Esquisse d'Ensemble de la Nomographie. (Mémorial des Sciences Mathématiques, No. 4.) By Maurice d'Ocagne. Paris, Gauthier-Villars, 1925. 68 pp.

This is one of the little volumes on interesting mathematical subjects now being published under the direction of Professor Henri Villat as Mémorial des Sciences Mathématiques. It covers in four chapters of twenty-eight sections most of what is known of nomographic theory. To the advanced student the treatment offers interest and charm for the presentation has authority, brevity and elegance. The engineer or physi-

cist seeking help in the actual construction of nomograms will be disappointed, for there are no applications. The reader recognizes at page 39 that the problem of determining under what conditions the equation in three variables $f(z_1,z_2,z_3)=0$ shall have the determinant form

$$\Delta_{123} = |f_i g_i h_i| = 0, \qquad (i = 1, 2, 3),$$

is fundamental and is referred to the literature for the solution. The generalized equation in six variables with corresponding nomograms consisting of three curve nets is also briefly considered. Chapter IV is devoted to nomograms with movable inscribed elements and affords suggestions for the study of nomograms depending on geometric invariants under displacements in the plane. There is a useful bibilography of thirty sources and the printing is almost without error. On page 49 there should be a factor 3 before the parenthesis in the fifth line from the bottom. On page 39, j and k should be deleted from the parenthesis line 14 from the bottom, and v should replace ν in equations 3), 4), and 5), page 30.

L. I. Hewes

Théorie Générale des Séries de Dirichlet. (Mémorial des Sciences Mathématiques, No. 17.) By G. Valiron. Paris, Gauthier-Villars, 1926. 56 pp.

The aim of the series to which this work belongs is to present in brief and compact form the most important results and outlines of methods in various mathematical subjects of current interest. This particular number on Dirichlet series gives an excellent survey of this interesting field which is now growing so rapidly.

One naturally thinks of comparing this book with the only other work devoted exclusively to this subject, namely *The General Theory of Dirichlet's Series* by Hardy and Riesz (Cambridge Tracts in Mathematics and Mathematical Physics, No. 18, published in 1915). They are somewhat similar in general plan and style. That there was a real need for a new book on the subject is seen from the fact that of the bibliography of nearly 150 references given by Valiron, over forty per cent were published since 1915 when the Hardy and Riesz work was issued.

A curious mistake is made on pages 31–32, in a brief discussion of the Hölder and Cesàro methods of summation of divergent series, where the names of these two methods are interchanged.

To those already interested in the subject and to those who wish know something of the results and methods without devoting too much time to it, this compact summary should prove very valuable.

L. L. Smail

NOTES

The April, 1927, number of the Annals of Mathematics ((2), vol. 28, No. 2) contains the following papers: *Some relations between a continuous curve and its subsets*, by H. M. Gehman; *Transformations of one principal equation into another*, by R. Garver; *On related maxima and minima*, by N. Miller; *On Fredholm's integral equations, whose kernels are analytic in a parameter*, by J. D. Tamarkin; *On the existence of the absolute minimum in space problems of the calculus of variations*, by L. M. Graves; *The mixed mean function*, by C. A. Shook; *The focal point for the problem of Lagrange with one variable end point*, by L. H. McFarlan; *Differential invariants of affinely connected manifolds*, by T. Y. Thomas and A. D. Michal; *Linear congruences in a general arithmetic*, by H. L. Olson; *Note on a differential equation*, by J. M. Thomas; *On the class of a covariant tensor*, by J. W. Alexander.

The Deutsche Mathematiker-Vereinigung, the Gesellschaft für angewandte Mathematik und Mechanik, the Deutsche Physikalische Gesellschaft, and the Gesellschaft für Technische Physik will meet together at Kissingen in September, 1927.

The International Geodetic and Geophysical Union will meet at Prague, September 3–10, 1927. Dr. William Bowie and Mr. W. D. Lambert, of the Coast and Geodetic Survey, have been appointed as delegates of the United States by the president of the National Academy of Sciences and the chairman of the National Research Council.

Professor Edmund B. Wilson has been elected president of the American Academy of Arts and Sciences.

The Reale Accademia dei Lincei has conferred its royal prize in mathematics on Professor Leonida Tonelli of the University of Bologna. One of the prizes founded by the (Italian) Ministry of Public Instruction has been awarded to Professor Enea Bortolotti, of the Reale Liceo Artistico di Bologna.

The Belgian Accademy announces the following subject for its prize in mathematics in 1928: *A contribution to the theory of quasi-analytic functions*. Competing memoirs should reach the Academy by August 1, 1928.

Dr. L. Prandtl, professor of applied mathematics at the University of Göttingen, has been awarded the gold medal of the Royal Aeronautical Society, in recognition of his work in aerodynamics.

The Franklin Institute has awarded Franklin Medals and certificates of honorary membership to Professor Max Planck, of the University of Berlin, and to Dr. G. E. Hale, honorary director of the Mount Wilson Observatory.

The University of Münster has conferred an honorary doctorate on Johann Georg Hagen, director of the Vatican Observatory, on the occasion of his eightieth birthday.

Professors A. S. Eddington, of Cambridge University, and G. H. Hardy, of Oxford University, have been elected corresponding members of the Bavarian Academy of Sciences.

Professors A. Einstein, A. A. Michelson, G. Mittag-Leffler, and W. Nernst, and Madame Curie have been elected honorary members of the Russian Academy of Sciences.

Awards from the Heckscher Fund at Cornell University have been made to Professor J. I. Hutchinson, for the salary of an assistant to spend part of his time on the study of properties of functions defined by certain Dirichlet series, and to Professor Arthur Ranum, for an assistant to spend part of his time on the study of the principles of duality in the differential geometry of surfaces and curves.

Dartmouth College has conferred honorary doctorates on President Max Mason, of the University of Chicago, Professor F. P. Brackett, of Pomona College, and Professor D. C. Miller, of the Case School of Applied Science.

The University of Wisconsin has conferred the honorary degree of doctor of science on Professor G. D. Birkhoff, of Harvard University.

Princeton University has conferred the honorary degree of doctor of science on Professor A. A. Michelson, of the University of Chicago.

Professor E. H. Moore, of the University of Chicago, has received the honorary degree of doctor of science from the University of Kansas and from Northwestern University.

President G. D. Olds, of Amherst College, has received an honorary doctorate of laws from Wesleyan University.

Brown University and the University of Rochester have conferred honorary doctorates on Professor M. I. Pupin, of Columbia University.

Yale University has conferred the honorary degree of doctor of science on Professor A. N. Whitehead, of Harvard University.

The University of Michigan has conferred the degree of doctor of science on Professor emeritus Alexander Ziwet, of that University.

Professor Svante Arrhenius, director of the Nobel Institute for Physical Chemistry, has retired.

Dr. Wolfgang Krull, of the University of Freiburg i. Br., has been promoted to an associate professorship of mathematics.

Dr. Josef Lense, of the University of Vienna, has been appointed associate professor of applied mathematics at the Munich Technical School.

Professor Arnold Sommerfeld, of the University of Munich, has been called to the chair of theoretical physics at the University of Berlin, as successor to Professor Max Planck, retired.

Professor A. Lafay, of the Ecole Polytechnique, has been appointed maître de conférences in mathematics at the Institut National Agro nomique.

Professor H. Villat, of the University of Strasbourg, has been appointed to the newly established chair of the mechanics of fluids at the Sorbonne.

Mr. A. S. Besicovitch, of the University of Leningrad, has been appointed university lecturer in mathematics at Cambridge University for a term of three years.

A fellowship in mathematics of $1,000 annually has recently been established at Brown University by Mr. H. D. Sharpe.

Assistant Professor L. C. Bagby, of the University of South Dakota, has been appointed professor of mathematics at the Linsly Institute of Technology.

Professor L. S. Dederick, of St. Stephen's College, has been appointed mathematician at the Aberdeen Proving Ground.

Assistant Professor W. W. Elliott has been promoted to a professorship of mathematics at Duke University.

Dr. P. A. Fraleigh has been appointed assistant professor of mathematics at the University of Vermont.

Assistant Professor G. I. Gavett has been promoted to an associate professorship of mathematics at the University of Washington.

Assistant Professor Howard Justice has been appointed associate professor at the University of Cincinnati.

Associate Professors J. H. Kindle and Edward S. Smith have been promoted to professorships of mathematics at the University of Cincinnati.

Professor P. H. Linehan, of the College of the City of New York, has been made Director of the Evening Session and of the Division of Vocational Studies.

Miss Muriel Metz has been promoted to an assistant professorship of mathematics at the University of Cincinnati.

Mr. T. H. Milne has been appointed assistant professor of mathematics at the University of Manitoba.

Dr. H. H. Neelley, of Yale University, has been appointed associate professor of mathematics at the Carnegie Institute of Technology.

Dr. C. A. Nelson, of Johns Hopkins University, has accepted a position at Rutgers University.

Dr. B. P. Reinsch, of the University of Illinois, has been appointed associate professor of mathematics at the Southern Methodist University.

Assistant Professor E. E. Whitford, of the College of the City of New York, has been promoted to an associate professorship of mathematics.

Dr. H. A. Zinszer, of the Mississippi State College for Women, has been appointed professor of mathematics at Hanover College.

The following appointments as instructor in mathematics are announced:

Columbia University, Dr. R. G. Archibald (associate), Mr. Joseph Feld, Dr. B. O. Koopman, Mr. H. W. Raudenbush, Dr. P. A. Smith, Mr. G. C. Vedova;

Kenyon College, Mr. Benedict Williams;

Northwestern University, Mr. H. L. Garabedian;

Yale University, Mr. T. C. Benton.

Dr. Wilhelm Ahrens, author of books on mathematical recreations, died at Rostock, April 23, 1927.

Dr. Felix Auerbach, professor of theoretical physics at the University of Jena, is dead.

Dr. Ludwig Burmester, formerly professor of mathematics at the Munich Technical School, is dead.

Dr. Viktor Eberhard, associate professor of mathematics at the University of Halle, is dead.

Dr. Ludwig Maurer, formerly professor of mathematics at the University of Tübingen, died January 10, 1927.

Dr. F. Thieme died March 9, 1926, at the age of seventy-three.

Dr. Anton Gmeiner, professor of mathematics at the University of Innsbruck, died January 11, 1927.

Dr. Ludwig Grossmann, of Vienna, is dead.

Dr. Alois Walter, professor of mathematics at the Montanistische Hochschule at Leoben, is dead.

Professor Giuseppe Bagnera, of the University of Rome, died May 13, 1927.

Mr. G. L. Cathcart, fellow of Trinity College, Dublin, and editor of Salmon's mathematical works, died March 26, 1927.

Professor W. G. Bullard, of Syracuse University, died February 16, 1927. Professor Bullard had been a member of the American Mathematical Society since 1899.

Professor W. W. Johnson, of the United States Naval Academy, died May 14, 1927, at the age of eighty-three. Professor Johnson had been a member of the American Mathematical Society since 1891.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

Betsch (C.). Fiktionen in der Mathematik. Stuttgart, Fr. Frommans Verlag, 1926. 24+372 pp.

Biswas (R. C.). The theory of equations and the complex variable. Calcutta, Chuckervertty, Chatterjee and Company, 1926. 8+269 pp.

Bowman (F.). Elementary algebra. Part II. New York and London, Longmans, 1927. 431 pp.

Brodetsky (S.). Sir Isaac Newton. A brief account of his life and work. London, Methuen, 1927. 12+161 pp.

Brown (J. T.) and Martin (A.). The elements of plane trigonometry. London, Harrap, 1927. 200 pp.

Burkamp (W.). Begriff und Beziehung. Leipzig, Meiner, 1927.

Charlier (C. V. L.). Naturwetenskapens matematiska principer av Isaac Newton. Lund, Gleerups Förlag, 1927.

Czaikowskyj (N.). Algebra. Part II. (In Ukrainian.) Prague, 1926.

Dickson (L. E.). Algebren und ihre Zahlentheorie. Mit einem Kapitel über Idealtheorie von A. Speiser. Zurich, Füssli, 1927. 307 pp.

Dombrowski (M. C. M.). Canon geometricus oder explizite definitorische Grundlagen der Geometrie. 2te neubearbeitete Auflage. Minsk, Verlag "Kultura," 1926. 8 pp.

Durell (C. V.). Mathematical tables (four figures). London, Bell, 1927. 40 pp.

Engel (F.). See Lie (S.).

Fladt (K.). Gewöhnliche Differentialgleichungen. (Mathematische Physikalische Bibliothek.) Leipzig, Teubner, 1927. 67 pp.

Forder (H. G.). The foundations of euclidean geometry. Cambridge, University Press, 1927. 12+349 pp.

Gheury de Bray (R.). See Reymond (A.).

Gibbs (R. W. M.) and Richards (G. E.). Mathematical tables. 2d edition, revised. London, Christophers, 1926. 16 pp.

Janet (M.). Les systèmes d'équations aux dérivées partielles. (Mémorial des Sciences Mathématiques, No. 21.) Paris, Gauthier-Villars, 1927.

Keyser (C. J.). Thinking about thinking. New York, Dutton, 1926. 91 pp.

Kleppisch (K.). Willkür oder mathematische Überlegung beim Bau der Cheops pyramide? München, Oldenbourg, 1927. 38 pp.

Landau (E.). Vorlesungen über Zahlentheorie. 3 Bände. 12+360+8+308+8+341 pp.

Lie (S.). Gesammelte Abhandlungen. 6ter Band: Abhandlungen über die Theorie der Transformationsgruppen, 2te Abteilung. Anmerkungen zum sechsten Band. Herausgegeben von F. Engel. Leipzig, Teubner, und Oslo, Aschehong, 1927. 24+752+186 pp.

Lodge (O.). Modern scientific ideas, especially the idea of discontinuity. London, Benn, 1927. 79 pp.

Losskij (N. O.). Handbuch der Logik. Autorisierte Übersetzung nach der 2te, verbesserte und vermehrte Auflage, von W. Sesemann. Berlin, Teubner, 1927.

Martin (A.). See Brown (J. T.).

Mullins (G. W.) and Smith (D. E.). Freshman mathematics. Boston, Ginn, 1927. 6+386 pp.

Pascal (E.). Repertorium der höheren Mathematik. 2te, völlig umgearbeitete Auflage herausgegeben von E. Salkowski und H. E. Timerding. 1ter Band: Analysis. 2ter Teilband. Leipzig, Teubner, 1927. 12+494 pp.

Patterson (C. H.). Problems in logic. New York, Macmillan, 1926. 12+331 pp.

Reymond (A.). History of the sciences in Greco-Roman antiquity. Translated by R. Gheury de Bray. London, Methuen, 1927. 10+245 pp.

Richards (G. E.). See Gibbs (R. W. M.).

Rothstock (W.). See Russell (B.).

Russell (B.). Unser Wissen von der Aussenwelt. Uebersetzt von W. Rothstock. Leipzig, Meiner, 1926. 331 pp.

Russell (B.). See Whitehead (A. N).

Salkowski (E.). See Pascal (E.).

Saltykow (N. N.). Nouvelles leçons sur la théorie des équations aux dérivées partielles. Belgrade, 1926. 4+214 pp.

Scarpitti (P.). Quadrature du cercle. Buenos Aires, 1927. 18 pp.

Seitz (B.). Sur l'arithnomie des nombres de Weierstrass généralisés et de quelques systémes de polytettarions complexes. Lausanne, Imprimerie de la Concorde, 1926.

Sesemann (W.). See Losskij (N. O.).

Simonart (F.). Leçons d'algèbre supérieure. Louvain, Librairie Universitaire, 1927. 298 pp.

Smith (D. E.). See Mullins (G. W.).

Speiser (A.). See Dickson (L. E.).

Strassmann (R.). Zur Theorie der π-adischen Zahlen. Das Gauss'sche Fundamentaltheorem und die in einem π-adischen Körper enthaltenen Einheitswurzeln. Göttingen, Dieterichsche Universitäts-Buchdruckerei, 1926. 124 pp.

Stuyvaert (M.). La bosse des mathématiques. Gand, Van Rysselberghe et Rombaut, 1927. 124 pp.

Thompson (A. J.). Logarithmica Britannica. Part 8: Numbers 80,000 to 90,000. Cambridge, University Press, 1927. 5+114 pp.

Timerding (H. E.). See Pascal (E.).

Tummers (J. H.). Het nut van het wijsgeerig denken in wis-en natuurkunde. Nijmegun, Dekker en van der Vegt, 1926. 23 pp.

Weinrich (H.). Die sokratische Methode und wir Mathematiker. Berlin, Salle, 1926. 50 pp.

Wenner (F.). Praktische Rechenbildkunde (Nomographie). Aachen, Aachener Verlag, 1926. 6+78 pp.

Whitehead (A. N.) and Russell (B.). Principia mathematica. 2d edition. Volume 3. Cambridge, University Press, 1927. 8+491 pp.

Wirtinger (W.). Allgemeine Infinitesimalgeometrie und Erfahrung. Leipzig, Teubner, 1927. 32 pp.

PART II. APPLIED MATHEMATICS

Alexander (T.) and Thomson (A.W.). The scientific design of masonry arches. London, Macmillan, 1927. 4+36 pp.

Amaldi (U.). See Levi-Civita (T.).

Anglas (J.). D'Euclide à Einstein. Relativité et connaissance. Paris, Librairie Stock, 1926. 121 pp.

Auerbach (F.). Die Grundbegriffe der modernen Naturlehre. Leipzig, Teubner, 1926. 128 pp.

Bennett (T. A.). See Youngson (P.).

Bisconcini (G.). Esercizi e complementi di meccanica razionale. Milano, Libreria Editrice Politecnica, 1927. 480 pp.

Boehm (F.). Versicherungsmathematik. II: Lebensversicherungsmathematik. Einführung in die technischen Grundlagen der Sozialversicherung. (Sammlung Göschen.) Berlin, de Gruyter, 1926.

Bohle (H.). Electrical photometry and illumination. A treatise on light and its distribution. 2d edition, revised and enlarged. London, Griffin, 1925. 16+427 pp.

Bouasse (H.). Emission, chaleur solaire, éclairage, théorie de l'émission. Paris, Delagrave, 1926. 500 pp.

Boulanger (A.). Les principes de la mécanique des ressorts. (Leçons choisies de mécanique.) Paris, Gauthier-Villars, 1927. 86 pp.

Boutaric (A.). Thermodynamique et chimie, d'après la 2e édition de l'ouvrage de P. Duhem. Paris, Hermann, 1926. 6+650 pp.

Bricard (R.). Leçons de cinématique. Tome II: Cinématique appliquée. Paris, Gauthier-Villars, 1927. 8+352 pp.

de Broglie (L.). Ondes et mouvements. Paris, Gauthier-Villars, 1926. 6+133 pp.

Brown (F. G. W.). Higher mathematics for students of engineering and science. London, Macmillan, 1926. 12+488 pp.

Bruhat (G.). Cours de thermodynamique. Paris, Masson, 1926. 408 pp.

Bryan (W. L.). See Carmichael (R. D.).

Carboni (S. O.). Assicurazioni su la vita umana. Roma, Athenaeum, 1926.

Carmichael (R. D.), Davis (H. T.), MacMillan (W. D.), and Hufford (M. E.). A debate on the theory of relativity. With an introduction by W. L. Bryan. Chicago, Open Court, 1927. 8+154 pp.

Couffon (R.). Transport de l'électricité. Paris, Armand Colin, 1926. 219 pp.

Crussard (L.). Ventilateurs et compresseurs. Paris, Baillière et Fils, 1926. 414 pp.

Dalton (O. M.). The Byzantine astrolabe at Brescia. London, Oxford University Press, 1926. 14 pp. +3 plates.

Darrow (K. K.). Introduction to contemporary physics. New York, Van Nostrand, 1926. 26+453 pp.

Davis (H. T.). See Carmichael (R. D.).

Digges (L.). See Digges (T.).

Digges (T.). The theodelitus and topographical instrument of Leonard Digges of University College, Oxford. Described by his son Thomas Digges in 1571. (Old Asmolean Reprints, 4.) Oxford, R. T. Gunther, 1927. 20+59 pp.

Duhem (P.). See Boutaric (A.).

Eleutheropulos (A.). Die exakten Grundlagen der Naturphilosophie. Stuttgart, Enke, 1926. 8+116 pp.

Eucken (A.). See Müller (J.).

Geckeler (J.). Ueber die Festigkeit achsensymmetrischer Schalen. Berlin Verein Deutscher Ingenieure, 1926. 52 pp.

Gehrcke (E.), herausgegebem von. Handbuch der physikalischen Optik. 1ter Band, 1te Hälfte, und 2ter Band, 1te Hälfte. Leipzig, Barth, 1926–1927. 470+8+418+5 pp.

Gianfranceschi (G.). La fisica dei corpuscoli. 3a edizione. Roma, Università Gregoriana, 1926. 8+267 pp.

Goodman (J.). Mechanics applied to engineering. 9th edition, revised. London, Longmans, 1926.

Graetz (L.), herausgegeben von. Handbuch der Elektrizität und des Magnetismus. Band 5, Teil 1. Leipzig, Barth, 1926. 5+262 pp.

Grüning (M.). Die Tragfähigkeit statisch unbestimmter Tragwerke aus Stahl bei beliebig häufig wiederholter Belastung. Berlin, Springer, 1926.

Hamm (A.). Hochfrequenzverstärker. (Bibliothek des Radio-Amateurs.) Berlin, Springer, 1926. 133 pp.

Hoppe (E.). Geschichte der Physik. Braunschweig, Vieweg, 1926. 8+536 pp.

Hufford (M. E.). See Carmichael (R. D).

Johnson (J. B.), and others. The theory and practice of modern framed structures. 10th edition, partly rewritten. Part I. London, Chapman and Hall, 1927.

Klutz (T.). Calcul graphique des poutres à section constante. Paris, La Construction de Ciment Armé, 1926. 53 pp.

Kraitchik (M.). Théorie mathématique des opérations financières. 2e édition, revue, corrigée et augmentée. Paris, Gauthier-Villars, 1927.

Krey (H.). Die Flutwelle in Flussmündungen und Meeresbuchten. Berlin, Versuchsanstalt für Wasserbau und Schiffbau, 1926. 59 pp.

Langlois (R.). Les machines asynchrones à champ tournant, à gabues, et à collecteurs. Paris, Dunod, 1926. 16+268 pp.

Lebert (E.). Interprétation géométrique et physique de la théorie d'Einstein. Vannes, Lafolye Frères, 1926.

Levi-Civita (T.) e Amaldi (U.). Lezioni de meccanica razionale. Volume 2, parte 2. Bologna, Zanichelli, 1927. 10+685 pp.

Liwschitz (M.). Die elektrischen Maschinen. Leipzig, Teubner, 1926. 8+337 pp.

Loeb (L. B.). Kinetic theory of gases. New York, McGraw-Hill, 1927. 555 pp.

LOMBARDI (L.). Corso teorico-pratico di elettrotecnica. Volume 1. Milano, Vallardi, 1926. 12+642 pp.

LORENTZ (H. A.). Lectures on theoretical physics delivered at the University of Leiden. Authorized translation by L. Silberstein and A. P. H. Trivelli. Volume 1. London, Macmillan, 1927. 11+195 pp.

LORENZ (H.). Lehrbuch der technischen Physik. 2te, vollständig neubearbeitete Auflage. Band 1, Teil 2: Mechanik räumlicher Gebilde. Berlin, Springer, 1926. 302 pp.

MACLEAN (J.). Graphs and statistics. A suggestion for a finishing course in mathematics. Bombay, Ramchandra Govind, 1926. 14+200 pp.

MACMILLAN (W. D.). See CARMICHAEL (R. D.).

MACPHERSON (H.). Modern astronomy. Its rise and progress. London, Oxford University Press, 1926. 196 pp.

MARCOTTE (E.). Les moteurs à explosion. Paris, Colin, 1926. 216 pp.

MAYER (M.). Die Sicherheit der Bauwerke und ihre Berechnung nach Grenzkräften anstatt nach zulässigen Spannungen. Berlin, Springer, 1926. 4+66 pp.

MAYOR (B.). Introduction à la statique graphique des systèmes de l'éspace. Lausanne, Librairie Payot, 1926. 78 pp.

VON MISES (R.). Fluglehre. Vorträge über Theorie und Berechnung der Flugzeuge in elementarer Darstellung. 3te, stark erweiterte Auflage. Berlin, Springer, 1926. 6+321 pp.

MÜLLER (J.) und POUILLET (C. S. M.). Lehrbuch der Physik. 11te Ausgabe. 3ter Band, 1te Hälfte: Physikalische, chemische, und technische Thermodynamik. Herausgegeben von A. Eucken. Braunschweig, Vieweg, 1926. 18+1185 pp.

OGG (A.). See PLANCK (M.).

OSTENFELD (A.). Die Deformationsmethode. Berlin, Springer, 1926. 5+118 pp.

PLANCK (M.). Treatise on thermodynamics. Translated with the author's sanction by A. Ogg. 3d edition from the 7th German edition. London, Longmans, 1927. 14+297 pp.

POUILLET (C. S. M.). See MÜLLER (J.).

RIETZ (H. L.). Mathematical statistics. (Carus Monographs, No. 3.) Chicago, Open Court, 1927. 12+181 pp.

ROOT (R. E.). The mathematics of engineering. Baltimore, Williams and Wilkins, 1927. 550 pp.

SAMTER (M.). Die technische Mechanik. Band 1. Charlottenburg, Kiepert, 1926. 6+184 pp.

SCHILLING (W.). Statik der Bodenkonstruktion der Schiffe. Berlin, Springer, 1925. 6+185 pp.

SCHLEICHER (F.). Kreisplatten auf elastischer Unterlage. Berlin, Springer, 1926. 10+147 pp.

SEE (T. J. J.). New dynamical wave theory of the tides. Washington, Government Printing Office, 1926.

SILBERSTEIN (L.). See LORENTZ (H. A.).

SLICHTER (W. I.). Principles underlying the design of electrical machinery. London, Chapman and Hall, 1926.

Staus (A.). Der Genauigkeitsgrad von Flügelmessungen bei Wasserkraftanlangen. Berlin, Springer, 1926.

Thomson (A. W.). See Alexander (T.).

Trivelli (A. P. H.). See Lorentz (H. A.).

Turner (D. M.). Makers of science: electricity and magnetism. London, Oxford University Press, 1927. 15+184 pp.

Turpain (E.). Eléments de thermodynamique. Paris, Gauthier-Villars, 1927. 170 pp.

Verein Deutscher Ingenieure. Hydraulische Probleme. Vorträge auf der Hydrauliktagung in Göttingen am 5. und 6. Juni 1925. Berlin, Verein Deutscher Ingenieure, 1926. 8+219 pp.

Vezes (M.). Lecons de chimie physique. Paris, Vuibert, 1927. 545 pp.

Wall (T. F.). Applied magnetism. London, Benn, 1927. 262 pp.

Werkmeister (P.). Vermessungskunde. 4te Auflage. Band 1. Berlin, de Gruyter, 1926. 10+275 pp.

Wotruba (R.). Der ein- und mehrphasige Wechselstrom. München, Oldenbourg, 1927. 6+86 pp.

Wulf (T.). Lehrbuch der Physik. Freiburg i. Br., Herder, 1926. 14+512pp.

Youngson (P.) and Bennett (T. A.). Practical mathematics for marine engineers: second class. Glasgow, Munro, 1926. 504 pp.

Yule (G. U.). An introduction to the theory of statistics. 8th edition, revised. London, Griffin, 1927. 15+415 pp.

Zipperer (L.). Technische Schwingungslehre. 1: Allgemeine Schwingungsgleichungen. (Sammlung Göschen.) Berlin, de Gruyter, 1927.

GROUP PHOTOGRAPH TAKEN AT THE MADISON MEETING, 1927.

1 Mo[illegible] L. J., 2. Graves, L. M., 3. Owens, Helen B., 4. Snyder, Mrs. Margaret, 5. Latimer, C. G., 6. Stouffer, E. B., 7. Carmichael, R. D., 8. Earl, James M., 9. Owens, F. W., 10. Dalaker, H. H., 11. Hotelling, H., 12. Yancey, B. F., 13. Blackwell, H. F., 14. Carmichael, G. N., Jr., 15. Ri[illegible] W. C., 16. Woo[illegible] W. K., 17. Anderson, W. E., 18. Gillespie, D. C., 19. Frame, G. C., 20. Grove, Vernon G., 21. Webber, D. V., 22. Fry, I. C., 23. Ludwick, J. V., 24. Ma[illegible] A. D., 25. Gill, E. P., 26. Cra[illegible] H. V., 27. Olsen, H. I., 28. Sta[illegible] J. A., 29. Lane, J. P., 30. Ta[illegible] J. L., 31. Dyer, L. L., 32. Archibald, R. C., 33. Hen[illegible] F., 34. Greentree, W. C., 35. Metcalf, N. H., 36. Hendrix, E. R., 37. Carter, W. D., 38. Greentree, Mrs., 39. Hildebrandt, T. H., 40. Dyer, Alice C., 41. Sh[illegible] H. E., 42. Dunlap, L. W., 43. Ewing, W. B., 44. Colgrove, Jane L., 45. Bray, H. L., 46. Lawson, Mayme I., 47. Mueller, Nathan, 48. Bell, E. L., 49. Hughes, Jewell, 50. Hart, W. L., 51. Hallock, J. R., 52. Wonder, Vera Pell, 53. Bray, R. W., 54. Colgrove, Elizabeth, 55. Craig, C. E., 56. Jackson, D., 57. Yeston, C. H., 58. Maria, A. J., 59. Merrill, A. S., 60. Kinsey, Lois, 61. McKinney, J. V., 62. Fleig, H. L., 63. Snyder, Virgil, 64. Gehr, M. C., 65. Carnie, C. C., 66. Brooks, W. C., 67. Carmichael, I. W., 68. Dresser, Arnold, 69. Richardson, R. C., D., 70. Spector, M. L., 71. Pettit, H. P., 72. Rabe, W. F., 73. Kennedy, A. J., 74. Crathorne, A. R., 75. Van Vleck, L. E., 76. Engelhorn, M. H., 77. Cockerham, N. E., 78. Sumner, C. S., 79. Pierce, I. A., 80. Bu[illegible] W. H., 81. Curtis, H. B., 82. Hornish, V. A., 83. March, H. W., 84. Birkhoff, G. D., 85. Thomas, Earl M., 86. Langer, R. E., 87. Carver, John J., 88. Hazlett, H. M., 89. Jones, F. M., 90. [illegible], L. B., 91. Sumner, E. D., 92. Hoyle, L., 93. Bunyan, Mrs. L. H., 94. Shewart, W. A., 95. Miser, W. L., 96. Bunyan, L. H., 97. Wall, H. S., 98. Hastings, M. L., 99. Jones, R. W., 100. Jones, Mr. I. A., 101. Bu[illegible] 102. Taylor, J. H., 103. Mickelson, E. L., 104. Adkins, L. K., 105. Barnett, I. A., 106. Barnett, Ethel, 107. Beckwith, Ethelwynn R., 108. Torrey, M. M.

THE THIRTY-THIRD SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

The thirty-third summer meeting and eleventh colloquium of the Society was held at the University of Wisconsin, Madison, Wisconsin, from Tuesday to Saturday, September 6–10, 1927, preceded by the summer meeting of the Mathematical Association of America. It will be remembered as one of the most notable occasions in the annals of the Society.

The colloquium lectures by Professors Bell and Wheeler were delivered on Tuesday, Wednesday, Thursday, and Saturday mornings, and Thursday evening. On Thursday afternoon and Friday morning the Society met in sections for the reading of papers, the Sections of Analysis and of Geometry meeting on Thursday and those of Algebra and Analysis and Miscellaneous Subjects on Friday. The joint dinner of the Society and Association with Professor C. S. Schlichter as toastmaster held at the Maple Bluff Golf Club on Wednesday evening was attended by 150 persons.

A reception was tendered the visiting mathematicians and their friends by the officials of the University, and a program of interesting excursions was carried out, and ample provision was made for sports. A picture of the attending mathematicians and their friends was taken, a copy of which is being distributed with this issue of the Bulletin.

The attending mathematicians and their friends were entertained at Chadbourne and Barnard Halls of the University; the meetings were held in Bascom Hall and the colloquium lectures in Sterling Hall.

Resolutions were passed expressing profound appreciation of the cordial hospitality of the University; thanks to the department of mathematics and in particular to the local members of the Committee on Arrangements, Professors E. B. Van Vleck (Chairman), Arnold Dresden, and Warren Weaver; to Professor R. W. Babcock, who was in

charge of the registration at Chadbourne Hall; and to others who contributed to the success of the gathering.

Many attending members recalled with pleasure the earlier summer meeting and colloquium held at the University of Wisconsin in 1913. On that occasion Professor L. E. Dickson gave a series of lectures entitled "On invariants and the theory of numbers," and Professor W. F. Osgood a series entitled "Topics in the theory of functions of several complex variables." The 1927 colloquium must therefore be known as the Second Madison Colloquium. A special report is given on page 663 of the present issue of the Bulletin.

The attendance included the following one hundred thirty-three members of the Society:

F. E. Allen, Altshiller-Court, W. E. Anderson, Archibald, R. W. Babcock, Barnett, Batchelder, W. O. Beal, E. R. Beckwith, E. T. Bell, Birkhoff, Blichfeldt, Bray, Brenke, Brink, E. W. Brown, H. E. Buchanan, L. H. Bunyan, Bussey, Cairns, C. C. Camp, Carlson, Carmichael, Chang, Chittenden, J. T. Colpitts, Comstock, Conkwright, C. F. Craig, H. V. Craig, Crathorne, Curry, Dalaker, H. T. Davis, Dean, Dickson, Dines, Dowling, Dresden, Emch, G. C. Evans, H. P. Evans, Feinler, W. B. Ford, Fry, Gaba, Garrett, Gergen, Gill, D. C. Gillespie, M. C. Graustein, W. C. Graustein, L. M. Graves, Griffiths, V. G. Grove, Guggenbühl, Harkin, W. L. Hart, W. W. Hart, Hartung, E. R. Hedrick, Hildebrandt, Hoersch, Hollcroft, Hosford, Hotelling, J. C. Hughes, Louis Ingold, Ingraham, Dunham Jackson, Jensen, Kearney, Kempner, Krathwohl, E. P. Lane, Langer, Latimer, Harry Levy, Logsdon, J. V. McKelvey, MacMillan, March, Maria, A. S. Merrill, Michal, Mickelson, Mills, Miser, E. J. Moulton, Olson, F. W. Owens, H. B. Owens, Palmer, Parkinson, Pettit, E. C. Phillips, Phipps, Pierce, Pogo, R. G. D. Richardson, H. L. Rietz, E. D. Roe, Roever, Roos, Roth, Runge, Sanger, Shewhart, Shohat, Sinclair, Skinner, Slaught, Slichter, Snedecor, Virgil Snyder, Stouffer, Swartzel, Tate, J. H. Taylor, E. M. Thomas, Torrey, Trjitzinsky, Van Vleck, Wall, J. H. Weaver, Warren Weaver, Westfall, A. P. Wheeler, Widder, Wolff, Worthington, Yanney, Yeaton.

The Secretary announced the following elections to membership in the Society:

To sustaining membership:

The Georgia Power Company, Atlanta, Ga.;
The University of Michigan, Ann Arbor, Mich.

To ordinary membership:

Dr. Clement Walker Andrews, Librarian, John Crerar Library;
Mr. Loring Beal Andrews, Harvard University;

Mr. John William Arnold, Western Union Telegraph Co., New York City;
Professor Herman T. R. Aude, Colgate University;
Mr. Benjamin Brownstein, tube mill engineer, Ellwood City, Pa.;
Mr. Arnold Chaimovitch, University of Chicago;
President Harry F. Chapell, Chapell Auditors, New York City;
Professor Edward Everett Colyer, Kansas State Teachers College, Hays;
Professor Walter H. Durfee, Hobart College;
Mr. Theodore Miller Edison, East Orange, N.J.;
Mr. Henry Ericson, Washington High School, Milwaukee;
Mr. John Graham Foley, engineer, New York Central Railroad;
Mr. Edwart T. Frankel, National Industrial Conference Board, New York City;
Mr. Albert R. Gallatin, of Smith and Gallatin, New York City;
Professor Joaquin Gallo, Observatorio Astronomico Nacional, Tacubaya, Mexico;
Professor Horace Newton Hubbs, Hobart College;
Mr. Eugene Law Ickes, geologist, Los Angeles;
Professor Richard P. Johnson, Carnegie Institute of Technology;
Miss Elizabeth E. Knight, Milwaukee State Normal School;
Mr. Joseph Harrison Kusner, University of Florida;
Mr. John Buchanan MacLeod, Chicago;
Dr. James D. Maddrill, Union Labor Life Insurance Company, Washington;
Mr. Phillip Newman, Columbia University;
Mr. William Ross Paul, Chicago;
Professor Thomas Reeve Rosebrugh, University of Toronto;
Mr. Samuel Barnell Rosenbaum, Milford School, Milford, Conn.;
Professor Anestis George Sarafoglou, Robert College, Constantinople;
Dr. Gyan Chand Sharma, University of Notre Dame;
Mr. Hartley L. Smith, New England Power Company, Worcester, Mass.;
Professor Louis Beaufort Stewart, University of Toronto;
Dr. Dirk Jan Struik, Massachusetts Institute of Technology;
Mr. Paul McCartney Swingle, Ohio State University;
Miss Helen Thompson, Psycho-Clinic, Yale University;
Mr. James Edgar Thompson, Pratt Institute;
Mr. Charles Wexler, Roxbury, Mass.;
Mr. George Washington Wishard, Norwood, Ohio;
Mr. Byron C. Wolverton, Friendship Telephone Company, Syracuse;
Mr. Samuel Harry Woolf, Harvard University.

Nominees of sustaining members:

Mr. Charles M. Biscay, Western and Southern Life Insurance Company;
Mr. Charles M. Biscay, Jr., Western and Southern Life Insurance Company;
Mr. Wyman A. Bristol, University of Pennsylvania;
Mr. D. K. Chenoweth, Western and Southern Life Insurance Company;
Mr. C. E. Nelson, Missouri State Life Insurance Company;
Mr. R. A. Ryan, Western and Southern Life Insurance Company;
Professor S. E. Stillwell, Western and Southern Life Insurance Company.

The following members of the London Mathematical Society have accepted membership under the reciprocity agreement:

Professor William E. H. Berwick, University College, Bangor, Wales;
Mr. Charles Henry Rowe, fellow of Trinity College, Dublin.

Twenty-three applications for membership were received.

On recommendation of the Council, the Society adopted two amendments of the by-laws. The first adds to the membership of the Council the three representatives of the Society on the Editorial Board of the American Journal of Mathematics, thus placing this periodical on the same basis as the Bulletin and the Transactions. The second replaces the Assistant Secretary by two Associate Secretaries, with the expectation that the extra official will be in the east.

It was announced that Professor E. W. Brown has accepted the invitation to give the fifth Josiah Willard Gibbs Lecture in connection with the 1927 Annual Meeting in Nashville.

The following appointments were announced: Committee on the Society Visiting Lectureship, Professors G. D. Birkhoff, G. A. Bliss and E. R. Hedrick; Committee on Arrangements for the Gibbs Lecture, Professors W. L. Miser (Chairman) and James McClure of Vanderbilt University and Mr. J. L. Wilson of Nashville; additional representatives of the Society on the augmented American Section of the International Mathematical Union, Professors G. A. Bliss, A. B. Coble, Arnold Dresden, E. V. Huntington, and C. H. Sisam.

On account of the change in his residence Professor Arnold Dresden presented his resignation as Assistant Secretary and a resolution of thanks was passed by the Society as follows:

"Resolved that we, the members of the American Mathematical Society, especially those who have been attending the Chicago or Western meetings of this Society, express to Professor Arnold Dresden our sincere appreciation of and gratitude for the many ways in which he has been of service to these meetings during the last eleven years. We feel that

his genuine interest in the welfare of the Society, his untiring efforts in its behalf, and the consummate skill with which he has conducted its affairs, including the arrangements for its meetings, in this section of the country have been of inestimable value to the growth of mathematical interest."

Professor M. H. Ingraham was appointed to fill the vacancy for the remainder of 1927.

Titles and abstracts of the papers read before sectional sessions of the Society follow below: the papers numbers 1 to 8 were read before the Section of Analysis, Professor G. C. Evans presiding; numbers 9 to 16 and 69 before the Section of Geometry, President Snyder presiding; numbers 17 to 30 and 66 to 68 before the Section of Algebra, Vice-President Bell presiding; and numbers 31 to 65 before the Section of Analysis and Miscellaneous Subjects, ex-president Birkhoff presiding. Mr. Heineman was introduced by Professor Dresden, Professor Besicovitch by Professor Tamarkin (his paper was read by Professor Langer), Professor Uspensky by the Secretary, and Mr. Dorroh by Professor R. L. Moore. The papers of the following authors were read by title: Ayres, Birkhoff (first paper), Buchanan (second paper), Evans (second paper), Kempner (first paper), Langford, Mears, Merriman, Miller, Moore, Rainich, Schelkunoff, Sharpe, Roos (second paper), Uspensky (second paper), Vandiver, Walsh, Webber, G. T. Whyburn, W. M. Whyburn, Widder and Gergen, Wilson.

1. Professor J. A. Shohat: *On certain developments in series of Tchebycheff polynomials.*

Consider the finite development $x^n = H_0\phi_0(p; x) + \cdots + H_n\phi_n(p; x)$, where $\phi_n(p; x)$, $n = 0, 1, 2, \cdots$, is a system of Tchebycheff polynomials corresponding to a given interval (a, b) with the characteristic function $p(x)$. In the case $(a, b) = (-1, 1)$, $p(x) \equiv 1$ (Legendre polynomials), all the coefficients H_i above, as is well known, are positive, a property leading to many interesting results. In the present paper the author extends this property of the H's to the general case of "symmetric Tchebycheff polynomials": $a = -b$ (finite or infinite), $p(x) \equiv p(-x)$. Application is made, in particular, to the discussion of the convergence of the development $f(x) \backsim \sum_{n=0}^{\infty} A_n\phi_n(p; x)$, where the $A_n \equiv \int_{-a}^{a} p(x)\phi_n(p; x)f(x)dx$ are all positive and $f(x)$ and $p(x)$ are subject to certain general conditions.

2. Professor L. L. Dines: *Linear inequalities in general analysis.*

This paper appears in full in the present number of this Bulletin.

3. Professor L. L. Dines: *A theorem on orthogonal sequences.*

For any positive constant e greater than unity, we denote by $\mathfrak{S}_e'$ and $\mathfrak{S}_e''$ the classes of all infinite real sequences $\{\sigma_i'\}$ and $\{\sigma_i''\}$ respectively, such that the series $\sum_i|\sigma_i'|^e$ and $\sum_i|\sigma_i''|^{e/(e-1)}$ converge. If $\{\sigma_i'\}$ and $\{\sigma_i''\}$ are sequences of these respective classes, then the series $\sum_i\sigma_i'\sigma_i''$ converges absolutely, as is well known. If the sum of this product series is zero, the two sequences are said to be *orthogonal.* We prove the following theorem: A necessary and sufficient condition that there exist in $\mathfrak{S}_e''$ a positive sequence orthogonal to each of a given finite set of sequences in $\mathfrak{S}_e'$ is that no linear combination of the given set be M-definite.

4. Professor R. E. Langer: *The boundary problem associated with a differential equation in which the coefficient of the parameter changes sign.*

This paper deals with the boundary problem associated with the differential system $u''(x)+\lambda x^\nu u(x)=0$, $u(-\alpha)=u(\beta)=0$, $\alpha, \beta>0$, in which ν is any real positive constant. The interest in the system centers on the fact that the interval considered contains the origin, so that the coefficient of the parameter changes sign if ν is an odd integer, and in general changes from real to complex. The method, that of asymptotic forms, has not heretofore been applied to the study of systems of this type. Asymptotic expressions for the characteristic values and functions are derived, and the theorem for the expansion of an arbitrary function as established is far more general than the existing theorems derived by other methods.

5. Professor W. B. Ford: *On the behavior of integral functions in distant portions of the plane.*

This paper will appear in full in an early number of this Bulletin.

6. Professor Dunham Jackson: *On the approximate representation of analytic functions.*

This paper will appear in full in an early number of this Bulletin.

7. Professor T. H. Hildebrandt: *Lebesgue integration in general analysis.*

This paper points out a method for defining a Lebesgue type of integration in the case of functionals on a linear real range to a complete linear function space. Riemann integration for such functionals has been defined by L. M. Graves (Transactions of this Society, vol. 29 (1927), pp. 166 ff.). The method of defining a Lebesgue integration is suggested by the validity

of Osgood's theorem in this situation, i.e., if a sequence of Riemann integrable functionals bounded in their totality converges to a Riemann integrable functional, then the limit of the integrals is the integral of the limit. Starting with a class of Riemann integrable functionals, e.g., the continuous functionals, and using convergent (or almost convergent) sequences of functionals, we can define integration for functionals not Riemann integrable.

8. Professor T. H. Hildebrandt: *Note on interchange of order of limits.*

This paper will appear in full in an early number of this Bulletin.

9. Mr. G. A. Parkinson: *A theory of parallelism in sub-spaces.*

In this paper the angle between two vectors at different points of a V_n is defined. On the basis of this definition, the notion of the parallel motion of a vector in V_{m_1}, a sub-space of V_n, with respect to V_m, also a sub-space of V_n, is developed. The special case in which V_{m_1} and V_m are applicable is discussed. Parallel curves of V_n are defined, and a notion of orders of parallelism between sub-spaces V_{m_1} and V_m is discussed.

10. Mr. S. A. Schelkunoff: *On certain properties of orthogonal and generalized orthogonal transformations.*

This paper gives an extension of Euler's formulas for transformation of coordinates, and certain properties of the coefficients, also a decomposition of a general rotation in an n-flat into a succession of "simple" rotations, and the determination of the angles of simple rotations. The generalized orthogonal transformations are defined by means of relations between the coefficients $a_k{}^i$: $a_k{}^i\bar{a}_k{}^j=1$ when $i=j$, and $a_k{}^i\bar{a}_k{}^j=0$ when $i\neq j$, where $\bar{a}_k{}^i$ is the conjugate of $a_k{}^i$. Certain properties of generalized orthogonants are then obtained. The remainder of the paper is concerned with quasi-metrical transformations of null-flats.

11. Professor T. R. Hollcroft: *Limits for double points of surfaces.*

Limits that can be reached but not exceeded are found for the number of distinct conic nodes of an algebraic surface of any order. For orders greater than seven, these limits are smaller than those previously found. Also the maximum number of binodes of each kind and unodes of each kind is found for any algebraic surface. It is further shown that all double points of whatever kind that an algebraic surface can have may be real. The maximum numbers of consecutive conic nodes, binodes and unodes of surfaces of given order are determined. Distinct conic nodes account for one invariant each, but when they combine to form a higher singularity, the number of conic nodes is greater than the number of invariants. A certain number of conic nodes may thus be considered independent and the rest dependent. For a given order of a multiple point, a limiting order

of the surface containing it is found such that for a smaller order of the surface the number of independent conic nodes in the multiple point exceeds the maximum number of distinct conic nodes a surface of that order can have. The existence of the surfaces is shown to depend upon that of plane curves with distinct double points.

12. Professor Nathan Altshiller-Court: *On four mutually orthogonal circles.*

Four mutually orthogonal circles are considered in their relations to the triangles formed by their centers and to the circumcircles of these triangles. Among others the following results are obtained. The square of the radius of any one of four given mutually orthogonal circles is equal to one half of the power of its center with respect to the circle passing through the centers of the remaining three circles. The algebraic sum of the squares of the radii of the given circles is equal to the square of the diameter of the circle passing through the centers of three of the given circles. Each of the given circles is the conjugate circle of the triangle formed by the centers of the remaining three circles, and, conversely, the conjugate circles of an orthocentric group of four triangles are mutually orthogonal. The trilinear pole of an axis of similitude of three of the given circles with respect to the triangle formed by their centers coincides with the pole of this line with respect to the fourth circle.

13. Professor J. H. Weaver: *On certain points associated with a triangle.*

The author has discussed the properties of four points associated with a triangle and defined as follows. Let triads of circles be drawn in such a way that the internal segments described on BC, CA, AB respectively contain angles (1) C, A, B; (2) B, C, A; (3) $2C$, $2A$, $2B$; (4) $2B$, $2C$, $2A$. These four triads of circles determine four points, P_1, P_2, P_3, P_4. Among the interesting properties of these four points the following may be noted: (1) the pedal triangles of the four points are similar; (2) the pairs of lines BP_1 and CP_2, CP_1 and AP_2, AP_1 and BP_2 are parallel; (3) if BP_2 and CP_1 intersect in A_1, CP_2 and AP_1 intersect in B_1, and BP_1 and AP_2 intersect in C_1, then the triangles ABC and $A_1B_1C_1$ are congruent and have the sides AB and A_1B_1, BC and B_1C_1, CA and C_1A_1 respectively parallel; moreover, the points P_1 and P_2 bear the same relation to each of the triangles ABC and $A_1B_1C_1$; (4) if the points P_3 and P_4 replace P_1 and P_2 in (3), then three points A_2, B_2, C_2 are determined. These three points and the points P_3 and P_4 lie on a circle.

14. Professor Harold Hotelling: *Manifolds of orthogonal ennuples.*

Regarding an orthogonal ennuple as a set of n mutually perpendicular non-sensed and indistinguishable lines through a point in n-space, the set of orthogonal ennuples at a point constitutes a manifold of $n(n-1)/2$ dimensions. This paper deals with certain analysis situs properties of these manifolds.

15. Professor F. R. Sharpe: *Superabundant nets of plane curves.*

In this paper, the author considers a general method of finding superabundant nets of plane curves which determine, by their intersections, involutions of the order of the grade. The involutions are first found on a rational surface F_n, by means of its intersections with the lines of a rational congruence C, having a fundamental line l. If l is r-fold on F_n, the involution is of order $n-r$. A net of surfaces N can be found such that any two surfaces of N meet in a line of C. The surfaces of N meet F_n in a net of curves. When F_n is mapped on a plane, the corresponding net of curves determines an involution of order $n-r$. By considering surfaces of orders 3 and 4, various new types of involutions of order 3 can be found, and similarly from surfaces of orders 4, 5, and 6, involutions of order 4.

16. Professor V. G. Grove: *Nets with equal W invariants.*

Nets with equal W invariants the author has called I nets. A conjugate I net is an isothermally conjugate net. By the introduction of the term "equal point invariants of the second kind," and by extending the idea of equal point invariants, many theorems on isothermally conjugate nets become true for I nets. Conjugate nets have equal point invariants of the second kind.

17. Professor G. A. Miller: *Substitutions which transform a regular group into its conjoint.*

This paper appears in full in the present number of this Bulletin.

18. Professor C. G. Latimer: *A note on a theorem of Hermite.*

If $|a_{ij}|$, $(i, j, =1, 2, 3)$, is the general symmetric third-order determinant, and A_{ij} is the cofactor of a_{ij}, Hermite showed that the form $\phi(x)=x_0^2+\sum_{i,j=1}^{3}A_{ij}x_ix_j$ repeats under multiplication. It may be shown that by a unitary transformation T, ϕ may be transformed into $\psi=x_0^2+bcu^2+acv^2+abw^2$, where a, b, c are rational functions of the a_{ij}. Then by use of a recent result of Bell's on the product of two such forms, and by application of the transformation T^{-1}, we obtain Hermite's formulas for the product $\phi(x)\cdot\phi(y)$. Our proof with that of Bell's result referred to above is substantially shorter than Hermite's original proof or that of Bachmann.

19. Professor C. G. Latimer: *On the representation of integers by certain indefinite quaternary forms.*

In this paper it is shown that if α is a positive integer $\leqq 163$ in the form $4k+3$, or the double of such an integer, and contains no square factor, then the form $f(x, y, z, w)=x^2+y^2-\alpha z^2-\alpha w^2$ represents all integers. The proof is by the method of descent, being a generalization of a method used by Dickson in obtaining similar results for the case $\alpha=-7$. It is shown

that if $f(X, Y, Z, W)=mn\neq 0$, then m is represented by f if the same is true of n. Repeated application of the above leads to the cases $n=\pm 1$, which are easily treated. If m is a prime, the existence of integers $X, Y, Z, W, n\neq 0$ such that $f(X, Y, Z, W)=mn$, is well known. Since f repeats under multiplication, and since f represents -1, for each of the above values of α, as may be readily verified, our result follows.

20. Professor L. E. Dickson: *All positive integers are sums of values of a quadratic function of x.*

This paper appears in full in the present number of this Bulletin.

21. Professor L. E. Dickson: *Generalizations of the theorem of Fermat and Cauchy on polygonal numbers.*

This paper will appear in full in an early number of this Bulletin.

22. Professor L. E. Dickson: *Extended polygonal numbers.*

This paper will appear in full in an early number of this Bulletin.

23. Professor L. E. Dickson: *Generalized polygonal numbers.*

These numbers include both the polygonal and extended polygonal numbers. Hence they are the values of $e(x)$ for all positive, negative, and zero integers x. Every integer is a sum of three generalized pentagonal numbers; also of three generalized hexagonal numbers; also of four generalized octagonal numbers; etc.

24. Professor A. J. Kempner: *On the use of certain types of formulas in the theory of prime numbers.*

A number of analytic expressions are known for the representation of prime numbers (by means of the gamma function, or infinite series, or definite integrals, or repeated limiting processes). None of these formulas has so far proved to be of value in the theory of prime numbers. A simple analysis of these formulas shows that they are practically all equivalent to Wilson's theorem, which is itself no deep lying theorem. It is therefore not to be expected that the mere formulation of such theorems, unless they be accompanied by a powerful analytic method of attack, will add to our knowledge in the theory of prime numbers. Such methods of attack are not known for any of the formulas.

25. Mr. W. E. Roth: *A solution of the matric equation $P(X)=A$.*

This paper is concerned with finding, by elementary methods, such solutions of the equation $P(X)=A$ as are expressible as polynomials in A. Here $P(X)$ is a polynomial in X with scalar coefficients, and A is a square matrix of order n, not necessarily non-singular. Such solutions were found by Frobenius (Berliner Sitzungsberichte, 1896) for the case

$X^2=A$; they are given by Dickson in his *Modern Algebraic Theories*, for the case $X^m=A$. In both cases, A is supposed to be non-singular. The equation $X^m=A$ has also been treated by Cecioni (Annali della Reale Scuola Normale, Pisa, 1909), and by Kreis (Vierteljahrschrift, Zurich, 1908); in his Zurich dissertation, the latter author had also studied the general equation $P(X)=A$. In all of the papers cited, use is made of the Weierstrass elementary divisors and associated normal forms; the method is not easy to follow and is complicated in its application.

26. Mr. E. R. Heineman: *Generalized Vandermonde determinants.*

A generalized Vandermonde determinant is obtained from the ordinary Vandermonde determinant by permitting the indices to take any set of values. The Vandermonde matrix is defined to be the Vandermonde determinant with the nth powers of its variables added as an extra row. By successively blocking out each of the first n rows of this matrix, we obtain n determinants which can be called secondary Vandermonde determinants. The ordinary Vandermonde determinant, which we get by omitting the last row of this matrix, will be called the principal Vandermonde determinant. It can be shown that every generalized Vandermonde determinant is expressible as a determinant-function of the principal and secondary Vandermonde determinants.

27. Professor H. S. Vandiver: *On the theorem of Kummer concerning power characters of units in a cyclotomic field.*

This paper presents, in a modern form, the argument used by Kummer in Crelle's Journal (vol. 44 (1852), pp. 121–130) for obtaining expressions giving the pth power characters of units in the field defined by a primitive pth root of unity, p being an odd prime. Extensive use of these results will be made in other papers by the author.

28. Professor H. S. Vandiver: *Summary of results and proofs concerning Fermat's last theorem.* (Third note.)

In this note, among other results a proof is indicated for the following theorem: Under the assumptions (1) none of the Bernoulli numbers $B_{vp}(v=1, 2, 3, \cdots, (p-3)/2)$ is divisible by p^3, and (2) the second factor of the class number of the field $k(\alpha)$, $\alpha=e^{2i\pi/p}$, is prime to p, it follows that $x^p+y^p+z^p=0$ is not satisfied in rational integers x, y, and z, prime to each other, none zero, if $xyz\equiv 0 \pmod{p}$, and p is an odd prime.

29. Professor H. S. Vandiver: *On an extension of the Bernoulli summation formula.*

In this note is considered a theorem of Glaisher's which expresses in terms of Bernoulli numbers the sum of nth powers of integers in an arithmetical progression.

30. Professor H. S. Vandiver: *On the norm-residue symbol in the theory of cyclotomic fields.*

This symbol (Hilbert, *Bericht*, p. 413) has always, apparently, been used to represent a summation. In the present paper it is shown how the summation may be carried out.

31. Professor C. C. Camp: *A multiple-parameter differential system which leads to a development of a function in p variables.*

The system $u_i' + (\sum_{j=1}^{p} \lambda_j a_{ij}) u_i = 0$, $i = 1, 2, \cdots, p$, with the boundary conditions $u_i(\pi) = u_i(-\pi)$, $i = 1, 2, \cdots, p$, where a_{ij} is an integrable function of x_i either identically zero or of constant sign and such that the determinant $|A_{ij}| \neq 0$, $2\pi A_{ij} \equiv \int_{-\pi}^{\pi} a_{ij} dx_i$, leads to a multiple expansion in which the coefficients involve the p-fold integral of $f |a_{ij}| \prod_{i=1}^{p} v_i^*$; v_i^*, $i = 1, 2, \cdots, p$, being a principal solution of the adjoint system. We make the change of parameters $\nu_i = \sum_{j=1}^{p} \lambda_j A_{ij}$ for the convergence proof, and in the first integrations by parts in the contour integrals we replace the elements of the first column of $|a_{ij}(s_i)|$ by $\sum_{j=1}^{p} \lambda_j a_{ij}(s_i)$. In certain subsequent integrations we have the cofactors of these elements as multipliers in the integrands. By replacing λ_k by its appropriate values $(\nu_i - \sum_{j=1, j \neq k}^{p} \lambda_j A_{ij})/A_{ik}$, $(i = 1, 2, \cdots, p; k = 2, 3, \cdots, p)$, in the several integrands of the exponential indices, we may evaluate the limits of the parts which do not contribute zero as the so-called mean value of f.

32. Professor A. S. Besicovitch: *Fundamental geometric properties of linearly measurable plane sets of points.*

The notion of a linearly measurable plane set of points was introduced in 1914 by Carathéodory. These sets (A) include as a special case rectifiable plane curves. The present paper develops a general theory of such sets. The notion of the density and of the tangent are introduced and discussed. The sets (A) are classified as regular and irregular. The regular sets (A) in many respects are analogous to the rectifiable curves. For instance, the tangent exists at almost all points of a regular set (A); to any regular set (A) of measure L and any positive number ϵ, there corresponds a set (finite or denumerable) of rectifiable curves, of total measure $< L + \epsilon$, which contains almost all the points of (A). On the other hand, no such properties exist for the irregular sets (A). They are entirely dissimilar to the rectifiable curves, which is shown by many examples. In one of these, an irregular set (A) is constructed, of measure > 0, whose projection on any straight line has measure 0.

33. Professor Harold Hotelling: *Correlations with incomplete data.*

The data for a correlation problem consist essentially of a matrix of observations. One row represents each variate, and one column each instance. Ordinary, partial, and multiple correlation coefficients are all

invariant under certain operations on the matrix. Frequently incompleteness of data leaves certain elements of the matrix unknown. In such cases statisticians have faced the dilemma of discarding every column containing an unknown element, which may so far reduce the number of instances as to rob the results of significance, or of using all possible instances for calculating each ordinary correlation, which may give impossible values for the multiple and partial correlations. But regarding the data as obtained by a random sampling of a normal distribution, it is possible to escape both horns of the dilemma in determining the most likely values, which are necessarily consistent, in the aggregate sampled. The equations for doing this are presented. By similar means an explicit solution is obtained for the following problem. Two variables are known a priori to be uncorrelated, but in a sample there is usually some correlation. What corrections should on this account be applied to the observed correlations of these variables with others, and to partial and multiple correlations involving them?

34. Professor Harold Hotelling: *Spaces of statistics and their metrization.*

An ordinary statistical graph is a geometric figure whose essential properties are invariant only under a group of transformations even more limited than that of euclidean geometry. However, certain statistical problems admit of geometric representation in a manner for which the concepts of differential geometry have significance. One way of metrizing the space of statistical variables is by means of the quadratic form appearing in the normal law of error. This becomes a differential form in various problems in evolution and in economics. Examples of the latter are given in a paper by the author in the September, 1927, number of the Journal of the American Statistical Association.

35. Professor A. J. Kempner: *On the shape of polynomial curves.* (Second paper.)

This is a continuation of a paper presented to the Society in 1919 under the same title. In the earlier paper, the problem treated was to prove the existence of polynomial curves $y=a_0x^n+\cdots+a_n$ (all quantities involved real) with a prescribed arrangement (arbitrary except for obvious restrictions) of the order of magnitude in which the successive maxima and minima are arranged. In that paper, points of inflection were not assumed on the curves. In the present paper, by considering them as being generated by letting successive maxima and minima coincide, with subsequent further continuous variation of the coefficients, points of inflection are taken into account.

36. Professor H. E. Buchanan: *On the oscillations of three finite masses near the equilateral triangle solutions.*

In this paper the author discusses the equations of variation near the equilateral triangle positions. The characteristic exponents are found, and

their nature depends on a certain relation between the masses. In case one of the masses is infinitesimal, these exponents reduce to those given by Moulton in his celestial mechanics. The instability of the equilateral triangle positions is proved, and two types of periodic orbits are shown to exist.

37. Professor H. E. Buchanan: *Note on a certain memoir of Liouville's.*

In a very interesting paper published in Connaissance des Temps, 1845, Liouville discusses the stability of the straight-line solutions of the three-body problem. His result depends on the solution of the equation

$$\alpha^4+(2n'^2-n^2)\alpha^2+(n'^2-n^2)(2n^2+n'^2)=0.$$

He shows that n'^2-n^2 is negative for all values of the masses, and consequently the straight line solutions are unstable because of the presence of a term of the type $e^{\alpha t}$, α real and positive. In the present note it is shown that Liouville's equation is exactly the same as an equation of Buchanan's in a paper in the American Journal of Mathematics (vol. 45 (1923)), p. 98, equation 12), where the periodic orbits near the straight-line solution are discussed, except that Liouville always keeps his three bodies in a straight line. It follows that Liouville's equation is the same as Moulton's *Periodic Orbits,* Carnegie Institution Publication No. 161, and Plummer's Monthly Notices, Royal Astronomical Society, vol. 62 (1901), if one of the masses is infinitesimal.

38. Professor L. M. Graves: *Discontinuous solutions in space problems of the calculus of variations.*

The extension of the Jacobi condition to the case of minimizing curves with corners, the so-called discontinuous solutions, was given by Carathéodory for problems in the plane. His condition is apparently not extensible to space problems, and the details of the proof are very complicated for the plane case. The new form of the condition of Carathéodory obtained by the present author states that the functional determinant of the family of "extremaloids" or broken extremals, whose zeros correspond to points of contact with the envelope of the family, shall not change sign at a corner of a minimizing curve. The statement and proof are practically as simple for n-dimensional problems as for the plane problem. When modified in the usual way, the new condition fits into the familiar set of sufficient conditions for a minimum.

39. Professor D. C. Gillespie: *On infinite determinants and their associated systems of linear equations.*

If in a normal determinant each element of the principal diagonal be diminished by unity, the resulting double array of elements are the terms of an absolutely convergent double series. The normal determinant converges; if its value is different from zero and the constant terms in the

associated system of equations form a bounded set, the equations have one and only one bounded solution. The striking fact that the solution obtained is the only bounded solution is, of course, due to the assumption made concerning the terms of the principal diagonal. The determinants considered in this paper have in common with normal determinants the property that the elements of the principal diagonal approach unity, but differ from them in that the absolute value of the elements in diagonals immediately below the principal diagonal are allowed to approach unity. The determinants are shown to converge, and a solution is obtained for the system of equations associated with each non-vanishing determinant, the solution being the only bounded one if a bounded solution exists. If no bounded solution exists, the solution found is the only solution in which the nth unknown is less in absolute value than the nth term of a certain sequence.

40. Professor G. Y. Rainich: *Deviation from normality of congruences of curves.*

For a congruence of "equidistant" curves in curved n-dimensional space certain expressions which vanish in the case of a normal congruence are introduced as giving in the general case a measure of deviation from normality in the neighborhood of a curve. Integrals of these expressions (which represent an antisymmetric tensor) over a surface cutting the curves of the congruence are introduced, as giving the *total deviation* for the corresponding portion of the congruence. Conditions are considered (for $n>3$) under which this total deviation vanishes over a closed surface. In the case for which $n=5$, the Kaluza-Klein interpretation of electromagnetism is obtained. The centrosymmetric case is worked out as an example of the general theory.

41. Professor D. V. Widder: *On the expansion of analytic functions of the complex variable in generalized Taylor's series.*

In a previous paper the representation of a real function in generalized Taylor's series was discussed. The methods there employed are not applicable to the complex case. In the present paper new methods are developed and the representation of an arbitrary analytic function is obtained. The result is comparable with a result of Birkhoff's given in the Comptes Rendus, 1917.

42. Mr. J. J. Gergen and Professor D. V. Widder: *On Taylor's series admitting the circle of convergence as a singular curve.*

This paper obtains a new criterion that the circle of convergence of a power series be a singular curve. The criterion is a generalization of one recently given by S. Mandelbrojt. By an application of the result, an elementary proof of a familiar theorem on lacunary series is given.

43. Professor R. L. Moore: *A separation theorem.*

In this paper the following theorems are proved. (I) If, in a plane S, M is a bounded continuous curve, K is a closed subset of M, and H is a subset of K and of some connected subset of $K+(S-M)$ but no point set of which H is a proper subset satisfies these conditions, then if the continuum E is a subset of $K-H$, there exists a simple closed curve lying wholly in $M-K$ and separating H from E. (II) If, in a plane S, K is a closed subset of a continuous curve M, and A and B are two points of M, then if A and B lie in a connected subset of $K+(S-M)$ they lie in a *closed* and connected subset of $K+(S-M)$. (III) In order that a bounded continuum M in a plane S should be a continuous curve, it is necessary and sufficient that if t is an arc and H is an open subset of $M+t$, and A and B are points of $S-H$ which are *weakly* separated by H, then H contains a simple closed curve that separates A from B. The first of these theorems may be regarded as an extension of a theorem of Zoretti's.

44. Mr. J. L. Dorroh: *Concerning a set of metrical hypotheses for geometry.*

In his paper *Sets of metrical hypotheses for geometry* (Transactions of this Society, vol. 9 (1908), pp. 487–512) R. L. Moore raises the question whether it follows from the set O of order axioms and the set C of congruence axioms employed therein that every segment has a mid-point. In the present paper this question is answered in the affirmative.

45. Dr. G. T. Whyburn: *On a problem of W. L. Ayres.*

The author shows that if a plane continuous curve M does not contain more than a finite number of simple closed curves of diameter greater than any preassigned positive number, then every connected subset of M is arcwise connected. This answers a question raised by W. L. Ayres in his paper *Concerning continuous curves of certain types.* (See abstract in this Bulletin, vol. 32 (1926), p. 307; the paper will appear in Fundamenta Mathematicae.) In proving this proposition it was shown that every continuous curve satisfying the hypothesis of this theorem has the property that each of its maximal cyclic curves contains only a finite number of simple closed curves. Then with the aid of the author's theorem that every connected subset of a continuous curve is arcwise connected provided that every connected subset of each of its maximal cyclic curves is arcwise connected (see my paper *Concerning the structure of a continuous curve*, presented to the Society Dec. 31, 1926; offered to the American Journal), the proof is easily completed.

46. Dr. G. T. Whyburn: *Concerning the cut points of continua.*

If X is any point set, let X^* denote a subset of X obtained by omitting from X some countable set of points. Let M be any plane continuum, and G the set of all the cut points of M. Then (1) if H is any uncountable subset

of G, there exist two points of H which are separated in M by uncountably many points of H; (2) there exists a set G^* (possibly null) such that for each point X of G^*, $M-X$ is the sum of two connected point sets; (3) every collection of mutually exclusive subcontinua of M each of which contains at least one point of G is countable; (4) for every subset E of G there exists an E^* no point of which belongs to any subcontinuum of M containing no other point of E; (5) there exists a G^* every point X of which is an end point of every continuum obtained by adding X to some maximal connected subset of $M-X$; (6) there exists a G^* such that M is connected im kleinen at every point of G^*; (7) there exists a G^* such that every subcontinuum N of M is connected im kleinen at every point of $G^* \cdot N$; (8) there exists a G^* every point of which is a point of Menger order two of M.

47. Dr. G. T. Whyburn: *Concerning certain kinds of continuous curves and other continua.*

The following theorems are proved. (1) Every connected and relatively open subset of an arcwise connected im kleinen point set is arcwise connected. (2) If H denotes the sum of the boundaries of all the complementary domains of a continuous curve M every subcontinuum of which is a continuous curve, then (a) H is arcwise connected, and (b) $M-H$ is totally disconnected. (3) If every maximal cyclic curve of a continuous curve M is a simple closed curve, then M is a Menger "regular curve" and, indeed, for any $\epsilon>0$, M is the sum of a finite number of continua each of diameter $<\epsilon$ and no two of which have more than one point in common. The case where M is the boundary of a domain is a special case of (3). (4) Every "regular point" (Menger) of a continuum M is regularly accessible from every complementary domain of M to the boundary of which it belongs. (5) Every cut point of a continuum M which is of Menger order two belongs to the boundary of only one complementary domain of M. (6) If every point of a continuum M which contains no domain is *regularly accessible from* $S-M$, then M is a Menger regular curve.

48. Dr. W. L. Ayres: *Concerning subsets of a continuous curve which can be connected through the complement of the continuous curve.*

A subset N of a plane continuous curve M is said to be connected through N and $S-M$ if there exists a subset H of $S-M$ such that $N+H$ is connected. (1) If K is a bounded subcontinuum of M, T is a totally disconnected cutting of K, and x and y are points of $K-T$ which cannot be connected through $K-T$ and $S-M$, then there exists a simple closed curve J which is a subset of M, separates x and y, and is such that $J \cdot K$ is a subset of T. (2) A bounded M-domain is simply connected with respect to M if and only if its M-boundary is connected through B and $S-M$. (3) If K is a closed cutting of M such that if H is any component of K then $K-H$ is not a cutting of M, then K is connected through K and $S-M$. (4) In order that a plane continuum K be a continuous curve it is sufficient

that if A is any arc, H is a closed subset of $K+A$, and x and y are two points of H which can be connected through H and $S-K$, then x and y can be connected through H and $S-K-A$ by a continuum. (R. L. Moore has shown the necessity of (4).)

49. Professor W. A. Wilson: *On irreducible cuts of the plane between two points.*

If F is an irreducible cut of the plane between two points, any component of the complement of F which has F as its frontier is called a *principal* component. A decomposition of a continuum C into two subcontinua is called *irreducible* if C is not the union of one of the subcontinua and a proper subcontinuum of the other. The following theorem is obtained. Let F be a bounded irreducible cut between two points which is decomposable and let $F=H+K$ be an irreducible decomposition of F into proper subcontinua. According as the number of principal components of the complement of F is finite or infinite, $H\cdot K$ is the sum of the same number or an infinite number of closed subsets, no pair of which have common points, and H and K are both irreducible between each pair of these sets. This is an analogue of C. Kuratowski's theorem regarding a completely irreducible cut of the plane, i.e., a cut of the plane which is the frontier of *every* component of its complement.

50. Dr. C. H. Langford: *An analysis of some general propositions.*

This paper appears in full in the present number of this Bulletin.

51. Professor J. L. Walsh: *On the approximation to harmonic functions by harmonic polynomials.*

This paper has appeared in the September-October number of this Bulletin.

52. Professor J. L. Walsh: *On the expansion of analytic functions in series of polynomials and of other analytic functions.*

This paper presents some extensions of the author's previous work on expansions of arbitrary functions in terms of polynomials (Transactions of this Society, vol. 26 (1924), pp. 155–170). In particular, there are considered the expansion of a discontinuous function, the analog of Gibbs's phenomenon, the analog of Abel's theorem and its converse, and equivalence of expansions not merely on a single curve but on a one-parameter family of curves.

53. Dr. G. M. Merriman: *On sufficient conditions for the convergence and Cesàro summability of the allied series of a double Fourier series.*

This paper has appeared in the September-October number of this Bulletin.

54. Miss Florence M. Mears: *Riesz summability for double series.*

This paper defines for double series a mean, $(R:\lambda, p; \mu, r)$, corresponding to the Riesz definition, (R, λ, p), for summing simple series. It includes theorems relative to the regularity and total regularity of the extended definition; it establishes a relation between methods of summation of the same type, (λ, μ), when either p or r, or both p and r, are changed; a relation between methods of summation of the same order, (p, r), when either λ or μ, or both λ and μ, are changed; certain necessary conditions for the Riesz summability of double series; theorems for the Dirichlet and Cauchy products of double series corresponding to those of Mertens, Cauchy and Abel for the Cauchy product of simple series; a sufficient condition for the summability of the product of two double series to the correct sum.

55. Dr. W. M. Whyburn: *Existence and oscillation theorems for non-linear differential systems of the second order.*

The method of successive approximations is used to establish a general existence theorem for the system of first-order differential equations $dy_i/dx = \sum_{j=1}^{n} A_{ij}(x, y_1, \cdots, y_n)y_j$, $(i=1, 2, \cdots, n)$. The second-order system $dy/dx = K(x, y, z; \lambda)z$, $dz/dx = G(x, y, z; \lambda)y$, is considered along with boundary conditions of the type $y(a, \lambda) = y(b, \lambda) = 0$. Existence and oscillation theorems are established for cases where K and G satisfy conditions similar to those imposed in treating the ordinary linear systems of the Sturmian type. A treatment of linear Sturmian systems is given in which several of the usual conditions on the coefficients are replaced by symmetric conditions on K and G. The methods of this paper are based on transformations of the type $y = u \cos v$, $z = u \sin v$.

56. Professor W. P. Webber: *On a generalization of the Weierstrassian theory of elliptic functions.*

Defining the variable u by the vector equation $u = i_1x_1 + i_2x_2 + \cdots + i_px_p$, where $i_1, i_2, \cdots, i_p$ are unit vectors in the directions of a system of mutually perpendicular axes $OX_1, OX_2, \cdots, OX_p$, and $x_1, x_2, \cdots, x_p$ are real coordinates of a point referred to these axes; defining w by the vector equation $w = 2i_1m_1\omega_1 + 2i_2m_2\omega_2 + \cdots + 2i_pm_p\omega_p$, where $m_1, m_2, \cdots, m_p$ take all integer values independently and where $\omega_1, \omega_2, \cdots, \omega_p$ are arbitrary real constants; and employing a generalization of ordinary vector algebra, the author constructs the function

$$F_p(u) = (1/u_p) + \sum{}' [1/(u-w)^p - 1/w^p],$$

which is valid and possesses p periods $2\omega_1, 2\omega_2, \cdots, 2\omega_p$; the accent on $\sum$ indicates that $w=0$ is omitted. Convergent series valid in the fundamental cell are obtained for the cases $p=3$ and $p=4$. Functions analogous to the classical functions σ, ζ, Z are given.

57. Professor G. D. Birkhoff: *Reversibility in dynamics.*

In this paper it is proved that a necessary and sufficient condition for complete stability of a periodic motion is the reversibility of the time t

in the differential equations, provided that the dependent variables are also subject to transformation.

58. Professor G. D. Birkhoff: *The periodic motions near a stable periodic motion of a dynamical system.*

On the basis of a very simple lemma concerning the invariant points of a transformation of the plane in the neighborhood of a given invariant point, the author shows that there exist infinitely many periodic motions in an arbitrarily small neighborhood of a stable periodic motion of a dynamical system with two degrees of freedom. This lemma is closely related to the extended form of Poincaré's last geometric theorem.

59. Professor G. C. Evans: *Note on a generalization of a theorem of Bôcher.*

Let $u(M)$ be a "potential function of its vector derivative or gradient" and satisfy Bôcher's equation $\int_s D_n u ds = 0$ "almost everywhere" on rectangles in the plane open region Σ. Then $u(M)$ has at most removable discontinuities in Σ, and when these are removed by changing the values of $u(M)$ at most in a set of measure zero, it becomes continuous with all its derivatives and satisfies Laplace's equation at every point of Σ. The theorem remains valid if the set of rectangles is replaced by any "translatable regulat net"; and a corresponding theorem holds in any number of dimensions. Since it is not assumed that the curves on which Bôcher's equation holds are level curves of a harmonic function, an indirect proof is necessary.

60. Professor G. C. Evans: *Poisson's equation in the plane and other equations of elliptic type.*

The equation considered is $\int_s D_n u \, ds = \Phi(s)$, where $\Phi(s)$ is the function of curves with regular discontinuities which corresponds to a given arbitrary completely additive function of point sets $\Phi(e)$, in an open finitely connected region Σ. The principal solution is a "potential function of its vector derivative or gradient," belongs to the class (ii) of the author's book on *Logarithmic Potential*, and takes on zero boundary values almost everywhere "in the narrow sense." Both types of the first boundary value problem are uniquely solvable in their respective classes of functions. The corresponding problems for the sphere may also be solved. By means of integral equations, other equations of elliptic type may be treated, as in the Cambridge Colloquium. The given equation more or less reduces to Poisson's equation if $\Phi(s)$ or $\Phi(e)$ is assumed to be absolutely continuous.

61. Dr. C. F. Roos: *A mathematical theory of depreciation and replacement.*

In as much as a machine is usually replaced by another machine before its useful life has ended, it is quite important to consider the problem of determining the time at which a machine in operation should be replaced by another machine whose operating expense is different so that a maximum profit is obtained for some period of time extending from an initial time t_1

through a replacement time ω to some final time t_2 greater than ω. When formulated as a problem in the calculus of variations, this replacement problem is a type of Lagrange problem with variable end points and discontinuous integrand. In this paper a general formulation is made of a replacement problem and a special example of an actual economic situation is solved in some detail.

62. Dr. C. F. Roos: *The problem of depreciation in the calculus of variations.*

This paper will appear in full in an early number of this Bulletin.

63. Professor W. J. Trjitzinsky: *Expansion in series of non-inverted factorials.*

Using Cauchy's result $f(z)=[1/(2\pi i)]\int_C [f(t)/(z-t)]dt$ and $\int_0^1 u^n(1-u)^v du=(n!)/[(v+1)\cdots(v+n+1)]$, we expand $1/(z-t)$ in a series of direct factorials in z uniformly convergent when $R(t)<-1$. It follows that if $f(z)$ is analytic on and outside a closed contour C situated to the left of $R(t)=-1$, then for $R(z)>0$,

$$f(z)=a_0+a_1 z+a_2 z(z-1)+\cdots+a_n z(z-1)\cdots(z-n+1)+\cdots,$$

where $a_n=-[1/(2\pi i)]\int_C [f(t)/(t(t-1)\cdots(t-n))]dt$, and $|a_n|<h/(n!)$. This can be generalized for any position of C.

64. Professor W. J. Trjitzinsky: *A class of infinite products and an application to the theory of the gamma function.*

We consider functions

$$La_i(z)=\frac{e^{-\alpha z}}{z}\prod_{n=1}^{\infty} e^{z/a_n}\left(1+\frac{z}{a_n}\right)^{-1},$$

where α is analogous to the Eulerian constant, and the a_n are under certain restrictions; these functions reduce to the gamma function when $a_n=n$. A multiplication theorem for products $La_i(z)La_i(z+p_1)\cdots La_i(z+p_n)$ is obtained, where the p_i satisfy certain inequalities. When $a_i=i$, this theorem yields a multiplication theorem for the gamma function that is disctinct from the classical theorem.

65. Dr. W. L. Ayres: *Concerning continuous curves in n dimensions.*

In this paper the following theorems which have been proved previously for two dimensions are proved for n dimensions: (1) in order that a point P of a continuous curve M be an end point of M it is necessary and sufficient that P be a non-cut point which lies on no simple closed curve of M; (2) in order that a continuous curve be cyclically connected, it is necessary and sufficient that it contain no cut point; (3) in order that there exist a point which separates the points A and B in the continuous curve M it is necessary and sufficient that every two arcs of M whose end points are A and B have an interior point of both in common.

66. Professor L. E. Dickson: *Complete solution of the Waring-Kamke problem on sums of integral values of quadratic functions.*

Let $q(x)$ be an integer $\geqq 0$ for every integer $x \geqq 0$. For every $s \geqq 5$, there is found the least integer e_s such that every positive integer is a sum of s values of $q(x)$ and e_s numbers 0 or 1. There is deduced the least value L of $s+e_s$ for $s \geqq 5$. Next, it is shown that $4+e_4 \geqq L$ except for specified special cases, and then $4+e_4 = L-1$. Except for these special cases, it is not necessary to carry out the very difficult work of finding the exact value of e_4. By the new method developed, it is relatively simple to find each e_s for $s \geqq 5$. The memoir giving the complete solution of this problem will appear in the American Journal.

67. Professor J. V. Uspensky: *On the number of representations of integers by certain ternary quadratic forms.*

Denoting by (a, b, c) the form $ax^2+by^2+cz^2$, the author gives the expression for the number of representations of integers by the following forms: $(a, b, c) = (1, 2, 3)$, $(1, 2, 6)$, $(2, 3, 6)$, $(1, 3, 6)$, $(1, 1, 3)$, $(1, 3, 3)$, $(1, 1, 6)$, $(1, 6, 6)$, $(1, 1, 5)$, $(1, 5, 5)$, $(1, 2, 5)$, $(1, 5, 10)$, $(1, 2, 10)$, $(2, 5, 10)$.

68. Professor J. V. Uspensky: *On the connections between numbers of representations of integers by two totally different quadratic forms.*

For certain couples of quadratic forms involving three and four variables without any apparent relationship to each other, there exist unexpected relations between the numbers of representations of integers by these forms. The author gives several instances of this rather remarkable phenomenon.

69. Professor Arnold Emch: *On the mapping of the sextuples of the symmetric substitution group G_6 in a plane upon a quadric.*

This paper appears in full in the present number of this Bulletin.

R. G. D. Richardson, *Secretary.*

THE SECOND MADISON COLLOQUIUM

The eleventh colloquium of the American Mathematical Society and the second Madison Colloquium was held at Madison, Wis., in conjunction with the thirty-third summer meeting of the Society from September 6-10, 1927, the lecturers being Professor E. T. Bell, California Institute of Technology, and Professor Anna Pell Wheeler, Bryn Mawr College. The first Madison Colloquium was held in 1913, the lecturers being Professors L. E. Dickson and W. F. Osgood.* In opening the colloquium, President Snyder made the observation that this was the first colloquium of the Society at which a woman had been invited to give a course of lectures. Also that geographically the East and West were represented in the speakers, the Middle West in the place of meeting, and the South in a recent addition to the volumes of the colloquium series.

The lectures were divided as is customary into two groups of five lectures each, and were given from Tuesday to Saturday in the auditorium of Sterling Hall. One hundred twenty-seven persons attended these lectures, the largest number registered for any colloquium so far held, though in comparison with the attendance at Ithaca in 1924 (122) the gradient seems to be on the decrease. The names of those attending the lectures follows.

L. K. Adkins, F. E. Allen, Altshiller-Court, W. E. Anderson, Archibald, R. W. Babcock, Barnett, L. Battig, E. R. Beckwith, E. T. Bell, Birkhoff, Blichfeldt, Bray, Brenke, Brink, E. W. Brown, H. E. Buchanan, L. H. Bunyan, Bussey, Cairns, C. C. Camp, Carlson, G. N. Carmichael, R. D. Carmichael, Chang, Chittenden, J. T. Colpitts, Conkwright, C. F. Craig, H. V. Craig, Crathorne, Curry, Dalaker, H. T. Davis, Dean, Dickson, Dines, Dowling, Dresden, J. M. Earl, G. C. Evans, H. P. Evans, Feinler, W. B. Ford, Fry, Gaba, Gergen, Gill, D. C. Gillespie, W. C. Graustein, M. C. Graustein, L. M. Graves, Griffiths, V. G. Grove, Guggenbühl, Harkin, E. Hart, W. L. Hart, Hartung, E. R. Hedrick, Hildebrandt,

* A resumé of the first ten colloquia of the Society was given in connection with the report of the second Ithaca Colloquium, this Bulletin, vol. 32 (1926), pp. 25–27.

Hoersch, Hollcroft, F. Hopkins, Hosford, Hotelling, J. C. Hughes, L. Ingold, Ingraham, D. Jackson, Jensen, Kearney, Kempner, Krathwohl, E. P. Lane, Langer, Latimer, Logsdon, J. V. McKelvey, March, Maria, A. S. Merrill, N. H. Mewaldt, Michal, Mickelson, Miser, E. J. Moulton, Olson, F. W. Owens, H. B. Owens, Palmer, Parkinson, Pettit, Pierce, R. G. D. Richardson, H. L. Rietz, W. C. Risselman, Roever, Roos, Roth, Runge, R. G. Sanger, Shewhart, Shohat, Sinclair, Skinner, Slaught, Slichter, V. Snyder, W. A. Spencer, Stouffer, Swartzel, Tate, J. H. Taylor, E. M. Thomas, Torrey, Trjitzinsky, J. V. Uspensky, Van Vleck, Wall, W. Weaver, Westfall, A. P. Wheeler, Widder, Worthington, Yanney, Yeaton.

Below are given the synopses of the lectures as prepared by the lecturers in the order in which they were given:

ALGEBRAIC ARITHMETIC

By Professor E. T. Bell

I. *Definition and scope of algebraic arithmetic.* Arithmetical theories as opposed to algebraic or analytic. The algebraic varieties useful in arithmetic; irregular fields, rings, rays, semigroups; the algebras of infinite vectors.

II. *The algebra of unique factorization.* Applications to an arithmetic of functions defined through any unique factorization law.

III. *The algebra of parity.* Definition of parity functions. Division of parity and equality of functions.

IV. *Algebraic arithmetic of multiply periodic functions.* Arithmetical identities between arbitrary functions having parity. Applications to theorems on representations of integers in higher forms. Applications to quadratic forms and class number relations. Singly infinite class number relations.

V. *Arithmetical structure.* Arithmetization of an arithmetizable theory. Solution in terms of symbolic logic of certain of the postulate systems of rational arithmetic.

THE THEORY OF QUADRATIC FORMS IN INFINITELY MANY VARIABLES AND APPLICATIONS

By Professor Anna Pell Wheeler

I. *Spaces of infinitely many dimensions.* Vectors. Fundamental quadratic form. Scalar product. Orthogonality.

Construction of coordinate systems corresponding to a given fundamental form. Properties of coordinate systems in euclidean spaces. Properties of coordinate systems in non-euclidean spaces. Connection between spaces of different types and classes of functions. General coordinate systems.

II. *Infinite matrices and linear transformations.* Matrices, bilinear forms, and linear transformations. Limitedness with respect to Hilbert space. Limitedness with respect to other spaces. Complete continuity in the Hilbert sense. Extension. Infinite determinants. Some sufficient conditions for the alternative theorems. The Fredholm determinant. Principal solutions of homogeneous equations.

III. *Reduction of quadratic and bilinear forms.* Nature of invariants of linear transformations in the symmetric case. A necessary condition for the existence of invariants in the symmetric case. Carleman's sufficient condition and extension in the symmetric case. Symmetrizable cases. Semi-symmetrizable cases. Simultaneous reduction of two quadratic forms. Solutions of more general linear equations.

IV. *Application to the theory of linear functional equations.* Integral equations with continuous symmetric kernels. Integral equations with singular symmetric kernels. Symmetrizable kernels. Semi-symmetrizable kernels. Integro-differential equations. Systems of integral equations. Linear functional equations of various types.

V. *Application to the theory of linear differential equations.* Ordinary linear differential equations, orthogonal and polar. Singular linear differential equations. Elliptic differential equations. Hyperbolic differential equations. The Bliss system of differential equations.

Both sets of lectures will be published in book form by the Society. Professor Bell's will appear in the fall of 1927, and Professor Wheeler's probably in 1928.

T. H. Hildebrandt

AN ANALYSIS OF SOME GENERAL PROPOSITIONS*

BY C. H. LANGFORD

1. *Introduction.* General propositions are commonly described as being those propositions which arise from matrices by generalization, that is, as being such propositions as can be derived from some matrix by the attachment of an applicative, "some" or "every," to each variable constituent of the matrix.† In this paper an analysis of general propositions is suggested, which results from a slightly different view of the relations of general propositions to matrices. It appears that the analysis which is suggested involves a generalization of the ordinary analysis.

2. *Unanalysed Propositions.* We may begin with a consideration of elementary matrices whose values are elementary functions of unanalysed propositions, such as, for example, $p \supset q$. Let t_0 denote any elementary proposition. Then we can write, for example, $(t_0).t_0 \vee \sim t_0$, that is, every elementary proposition is true or false. If p_0, q_0 denote elementary propositions, then, of course, $p_0 \supset q_0$ denotes elementary propositions; but this latter, more complex function denotes a narrower range of propositions than does t_0, since whatever is an elementary proposition of the form $p_0 \supset q_0$ must be an elementary proposition of the form t_0, but not conversely. Let t_1 denote any elementary matrix. Then t_1 denotes what denotes elementary propositions; t_1 denotes t_0, $p_0 \supset q_0$, and the like, which denote elementary propositions. Now we can form such functions as $p_1 \supset q_1$, which denote a narrower range of matrices than does t_1.

Since elementary matrices are neither true nor false, we cannot say $(t_1).t_1 \vee \sim t_1$; but we can say

$$(t_1):(t_0).t_0 \vee \sim t_0,$$

* Presented to the Society, September 9, 1927.

† See *Principia Mathematica*, second edition, vol. 1, p. xxiii.

that is, every value of every matrix is true or false. Variables such as t_0, t_1 fall into a hierarchy of values and values of values. We can write such propositions as $(t_1):(\exists t_0).t_0$, that is, some value of every matrix is true (which is false), and $(\exists t_1):(t_0).t_0$, that is, all values of some matrix are true (which is true); and we have, of course,

$$(t_1)(t_0).t_0.\vee.(\exists t_1)(\exists t_0).\sim t_0,$$
$$(t_1)(\exists t_0).t_0.\vee.(\exists t_1)(t_0).\sim t_0,$$

and the like. Consider the proposition

$$(t_1).(t_0)t_0\vee(\exists t_0)\sim t_0,$$

that is, every matrix is such that all of its values are true or at least one of its values is false. This proposition, although a consequence of the proposition $(p).p\vee\sim p$, cannot be stated without the use of some variable such as t_1. We have also, of course,

$$(t_2):(t_1)(\exists t_0).t_0.\vee.(\exists t_1)(t_0).\sim t_0,$$

which is different from

$$(t_1)(\exists t_0).t_0.\vee.(\exists t_1)(t_0).\sim t_0;$$

and it would seem that the proposition

$$(\exists t_2):(t_1)(\exists t_0).t_0.\vee.(\exists t_1)(t_0).\sim t_0$$

is not equivalent to any proposition which can be stated without the use of some variable such as t_2.

3. *Values of Functions and Species of Functions.* There are two ways in which inferences can be drawn from universal propositions of the kind with which we are concerned: we can replace a universally quantified function by one of the values which it denotes, and we can replace a universally quantified function, as genus, by a more determinate function, as species.* Thus,

$$(t_1).(\exists t_0)t_0\vee(t_0)\sim t_0$$

* See *Principia Mathematica*, second edition, vol. I, p. xxiii.

implies

$$\exists(p_0 \vee q_0) . p_0 \vee q_0 . \vee . (p_0 \vee q_0) . \sim p_0 . \sim q_0,$$

that is,

$$(\exists p_0, q_0) . p_0 \vee q_0 . \vee . (p_0, q_0) . \sim p_0 . \sim q_0.$$

Again,

$$(t_1)(\exists t_0) . t_0 \vee \sim t_0$$

implies

$$(p_1 \vee q_1)\exists(p_0 \vee q_0) : p_0 \vee q_0 . \supset . p_0 \vee q_0,$$

that is,

$$(p_1, q_1)(\exists p_0, q_0) : p_0 \vee q_0 . \supset . p_0 \vee q_0.$$

It is to be noted that *genus* and *species* are to be understood in intension: being a function of the form $p_1 \vee q_1$ entails being a function of the form t_1; and this is the ground of the implication. There are two ways in which particular propositions can be inferred: we can infer a particular proposition from an instance of the function which is quantified particularly, and we can infer a particular proposition from a more determinate particular proposition. Thus, $(t_0) . t_0 \vee \sim t_0$ implies $(\exists t_1)(t_0) . t_0$; and $\exists(p_0 . q_0) . p_0 . q_0$ implies $(\exists t_0) . t_0$.

4. *Analysed Propositions.* Heretofore we have been concerned with matrices whose values are elementary functions of unanalysed propositions, and we have illustrated the way in which a hierarchy of functions can be formed in this connection. Thus we may have

$$p_2 \vee q_2,$$

which has as a typical value

$$p_1' . p_1'' . \vee . q_1' . q_1'',$$

which has as a typical value

$$r_0' \supset s_0' . r_0'' \supset s_0'' . \vee . t_0' \supset u_0' . t_0'' \supset u_0'',$$

which has as a value any elementary proposition of this form.* I wish to explain how a similar hierarchy can be formed in connection with analysed propositions. A propo-

* See *Principia Mathematica*, loc. cit., p. xxxi, etc.

sition is a value of a function if the function can be obtained by replacing constituents of the proposition by appropriate variables. We are to consider propositions which are values of relational functions, such as the proposition "a gives b to c." This proposition is a value of each of the functions "a gives b to x," $f(a, b, c)$, $f(a)$, $f(x)$, $f(x, y, z)$, among others. Commonly, one and the same proposition is open to various analyses. If we consider an appropriate constituent of a relational proposition, two entities appear which can be replaced by variables, namely this constituent and the remainder of the proposition. It is, however, often difficult to decide what are the constituents of a proposition which can be replaced by variables.

Now a proposition of the form Rab is a value of the function Ray; and it is clear that Ray can be obtained from Rxy by replacing x by a. We may indicate this order of denoting by writing Rx_1y_0; so that Rx_1y_0 denotes Ray_0, which denotes Rab. Subscripts indicate parameters of various orders, and accordingly they indicate the order of denoting; so that the subscript which attaches to a variable determines the point in the hierarchy at which the variable takes values. Thus a function $f(x, y, z)$ is ambiguous as regards the parametric order of its variable constituents. We may have $f_1(x_0, y_0, z_0)$, which denotes "x_0 gives y_0 to z_0," which denotes "a gives b to c"; or, we may have $f_0(x_1, y_1, z_1)$, which denotes $f_0(a, b, c)$, which denotes "a gives b to c"; and so on. It is clear that, as a result of this analysis, we can attach an applicative, "some" or "every," to the entire function at each stage of the hierarchy. If we apply "some" to $f_1(x_0, y_0, z_0)$ and "every" to $f_0(x_0, y_0, z_0)$, we have

$$\exists f_1(x_0, y_0, z_0) : (f_0(x_0, y_0, z_0)) . f_0(x_0, y_0, z_0),$$

which corresponds to $(\exists f) : (x, y, z) . f(x, y, z)$ in the ordinary analysis; whereas, if we apply "some" to $f_0(x_1, y_1, z_1)$ and "every" to $f_0(x_0, y_0, z_0)$, we have

$$\exists f_0(x_1, y_1, z_1) : (f_0(x_0, y_0, z_0)) . f_0(x_0, y_0, z_0),$$

which corresponds to $(\exists x, y, z) : (f) . f(x, y, z)$.

Now although the proposition $(f_1x_0):(f_0x_0).f_0x_0$ corresponds to $(f):(x).fx$, there is an important difference between these propositions which I wish to point out. When we say $(f):(x).fx$, it is presupposed that each value of x can be combined with every value of f to form a significant proposition; whereas, no such presupposition as this is involved when we write $(f_1x_0):(f_0x_0).f_0x_0$. If there should be values of f_0, say f', f'', and values of x_0, say x', x'', such that $f'x'$ and $f''x''$, but such that $f'x''$ and $f''x'$ are non-significant, this fact would not render $(f_1x_0):(f_0x_0).f_0x_0$ either false or non-significant. A matrix is to be regarded as taking such values as exhibit the form of the matrix; and the values taken by a variable constituent of a matrix depend upon the matrix in which the variable occurs. It is therefore possible that $(f_1x_0):(f_0x_0).f_0x_0$ should have a wider reference than $(f):(x).fx$.

5. *Analysed and Unanalysed Propositions.* I wish now to consider functions whose values are elementary functions of analysed propositions, and to inquire how, if at all, the denotative hierarchy for elementary functions of unanalysed propositions can be combined with the hierarchy for analysed propositions. We have such propositions as $(f_1x_0)(\exists f_0x_0).f_0x_0$ and $(\exists f_0x_1)(f_0x_0).f_0x_0$. Now it appears that we can extend the range of t_0, t_1, $\cdots$ so as to include such functions as f_0x_0, f_1x_0, $\cdots$. This can be seen to be possible by noting that f_0x_0 is a species of t_0, and that f_1x_0 is a species of t_1. An analysed function will have the order of its constituents of highest parametric order. It is clear that, for example, $(f_0x_0).f_0x_0 \vee \sim f_0x_0$ follows from $(t_0).t_0 \vee \sim t_0$, and that $(f_0x_0)f_0x_0 \vee (\exists f_0x_0)\sim f_0x_0$ follows from $(t_0).t_0 \vee (\exists t_0)\sim t_0$; and also that $(\exists f_0x_0)f_0x_0$ entails $(\exists t_0)t_0$. Consider the proposition

$$(a) \qquad (t_1).(\exists t_0)t_0 \vee (t_0)\sim t_0.$$

If we choose f_1x_0 as a species of t_1, (a) is seen to entail

$$(f).(\exists x)fx \vee (x)\sim fx\ ;$$

but if we choose f_0x_1 as a species of t_1, (a) is seen to entail

$$(x).(\exists f)fx \vee (f)\sim fx.$$

Functions of any number of variables can replace t_1, but, in any case, doubly-quantified propositions result. Of course, we can assign values to t_1, instead of replacing it by species. We can, for example, substitute fx_0 (where f is constant), so that we have

$$(\exists x)fx \vee (x)\sim fx\ ;$$

or, if we substitute f_0a, we have

$$(\exists f)fa \vee (f)\sim fa.$$

This analysis gives us a certain advantage as against the ordinary analysis. In the ordinary analysis, we can write $(p).p \vee \sim p$, and derive, as a species, $(f,\ x).fx \vee \sim fx$; but this is confined to singly-quantified propositions, that is, to propositions involving a single applicative, "some" or "every"; whereas, we are now able to write, say, $(p_n) \cdot \cdot \cdot (\exists p_0).p_0 \vee \sim p_0$, which is a proposition involving $n+1$ applicatives.

6. *Multiply-Quantified Propositions.* Let $F(t_0)$ denote an elementary function of t_0. Then it is clear that $(t_0).F(t_0)$ implies $(t_1):(t_0).F(t_0)$. Similarly, $(t_1):(t_0).F(t_0)$ implies $(t_2):.(t_1):(t_0).F(t_0)$. Now $(\exists t_0).F(t_0)$ can be obtained from $(t_0).F(t_0)$; and generally, in a function of the form $(t_n) \cdot \cdot \cdot F(t_0)$, we can turn any universal variable into a particular variable. Accordingly, any proposition on $F(t_0)$, of whatever degree of quantification, can be obtained from $(t_0).F(t_0)$.

In the argument of the last paragraph, we have used $F(t_0)$ to denote an elementary function of t_0; but I should like to point out that it is not at all necessary to employ such expressions as $F(t_0)$. The proposition $(\exists F):(t_0).F(t_0)$, for example, expresses what is expressed by $(\exists t_1):(t_0).t_0$. We can express the proposition

$$(F):(t_0).F(t_0).\supset.(t_1)(t_0).F(t_0)$$

in the following way. Note that $(\exists t_0).t_0$ is equivalent to $(\exists t_1)(\exists t_0).t_0$; so that we can write

$$(t_2):(\exists t_1)(\exists t_0).\sim t_0.\vee.(t_1)(t_0).t_0.$$

Mr. Russell has shown how elementary functions of propositions that are not elementary can be derived from elementary matrices.* Thus, $(x).fx \vee \sim fx$ implies $(x)(\exists y).fx \vee \sim fy$, which is equivalent to $(x)fx \vee (\exists y)\sim fy$. In precisely the same way,

$$(t_1)(t_0).t_0 \vee \sim t_0$$

implies

$$(t_1):(t_0)(\exists t_0').t_0 \vee \sim t_0',$$

which is equivalent to $(t_1).(t_0)t_0 \vee (\exists t_0)\sim t_0$, which implies

$$(t_1)(\exists u_1).(u_0)u_0 \vee (\exists t_0)\sim t_0,$$

which is equivalent to

$$(\exists t_1)(t_0)t_0 \vee (t_1)(\exists t_0)\sim t_0.$$

It is clear that elementary functions of propositions of the first and second orders can be derived in this way, whatever the degree of quantification of these propositions. Moreover, from $(t_1).(\exists t_0)t_0$ we derive both $(\exists x)fx$ and $(\exists f)fa$, although one of these latter propositions would be said to be of the first order and the other of the second order. Furthermore, in a proposition such as $(t_0)t_0$ there seems to be no reason for restricting t_0 to elementary propositions; it can denote propositions of the first and second orders as well. Let "$(\exists x).a$ gives b to x" be denoted by $f(a, b)$. Then

$$(\exists a, b)f(a, b) \vee (a, b)\sim f(a, b)$$

results from

$$(t_1).(\exists t_0)t_0 \vee (t_0)\sim t_0.$$

It does not matter that f is not elementary, since it occurs as an unanalysed constituent.

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* See *Principia Mathematica*, loc. cit., *8.

ON THE POLYNOMIAL CONVERGENTS OF POWER SERIES*

BY W. M. WHYBURN

In this paper we will consider the power series

$$\sum_{i=0}^{i=\infty} a_i z^i$$

with a unit radius of convergence. M. B. Porter† defined a set of convergents for this series in such a way that the set had many interesting properties.‡ He proved that at least one point of the unit circle was a limit point of the zeros of these convergents, and that in the neighborhood of every point of this circle some of the convergents took on values whose moduli were arbitrarily small. Jentzsch§ has shown that every point of the unit circle is a limit point of the zeros of the complete set $\{f_n(z)\}$ of convergents defined by

$$f_n(z) = \sum_{i=0}^{i=n} a_i z^i$$

for all positive integral values of n. It is the purpose of the present paper to show that the set of convergents defined by Porter has every point of the unit circle as a limit point of its zeros. As a special case of this result we get the theorem of Jentzsch. Other properties of these convergents are developed also.

Let $f(z)$ be the analytic function defined by the above power series inside of the circle $|z| = 1$ and let z_0 be a point

* Presented to the Society, April 15, 1927.

† Annals of Mathematics, vol. 8 (1906–7), pp. 189–192.

‡ Of extreme importance is the fact that Porter embodied in this set of convergents all of the properties that were later assigned by Montel (see his *Leçons sur les Séries de Polynomes etc.*, Paris, 1910) to his *normal families* of functions.

§ Acta Mathematica, vol. 41 (1917), pp. 253–270.

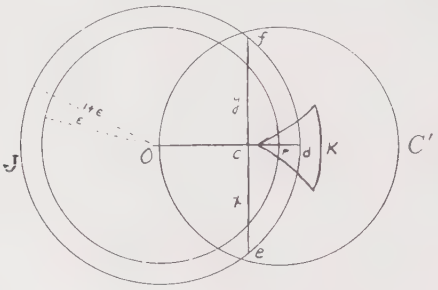
such that $|z_0|>1$. Let $\{P_{n_i}(z)\}$ be the set of convergents defined by

$$P_{n_i}(z) = a_{n_i}z^{n_i}[a_0z^{-n_i}/a_{n_i} + a_1z^{-n_i+1}/a_{n_i} + \cdots + 1],$$

where $|a_{n_i}z_0^{n_i}|$ is greater than any of the expressions $|a_jz_0^j|$, for $j=0, 1, 2, \cdots, n_i-1$.

THEOREM I. *Every point of the circle* $|z|=1$ *is a limit point of zeros of the polynomials of the set* $\{P_{n_i}(z)\}$.

PROOF. Assume the theorem false; then there exists a point p on the unit circle and a circle C' with its center at p such that there are no zeros of $f(z)$ or $\{P_{n_i}(z)\}$ within or on C'. Draw the chord xy within C' cutting the arc of $|z|=1$



that lies within C' in such a way that the point p is the midpoint of the intercepted sub-arc of this arc. Choose a positive number ϵ so that the circle $J: |z|=1+\epsilon$ cuts this chord (produced) at two points e and f, both of which lie within C'. Porter* has shown that we can pick a sub-sequence $\{P_{n_j}(z)\}$ of $\{P_{n_i}(z)\}$ such that for all n_j greater than a fixed index N_1, $P_{n_j}(z)$ has all of its zeros except a fixed finite number, m', inside of the circle J. By Hurwitz'† theorem we can determine an index N_2 so that for all $n_j>N_2$, the m' zeros of $P_{n_j}(z)$ that are not within J will lie within a set G of circles whose centers are at m' fixed points‡ and whose

* Loc. cit., p. 191.

† Acta Mathematica, vol. 40 (1916), pp. 179–183.

‡ These points are the zeros of the analytic function defined by $\{\overline{P}_{n_j}(z)\}$ outside of J, where $P_{n_i}(z)=a_{n_i}z^{n_i}|\overline{P}_{n_i}(z)|$.

radii are equal to a given positive constant ϵ'. We fix ϵ' so that none of the circles of the set G will have anything in common with the region C, where C lies within C' and is bounded by the arc edf and the chord ecf (c and d are the points in which the ray Op cuts the chord ef and the arc ef, respectively). Since $f(c)$ is different from zero, let $|f(c)|=2k$ and since $\lim_{n_j\to\infty}P_{n_j}(c)=f(c)$ uniformly, choose an index N_3 so that for every $n_j>N_3$, $|P_{n_j}(c)|>k$.

We determine* a region K that lies within C', contains p and infinitely many points outside of J, has nothing in common with any of the circles of G, and such that the distance from any point of the interior of J that does not belong to C or the boundary of C to any point of K is as great as the distance from that point to c. Let a positive number m be chosen so that the distance from any point of K to any of the circles of the finite set G is not less than m, and let another positive number M be chosen so that none of the above distances are greater than M.

By Porter's theorem, we can find an index N that is greater than N_1, N_2, N_3, such that $|P_N(s)|<m^{m'}\,k/(4M^{m'})$, where $P_N(z)$ belongs to $\{P_{n_j}(z)\}$ and s is a point of K. Let z_1, z_2, $\cdots$, $z_{m'}$ be the zeros of $P_N(z)$ that are not within J and let $z_{m'}+1$, $\cdots$, z_N be the remaining zeros of this function;

$$P_N(z) = a_0(z-z_1)(z-z_2)\cdots(z-z_{m'})\cdots(z-z_N).$$

$$\begin{aligned}(1)\qquad &|P_N(s)|\\ &=|a_0|\,|(s-z_1)|\cdots|(s-z_{m'})|\,|(s-z_{m'+1})|\cdots|(s-z_N)|\\ &\geqq m^{m'}|a_0|\,|(s-z_{m'+1})|\cdots|(s-z_N)|\end{aligned}$$

$$(2)\qquad |P_N(c)|\leqq M^{m'}|a_0|\,|(c-z_{m'+1})|\cdots|(c-z_N)|.$$

Since s is in K and $z_{m'+1}$, $\cdots$, z_N are all in J, we have

$$(3)\quad |(s-z_{m'+1})|\cdots|(s-z_N)|\geqq|(c-z_{m'+1})|\cdots|(c-z_N)|$$

* Such a region can be constructed by using the boundaries of the set of circles through c with centers within J but not within C'.

and upon combining (1), (2), and (3), we get

$$|P_N(s)| \geqq \frac{m^{m'}}{M^{m'}} |P_N(c)| > m^{m'}k/M^{m'},$$

since $|P_N(c)| > k$. This contradicts the fact that $|P_N(s)| < m^{m'}k/(4M^{m'})$ and thus yields our theorem.

COROLLARY I. *Every point of the circle* $|z| = 1$ *is a limit point of the zeros of* $\{f(z)\}$.

Porter* gave an example which exhibited a set of convergents that converged over a region that extended outside of the circle of convergence of the power series. This example shows that the property that we have established for our convergents does not necessarily belong to every infinite subset of $\{f_n(z)\}$.

In conclusion, we point out that in the neighborhood of the circle $|z| = 1$ the moduli of some of the convergents of the set $\{P_{n_i}(z)\}$ will take on any arbitrarily assigned positive value. In this respect the circle of convergence behaves very much like an essential singularity. However, there is one point of difference in that there are no Picard numbers associated with the circle of convergence.

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* Loc. cit., p. 191. Ostrowski (Journal of London Society, vol. 1 (1926), part 4) has recently made an extensive study of this "over-convergence." Ostrowski gives Jentzsch credit for having called attention to this property while in reality this credit should go to Porter, whose work antedates that of Jentzsch by ten years.

THE BINOMIAL QUARTIC AS A NORMAL FORM*

BY RAYMOND GARVER

From the standpoint of the Tschirnhausen transformation, the quartic equation may be said to occupy a distinct position, between the quadratic and cubic on one hand, and the higher degree equations on the other. For the lower degree equations can be reduced easily to binomial form by simple linear or quadratic transformations. Such reduction is not, in general, possible; but, for equations of degree not less than five, the second, third, and fourth terms can be removed by a fourth degree transformation without solving any equations of degree greater than the third.† (For the quintic such a transformation leads to the Bring-Jerrard normal form $x^5+a_4x+a_5=0$.) Now these two problems, reduction to binomial form, and removal of three intermediate terms, become equivalent for, and only for, the quartic equation. Hence the question of the reducibility of the quartic to the form $x^4+a_4=0$ is one of some interest.

It is worth while mentioning that none of the known methods for removing three terms apply to the quartic. They require, as mentioned above, fourth degree transformations, and fail when the given equation is of only that degree.‡

An examination of the literature shows that the question of the reducibility of the general quartic to binomial form by means of a Tschirnhausen transformation has been considered

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* Presented to the Society, October 29, 1927.

† See Dickson, *Modern Algebraic Theories*, 1926, Chap. XII; or almost any higher algebra.

‡ Salmon, *Modern Higher Algebra*, 4th ed., 1885, p. 250, and Matthiessen, *Grundzüge der Antiken und Modernen Algebra*, 1878, p. 125, state *incorrectly* that three terms can be removed from a quintic (or higher degree equation) by a third degree transformation, without solving any equations of degree greater than three.

by two men, and that they have stated different results. Lagrange, in his well known *Réflexions sur la Résolution Algébrique des Équations*,* gives an à priori proof that the reduction is possible. If we consider

$$x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0, \tag{1}$$

and the transformation

$$y = x^3 + k_2x^2 + k_3x + k_4, \tag{2}$$

the conditions that the new equation in y lack all intermediate terms are, as well known,

$$\sum y = 0, \quad \sum y^2 = 0, \quad \sum y^3 = 0,$$

where the summations extend over the four roots of (1). The equations of condition are of degree 1, 2, and 3 respectively in the parameters k_2, k_3, k_4, and if we eliminate any two of them, say k_3 and k_4, we shall arrive at a sixth degree equation. What Lagrange does, by an elegant though somewhat lengthy process, is to show that this sixth degree equation (when the original equation is a quartic) factors into three quadratics whose coefficients are themselves roots of cubic equations.

Over a hundred years later Sylvester, apparently unaware of this part of Lagrange's work, stated the opposite conclusion. On page 548 of his paper *On the so-called Tschirnhausen transformation*†, he says "Five is the minimum degree of equation from which three terms can be removed without solving an equation above the third degree," and a similar statement had been made earlier by Hamilton.‡ However, the proof of Sylvester is incomplete, or rather he is considering a slightly different problem, that of deter-

* Published in 1770–71. See his Oeuvres, vol. III, Paris, 1869, especially pp. 284–295.

† Journal für Mathematik, vol. 100 (1887), pp. 465–86. Collected Papers, vol. 4, Cambridge, 1912, pp. 531–49.

‡ Sixth Report of the British Association for the Advancement of Science, 1837, pp. 295–348.

mining in general how many parameters must be present to insure a solution of a system of equations without introducing any elevation of degree. He does not consider the possibility, in special cases, of a factorization such as the one established by Lagrange for the quartic.

I wish to indicate a reduction of the quartic to binomial form entirely different in form from that of Lagrange, and which, making use of some rather lengthy calculations which have been already worked out, seems much simpler. It makes use of a special type of Tschirnhausen transformation introduced by Hermite* and used later by Cayley,† in which the coefficients of the transformed equation are invariants. The use of the special transformation is by no means essential, but is very convenient because the transformed equation has been calculated, and been put in convenient form. Briefly, if we take our quartic in the form

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0\,, \tag{3}$$

and apply the transformation

$$y = (ax + b)B + (ax^2 + 4bx + 3c)C + (ax^3 + 4bx^2 + 6cx + 3d)D, \tag{4}$$

then the second coefficient of the transformed equation will be zero, and the other coefficients will be invariants of the two forms $(a, b, c, d, e)(X, Y)^4$, $(B, C, D)(Y, -X)^2$.‡ The transformed equation has been computed by Cayley and Salmon;§ we do not need its explicit form. However we do need a result stated by Salmon;‖ namely, that the invariant

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* Comptes Rendus, vol. 46 (1858), p. 961; Oeuvres, vol. 2, Paris, 1908, pp. 30–37.

† *On Tschirnhausen's transformation*, Collected Papers, vol. 4, Cambridge, 1891, pp. 375–94. See also Journal für Mathematik, vol. 58 (1861), pp. 263–69.

‡ Cayley, loc. cit.

§ Their notations are different; I have followed Cayley above, but have left (5) in Salmon's notation.

‖ Loc. cit., p. 254.

$ace+2bcd-ad^2-eb^2-c^3$, which he calls T, becomes, for the new equation,

$$T\phi^3 + S^2\Delta\phi^2 + 9\Delta^2 ST\phi + \Delta^3(54T^2 - S^3), \tag{5}$$

where

$$S = ae - 4bd + 3c^2, \qquad \Delta = (2/3)(BD - C)^2, \tag{6}$$

$$\phi = aB^2 + 4bBC + c(2BD + 4C^2) + 4dCD + eD^2.$$

If we denote (5) by T_y, the condition $T_y=0$ is a cubic in ϕ/Δ with known coefficients, which can then be solved. We then have to satisfy an equation of form $\phi/\Delta=r$, which, by (6), can be done by suitable choice of B, C, D. We then have the general quartic reduced to one in which the invariant T is zero.

Now it is known* that any quartic whose invariant T (known in algebraic geometry as the catalecticant) vanishes, can be reduced, by a linear fractional transformation (which is equivalent to a Tschirnhausen transformation), to binomial form. I have carried out the calculation, without any reference to the geometrical side of the problem, and while seemingly rather difficult it can be put in convenient form. Finally, since the product of two Tschirnhausen transformations is itself a Tschirnhausen transformation, we have the desired reduction.

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* Winger, *Introduction to Projective Geometry*, 1923, p. 209.

THE PROBABILITY LAW FOR THE INTENSITY OF A TRIAL PERIOD, WITH DATA SUBJECT TO THE GAUSSIAN LAW*

BY E. L. DODD

1. *Introduction.* Making use of Lord Rayleigh's† probability law for the resultant of n vibrations of fixed and equal amplitude but of chance phase, Schuster‡ found a probability law for the ratio of the square-root of the intensity of a trial period to the constant term of the Fourier development, and suggested the use of this as a criterion for the possible fortuitous nature of results apparently supporting the existence of periods under investigation. However, as periodicities in a sequence may be invariant to a change of origin, it seems desirable to have a criterion based upon intensity alone, which is thus invariant. Such a criterion will be derived in this paper. Some criterion—as Schuster pointed out—is indispensable in periodogram analysis. It signifies little that one trial period is more probable than another if all the apparent periodicities could be easily the result of chance. The question, then, is this : What fluctuations in the intensity may be expected when the data are subject to chance,—or, more specifically, to the Gaussian law?

2. *The Intensity of a Period and its Probability Law.* Suppose that the probability $p(x)$ that X_r will take on a value $\leqq x$ is given by

$$(1) \quad p(x) = \frac{h}{\pi^{1/2}} \int_{-\infty}^{x} e^{-h^2(t-b)^2} dt, \quad (r = 1, 2, \cdots, km\ ;\ km = n),$$

* Presented to the Society, May 7, 1927.

† *On the resultant of a large number of vibrations of the same pitch and of arbitrary phase,* Philosophical Magazine, (5), vol. 10 (1880), pp. 73–78.

‡ *On lunar and solar periodicities of earthquakes,* Proceedings of the Royal Society of London, vol. 61 (1897), pp. 455–465.

with "precision" h. In testing for a period extending over k variates, suppose the data written in m rows of k elements each:

$$\begin{array}{lcccc} & X_1 & X_2 & \cdots & X_k \\ (2) & X_{k+1} & X_{k+2} & \cdots & X_{2k} \\ & \cdot & \cdot & \cdots & \cdot \\ \hline (3)\ \text{Averages} & Y_1 & Y_2 & \cdots & Y_k, \end{array}$$

where

$$Y_j = \frac{1}{m} \sum_{r=0}^{m-1} X_{j+kr}.$$

Taking $\theta = 2\pi/k$, set

$$S = S(k) = \sum_{r=0}^{k-1} Y_{r+1} \sin r\theta, \quad C = C(k) = \sum_{r=0}^{k-1} Y_{r+1} \cos r\theta,$$

and

$$(4) \qquad J = S^2 + C^2, \quad I = \frac{4}{k^2} J.$$

This quantity I will be called the intensity because, if, instead of supposing X_r governed by (1), we let $X_r = c \sin(r\theta + \alpha)$, thus making it strictly periodic, then $I = c^2$, when $k > 2$.

We note that J is a quadratic form, with determinant of elements

$$(5) \qquad a_{rs} = \cos (r - s)\theta, \qquad (r, s = 1, 2, \cdots, k).$$

By an orthogonal transformation, J may be reduced to a sum of squares, whose coefficients are the roots λ of the equation $F(\lambda) = 0$, obtained by subtracting λ from the elements of the main diagonal of the preceding determinant and equating this new determinant to zero. When $k > 2$, there are $k - 2$ zero roots ; since, for any r and θ,

$$(6) \qquad \cos (r - 2)\theta - 2 \cos \theta \cdot \cos (r - 1)\theta + \cos r\theta = 0.$$

Indeed,* to the kth column add the $(k-2)$th column, and subtract the $(k-1)$th column multiplied by $2 \cos \theta$. The kth

*For this suggestion, the author is indebted to Miss Edna McCormick.

column becomes divisible by λ. Then take $r=(k-1)$, $(k-2)$, $\cdots$, 3. But

$$(7)\qquad \left\{\begin{array}{l}(k-1)\cos^2\theta+(k-2)\cos^2 2\theta+\cdots \\ \qquad\qquad +\cos^2(k-1)\theta=\dfrac{k(k-2)}{4}, \quad \theta=\dfrac{2\pi}{k},\end{array}\right.$$

is valid for $k>2$, and with the aid of this, it can be proved that

$$(8)\qquad F(\lambda)=(-1)^k\left\{\lambda^k-k\lambda^{k-1}+\frac{k^2}{4}\lambda^{k-2}\right\}=0,$$

which has $k/2$ as a double root.

In (2), let us suppose the origin changed, and that X_r is now the new value obtained by subtracting b from the original X_r. This leaves S, C, J, and I unchanged, and its effect upon (1) is expressed by setting $b=0$. Using the corresponding new values of Y_r in (3), then, the probability that $y_r<Y_r<y_r+dy_r$ is

$$(9)\qquad \frac{h_1}{\pi^{1/2}}e^{-h_1^2y_r^2}dy_r, \qquad\qquad h_1=hm^{1/2},$$

since by (3), each Y_r is the average of m values of X. By an orthogonal transformation, which does not change ΣY_r^2, we may now by (8) reduce J to

$$(10)\qquad J=\frac{k}{2}U^2+\frac{k}{2}V^2=u^2+v^2,$$

where U and V follow the normal law with the same precision h_1, and u and v with precision $h_1(2/k)^{1/2}$. Then the probability that $z<J<z+dz$ is given by*

$$(11)\qquad h_2^2e^{-zh_2^2}dz, \qquad\qquad h_2^2=2h_1^2/k=2mh^2/k.$$

To obtain the law for I in (5), we need only replace h_2^2 by

$$(12)\qquad H^2=h_2^2k^2/4=kmh^2/2=nh^2/2.$$

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* See Czuber, *Theorie der Beobachtungsfehler*, p. 148. Or Helmert, *Ueber die Wahrscheinlichkeit der Potenzsummen der Beobachtungsfehler*. Zeitschrift für Mathematik und Physik, vol. 21 (1876), pp. 192–218.

THEOREM. *If each of $n=km$ independent variates X_r is subject to the Gaussian law* (1) *with precision h, and if from the averages Y_r of the columns of X's in* (2), *I is formed in accordance with* (4); *then, when $k>2$, the probability $P(z)$ that $I \geqq z$, is given by*

$$P(z) = e^{-nh^2z/2}. \tag{13}$$

In the case of $k=2$,

$$P(z) = \frac{2}{\pi^{1/2}} \int_{z'}^{\infty} e^{-t^2} dt, \qquad z' = \frac{h(nz)^{1/2}}{2}. \tag{14}$$

For the case of $k=2$, the result is easily obtained by rotating axes through 45°, noting that $I=(Y_1-Y_2)^2$.

3. *Application and Discussion.* The averaging process (3) preserves intact a period extending over k items, but tends to smooth out accidental variations and other periods. But this partial smoothing of accidental variations is by no means obliteration. When the value of I in (4) is computed, we ask: is it highly improbable that so large a value would be found if the data contained only chance fluctuations? For example, suppose that the standard deviation σ for 1000 variates under examination is 5, and by the usual rule we take $h^2=1/(2\sigma^2)=0.02$, $nh^2/2=10$. For some particular k, say $k=8$, suppose that $I=1$. By (13) the probability that a chance normal distribution would give as large a value as 1 is $P(1)=e^{-10}=0.000,05$. Normality granted, the evidence would support the belief that a period covering approximately 8 items exists in the data. The exact determination of this period is beyond the scope of this paper.

In the proof, it has not been assumed that S and C are independent. Such an assumption would, indeed, have led more quickly to the result (13). A slight plausibility for this assumption arises from the fact that the "expected" or "mean" value of SC is zero, if independently the variates are subject to the same arbitrary probability law with finite second moment.

THE UNIVERSITY OF TEXAS

CONCERNING CONNECTED AND REGULAR POINT SETS*

BY G. T. WHYBURN

In this paper an extension will be given of a theorem of the author's† which states that if A and B are any two points of a continuous curve M, and K denotes the set of all those points of M which separate‡ A and B in M, then $K+A+B$ is a closed set of points. It will be shown that if A and B are any two points of any connected and regular (connected im kleinen) point set M, and K denotes the set of all those points of M which separate A and B in M, then $K+A+B$ is a closed and bounded set of points. This extended theorem is applied to show that a simple continuous arc may be defined as a connected and regular point set which is irreducibly connected§ between some two of its points.

THEOREM 1. *If A and B are any two points of a connected and regular point set M, and K denotes the set of all those points of M which separate A and B in M, then $K+A+B$ is a closed and bounded set of points.*

PROOF. I shall first show that $K+A+B$ is closed. Suppose, on the contrary, that there exists a point P which does not belong to $K+A+B$ but which is a limit point of K. Then there exists a sequence of points $S=Y_1, Y_2, Y_3, \cdots,$

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* Presented to the Society, San Francisco Section, June 18, 1927.

† G. T. Whyburn, *Some properties of continuous curves*, this Bulletin, vol. 33 (1927), pp. 305–308.

‡ The points A and B of the connected point set M are said to be separated in M by the point X of M provided that $M-X$ is the sum of two mutually separated point sets containing A and B respectively.

§ A connected point set M is said to be irreducibly connected between two of its points A and B provided that no proper connected subset of M contains both A and B. See N. J. Lennes, American Journal of Mathematics, vol. 33 (1911), p. 308. See also Knaster and Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), p. 206.

belonging to K and having P as its sequential limit point. Each point Y_i of this sequence separates M into two mutually separated sets $M_a(Y_i)$ and $M_b(Y_i)$ containing A and B respectively. Either there exists a subsequence W of S such that if Y is any element of W, then $M_b(Y)$ contains at least one point of W, or there does not exist any such sequence.

CASE I. Suppose there exists a subsequence W of S such that if Y is any point of W, then $M_b(Y)$ contains at least one point of W. Let X_1 denote one of the elements of W. Then $M_b(X_1)$ contains at least one point of W. Let X_2 denote one such point of W which belongs to $M_b(X_1)$. Likewise $M_b(X_2)$ contains at least one point X_3 of W, and $M_b(X_3)$ contains at least one point X_4 of W, and so on. This process may be continued indefinitely, giving an infinite subsequence $V=X_1, X_2, X_3, \cdots$, of S. Now since for each positive integer i, $M_a(X_i)+X_i$ is connected and does not contain X_{i+1} [for X_{i+1} belongs to $M_b(X_i)$], therefore $M_a(X_i)+X_i$ is a subset of $M_a(X_{i+1})$. It follows that the points of the sequence V are all distinct, and hence P is the sequential limit point of this sequence.

Now let

$$E = \sum_{i=1}^{\infty} M_a(X_i), \qquad F = M - E.$$

Since for each positive integer i, $M=M_a(X_i)+X_i+M_b(X_i)$, and $M_a(X_i)+X_i$ is a subset of $M_a(X_{i+1})$, it is clear that $U=\sum_{i=1}^{\infty} X_i$ is a subset of E and that F is identical with the set of points common to all of the sets $M_b(X_i)$. Now no point of F, save possibly P (in case P belongs to M), is a limit point of E. For suppose some point Q of F, distinct from P, is a limit point of E. Then since Q is neither a point nor a limit point of U, there exists a circle C having Q as center and which neither contains nor encloses any point of U. But since M is regular at Q, and Q is a limit point of E, there exists a point Z of E which can be joined to Q by a connected subset I of M lying wholly within C. There exists an integer

k such that Z belongs to $M_a(X_k)$. But Q belongs to every set $M_b(X_i)$ and hence belongs to $M_b(X_k)$. But since I is connected and does not contain X_k, it follows that $M_a(X_k)$ and $M_b(X_k)$ are not mutually separated, contrary to hypothesis. Therefore no point of F, save possibly P, is a limit point of E. Furthermore, no point whatever of E is a limit point of F. For if Q is any point of E, then for some positive integer j, $M_a(X_j)$ contains Q, and since $M_b(X_j)$ contains F, and $M_a(X_j)$ and $M_b(X_j)$ are mutually separated, it follows that Q is not a limit point of F.

Now $M=E+F$, and if P does not belong to M, then E and F are mutually separated sets, contrary to the fact that M is connected. And if P does belong to M, then $M-P=E+(F-P)$ is the sum of two mutually separated sets E and $F-P$ containing A and B respectively. Hence P separates A and B in M and therefore belongs to K, contrary to supposition. Thus, in this case, the supposition that $K+A+B$ is not closed leads to a contradiction.

Case II. Suppose S contains no subsequence W such that if Y is any element of W, then $M_b(Y)$ contains at least one point of W. Then no matter what subsequence W of S we take, W contains some point Y such that $M_a(Y)$ contains all the points of W except Y. Hence the sequence S itself contains a point X_1 such that $M_a(X_1)$ contains all the points of S except X_1. Let S_1 be the subsequence of S obtained by omitting the point X_1 from S. Then S_1 contains a point X_2 such that $M_a(X_2)$ contains all the points of S_1 except X_2. Let S_2 be the sequence obtained by omitting from S_1 the point X_2. Then S_2 contains a point X_3 such that $M_a(X_3)$ contains all the elements of S_2 except X_3. This process may be continued indefinitely, giving an infinite sequence of points $V=X_1, X_2, X_3, \cdots$, which is a subsequence of S. For each positive integer i, every point of the sequence S_i belongs to the set $M_a(X_i)$. Then since, for each i, $M_b(X_i)+X_i$ is connected and does not contain X_{i+1} [for X_{i+1} is an element of S_i, and hence belongs to $M_a(X_i)$], therefore $M_b(X_i)+X_i$ is a subset of $M_b(X_{i+1})$. It follows that the

points of the sequence V are all distinct, and hence P is the sequential limit point of this sequence.

Let

$$E = \sum_{i=1}^{\infty} M_b(X_i), \qquad F = M - E.$$

Then by an argument similar to that given under Case I, with the sets $M_a(X_i)$ and $M_b(X_i)$ interchanged, it is shown that in this case the supposition that $K+A+B$ is not closed leads to a contradiction. Therefore the set $K+A+B$ is closed.

I shall now show that the set $K+A+B$ is bounded. If K exists at all, it is clear that there exists a point O not belonging to M. Let T be an inversion of the plane about the point O. As M is regular, then $T(M)$ also is regular. Now $T(K)$ is identically the set of points in $T(M)$ which separate $T(A)$ and $T(B)$ in $T(M)$. Therefore, by the proof given above, $T(K)+T(A)+T(B)$ is a closed set of points. Since this set of points does not contain O, there exists a circle $T(C)$ with O as center and neither containing nor enclosing any point of the set $T(K)+T(A)+T(B)$. Then since the exterior of the circle $T(C)$ is the image of the interior of the circle C, the set of points $K+A+B$ lies wholly within the circle C and therefore is bounded. This completes the proof of the theorem.

THEOREM 2. *If the connected and regular point set M is irreducibly connected between some two of its points A and B, then M is a simple continuous arc from A to B.*

PROOF. Let K denote the set of all those points of M which separate A and B in M. Then since M is irreducibly connected between A and B, it follows by a theorem due to Knaster and Kuratowski* that $K=M-(A+B)$. Hence $K+A+B=M$. But by Theorem 1, $K+A+B$ is closed and bounded. Therefore M is closed and bounded and irre-

* *Sur les ensembles connexes*, loc. cit., p. 219, Theorem 19.

ducibly connected between A and B, and hence satisfies all the conditions of Lennes'* definition of a simple continuous arc from A to B.

The chief interest in the above definition of a simple continuous arc lies in the fact that in its statement nothing is said concerning the closure or the boundedness of the set in question. This result for the case of the arc is related to results obtained by R. L. Wilder† for the case of a simple closed curve. Wilder found that definitions of a simple closed curve could be given in which the closed and bounded conditions were replaced, or modified, by adding the condition of regularity.

THE UNIVERSITY OF TEXAS

NOTE ON THE FOURIER DEVELOPMENT OF CONTINUOUS FUNCTIONS‡

BY A. H. COPELAND

If we have given a function, $f(t)$, which is continuous and periodic, this function is not necessarily developable in a Fourier series, but by a monotone change of variable it becomes so. We shall prove in fact the following theorem.

THEOREM. *If a function, $f(t)$, is continuous and periodic with period $b-a$, then there exists a monotone continuous function, $t=t(\theta)$, transforming the interval $(0, 2\pi)$ into the interval*

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* N. J. Lennes, loc. cit. Hallett has shown that the condition of boundedness in Lennes' definition is superfluous. See G. H. Hallett, Jr., *Concerning the definition of a simple continuous arc*, this Bulletin, vol. 25 (1919), pp. 325–326. The same result was published two years later by Knaster and Kuratowski in their paper *Sur les ensembles connexes*, loc. cit., p. 224, Theorem 27.

† See an abstract of a paper *On the definition of simple closed curve*, this Bulletin, vol. 32 (1926), p. 123. See also p. 591.

‡ Presented to the Society, April 16, 1927.

(a, b), such that $f[t(\theta)] = F(\theta)$ is developable in a uniformly convergent Fourier series in the interval $(0, 2\pi)$.*

Such a transformation does not alter the general character of the function $f(t)$.

The proof of this theorem depends upon a theorem of Fejér† to the effect that if a power series converges to a function $w(z)$ within the unit circle $|z| = 1$, and if $w(z)$ maps conformally the interior of the unit circle onto the interior of a region bounded by a closed Jordan curve, then the power series converges uniformly on the boundary of that circle.

We shall also have need of the following lemma.

LEMMA. *Given a function $\phi(t)$ which is continuous for $a \leqq t \leqq b$, and such that $\phi(a) = \phi(b) = 0$, then there exists a continuous function, $\sigma(t)$, which is of limited variation and such that $\sigma(a) = \sigma(b) = 0$ and $\phi(t) - \sigma(t) \geqq 0$ when $a < t < b$.*

Let m be the minimum of $\phi(t)$ in the interval (a, b). If $m \geqq 0$ we can let $\sigma(t) = 0$. Let us, then, consider the case $m < 0$. Let α and β, respectively, be the lower and the upper bounds of the set of roots of the equation $\phi(t) = m$. Let $\sigma(t)$ be the minimum of $\phi(t)$ in the interval (a, t) when $a \leqq t \leqq \alpha$, let $\sigma(t) = m$ when $\alpha < t < \beta$, and let $\sigma(t)$ be the minimum of $\phi(t)$ in the interval (t, b) when $\beta \leqq t \leqq b$. Then $\sigma(t)$ is seen to be continuous and to satisfy the conditions $\sigma(a) = \sigma(b) = 0$ and $\phi(t) - \sigma(t) \geqq 0$ for $a \leqq t \leqq b$. Moreover, $\sigma(t)$ is monotone in the intervals (a, α) and $\beta, b)$; and hence it is of limited variation in the interval (a, b).

If, instead of the function $\sigma(t)$, we take the function $\rho(t) = \sigma(t) - \sin[\pi(t-a)/(b-a)]$, then $\phi(t) - \rho(t) > 0$ when $a < t < b$. Furthermore $\rho(t)$ is continuous and of limited variation, and $\rho(a) = \rho(b) = 0$.

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* Suggested by G. C. Evans, *The Logarithmic Potential*, Colloquium Series, American Mathematical Society (in press). Chap. VII.

† See L. Fejér, *Über die Konvergenz der Potenzreihe an der Konvergenzgrenze in Fällen der konformen Abbildung auf die schlichte Ebene*, Mathematische Abhandlungen Hermann Amandus Schwarz zu seinem fünfzigjährigen Doktorjubiläum gewidmet.

Let us consider the function $f(t)$. Without loss of generality we may assume that $f(0)=f(2p)=0$, where $2p$ is the period of this function. Let

$$\phi(t) = f(t) - \frac{f(p)}{p}v(t),$$

where

$$v(t) = t, \quad \text{if } 0 \leqq t \leqq p,$$
$$v(t) = 2p - t, \quad \text{if } p \leqq t \leqq 2p.$$

Then $\phi(0)=\phi(p)=\phi(2p)=0$. It follows from the preceding lemma that there exists a function $\rho(t)$ which is of limited variation, continuous, and such that $u(t)=\phi(t)-\rho(t)$ is zero when $t=0$, p, or $2p$, and $u(t)$ is otherwise positive in the interval $(0, 2p)$.

Next let $u(t)=u_1(t)+u_2(t)$, where

$$\begin{cases} u_1(t) = u(t), & \text{if } 0 \leqq t \leqq p, \\ u_1(t) = 0, & \text{if } p \leqq t \leqq 2p; \end{cases}$$

$$\begin{cases} u_2(t) = 0, & \text{if } 0 \leqq t \leqq p, \\ u_2(t) = u(t), & \text{if } p \leqq t \leqq 2p. \end{cases}$$

The pairs of functions $[u_1(t), v(t)]$ and $[u_2(t), v(t)]$ define two closed Jordan curves, C_1 and C_2.

The region interior to C_1 can be mapped conformally onto the interior of the unit circle, $|z|=1$. Between the points of C_1 and the points of the unit circle, there is a one-to-one and continuous correspondence preserving cyclic order.* That is, there exists a continuous monotone function, $t=t_1(\theta)$, relating the points, $z=e^{i\theta}$, of the unit circle to the points, $w=u_1(t)+iv(t)$, of the curve C_1. In particular, the mapping may be accomplished in such a manner that the points $\theta=0$, $\theta=\pi$ correspond respectively to the points $t=0$, $t=p$. Then $t_1(0)=0$ and $t_1(\pi)=p$.

* See C. Carathéodory, *Über die gegenseitige Beziehung der Ränder bei der konformen Abbildung des Inneren einer Jordanschen Kurve auf einen Kreis*, Mathematische Annalen, vol. 73 (1913), pp. 305–320.

Let $w=w_1(z)$ be the mapping function. Then $w_1(z)$ can be expanded in a power series about the origin. This series converges when $|z|<1$. Furthermore, by the theorem of Fejér, this series converges uniformly on the boundary of the circle $|z|=1$. It follows that the function $U_1(\theta)=u_1[t_1(\theta)]$ is developable in a uniformly convergent Fourier series.

In a like manner it can be seen that there exists a function $t=t_2(\theta)$ which is monotone, continuous, and such that $t_2(0)=0$ and $t_2(\pi)=p$, and also such that the function $U_2(\theta)=u_2[t_2(\theta)]$ is developable in a uniformly convergent Fourier series.

The sum $U_1(\theta)+U_2(\theta)$ defines a function $U(\theta)$ which is developable in a uniformly convergent Fourier series in the interval $(0, 2\pi)$, and we shall show that there exists a monotone continuous function $\theta=\theta(t)$ such that $U[\theta(t)]\equiv u(t)$. To do this, let us define a function $t(\theta)$ by the equations

$$\begin{cases} t(\theta) = t_1(\theta), & \text{if } 0 \leqq \theta \leqq \pi, \\ t(\theta) = t_2(\theta), & \text{if } \pi \leqq \theta \leqq 2\pi, \end{cases}$$

and let the inverse of $t=t(\theta)$ be $\theta=\theta(t)$. Then these two functions are continuous and monotone. Moreover

$$U_1[\theta(t)] + U_2[\theta(t)] = u_1(t), \quad \text{when } 0 \leqq t \leqq p,$$
$$U_1[\theta(t)] + U_2[\theta(t)] = u_2(t), \quad \text{when } p \leqq t \leqq 2p,$$

since $U_1(\theta)=0$ in the interval $(\pi, 2\pi)$, and $U_2(\theta)=0$ in the interval $(0, \pi)$. Thus $U_1[\theta(t)]+U_2[\theta(t)]\equiv u(t)$. Furthermore the functions $v[t(\theta)]$ and $\rho[t(\theta)]$ are of limited variation with respect to θ. Therefore

$$F(\theta) = U_1(\theta) + U_2(\theta) + \rho[t(\theta)] + \frac{f(p)}{p} v[t(\theta)]$$

is developable in a uniformly convergent Fourier series. But $F(\theta)=f[t(\theta)]$; hence the theorem is established.

The Rice Institute

ZEROS OF A FUNCTION AND OF ITS DERIVATIVE*

BY W. J. TRJITZINSKY

Macdonald has proved the following theorem.† Let $f(z)=u(x, y)+iv(x, y)$ be a function of z analytic throughout the interior of a single closed curve C, defined by the equation $|f(z)|=M$, where M is a constant. Let C be an ordinary curve in the sense that if ψ be the angle which the tangent to C makes with the x-axis, and if the point of contact of the tangent describes the curve C, ψ will increase by 2π. Then the number of zeros of $f(z)$ in this region exceeds the number of zeros of the derivative $f'(z)$ by unity. Our purpose is to generalize this result by showing the conclusion holds true even when M is not a constant, but a function $M=M(x, y)$ analytic and greater than zero within sufficiently large intervals, and C is a curve single, closed, and ordinary (in the sense mentioned above), and satisfying the equation $|f(z)|=M(x, y)$, provided that the point $(x=x(\theta), y=y(\theta))$ describes C once when θ changes from 0 to 2π; where $x(\theta)$ and $y(\theta)$ are periodic functions satisfying the equations

$$(1) \qquad \frac{u(x,y)}{M(x,y)}=\cos\theta, \qquad \frac{v(x,y)}{M(x,y)}=\sin\theta.$$

The solvability of these equations implies that the Jacobian of the left members does not vanish identically.

On C, let $f(z)=M(x, y)e^{i\theta}$. Then there results a pair of equations (1) whose solutions are $x_i=x_i(\theta)$ and $y_i=y_i(\theta)$ of period 2π in θ. By hypothesis, among them there is at least one solution, say $x=x(\theta)$, $y=y(\theta)$, representing the curve C, and the point $(x=x(\theta), y=y(\theta))$ describes C once as θ varies from 0 to 2π.

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* Presented to the Society, San Francisco Section, June 18, 1927.

† Proceedings of the London Society, vol. 29 (1898), pp. 576–577; Proceedings of the London Society, (2), vol. 15 (1916), pp. 227–242.

Letting $M(x(\theta), y(\theta)) = N(\theta)$, we observe that $N(\theta)$ as well as $N'(\theta)$ is of period 2π, and for all θ, $N(\theta) > 0$, so that when z describes C and θ varies from 0 to 2π, the complex quantity $N'(\theta) + iN(\theta)$ describes a closed curve entirely above the real axis; its modulus will return to the initial value and the variation of its argument will be zero. Now on C we have

$$f(z) = N(\theta) \cdot e^{i\theta},$$

$$f'(z) = e^{i\theta} \cdot \frac{d\theta}{dz} \cdot [N'(\theta) + iN(\theta)],$$

$$f''(z) = e^{i\theta} \cdot f\left\{ [N'(\theta) + iN(\theta)] \cdot \left[\frac{d^2\theta}{dz^2} + i\left(\frac{d\theta}{dz}\right)^2 \right] + [N''(\theta) + iN'(\theta)]\left(\frac{d\theta}{dz}\right)^2 \right\}.$$

The excess e of the number of zeros of $f(z)$ over the number of zeros of $f'(z)$ within C is

$$e = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz - \frac{1}{2\pi i}\int_C \frac{f''(z)}{f'(z)}dz$$

$$= -\frac{1}{2\pi i}\int_C \frac{\left(\frac{d^2\theta}{dz^2}\right)}{\left(\frac{d\theta}{dz}\right)}dz - \frac{1}{2\pi i}\int_C \frac{d\theta}{dz}\left[\frac{N''(\theta) + N'(\theta)}{N'(\theta) + iN(\theta)} - \frac{N'(\theta)}{N(\theta)}\right]dz.$$

The quantity

$$-\frac{1}{2\pi i}\int_C \frac{\left(\frac{d^2\theta}{dz^2}\right)}{\left(\frac{d\theta}{dz}\right)} \cdot dz = 1,$$

as can be seen from Whittaker's* presentation of the theorem cited. Hence we have

* Whittaker and Watson, *Modern Analysis*, 3d ed., p. 121.

$$e = 1 - \frac{1}{2\pi i}\int_C \left(\frac{d\theta}{dz}\right)\cdot\left[\frac{N''(\theta) + iN'(\theta)}{N'(\theta) + iN(\theta)} - \frac{N'(\theta)}{N(\theta)}\right]dz$$

$$= 1 - \frac{1}{2\pi i}\int_0^{2\pi}\left[\frac{N''(\theta) + iN'(\theta)}{N'(\theta) + iN(\theta)} - \frac{N'(\theta)}{N(\theta)}\right]d\theta,$$

since θ varies from 0 to 2π when z describes C. Moreover

$$e = 1 - \frac{1}{2\pi i}\cdot\left\{\log\left[N'(\theta) + iN(\theta)\right]\right\}_0^{2\pi} + \frac{1}{2\pi i}\left[\log N(\theta)\right]_0^{2\pi}.$$

We know that $N(\theta)$ is real; hence $\left[\log N(\theta)\right]_0^{2\pi} = 0$. On the other hand, the variation of the argument of $[N'(\theta) + iN(\theta)]$ as θ changes from 0 to 2π is zero, so that $\log\left[(N'(\theta) + iN(\theta))\right]_0^{2\pi} = 0$. Hence $e = 1$. This proves the theorem.

THE UNIVERSITY OF TEXAS

LINEAR INEQUALITIES IN GENERAL ANALYSIS*

BY L. L. DINES

1. *Introduction.* In his studies in general analysis, E. H. Moore has developed† a theory of the linear functional equation

$$\xi + J_\kappa\xi = \eta.$$

Here ξ and η denote functions (the latter given, the former to be determined) belonging to a class $\mathfrak{M}$ of real-valued functions on a general range $\mathfrak{P}$. The kernel function κ belongs to a class $\mathfrak{K}$ which is well defined in terms of the fundamental class $\mathfrak{M}$. A sufficient foundation for the theory is laid by means of postulates upon the class $\mathfrak{M}$ and the functional operation J.

The purpose of the present paper is to consider the linear *inequality*

$$\xi + J_\kappa\xi > 0, \tag{1}$$

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* Presented to the Society, September 8, 1927.

† *On the foundations of the theory of linear integral equations*, this Bulletin, vol. 18, pp. 334–362.

a type which includes the *non*-homogeneous inequality in a sense indicated in § 5.

The most interesting feature is the relationship between the inequality (1) and the *adjoint equation*

$$\mu + J\mu\kappa = 0,$$

as expressed in the theorem of § 4. For the proof of this theorem, two postulates are introduced in addition to those used by Professor Moore.

2. *The Basis.* As a basis we assume Moore's $\sum_5$:*

$$(\mathfrak{A} ; \mathfrak{P} ; \mathfrak{M} ; \mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_* ; J \text{ on } \mathfrak{K} \text{ to } \mathfrak{A}),$$

where $\mathfrak{A}$ is the class of real numbers; $\mathfrak{P}$ is a general class of elements p; $\mathfrak{M}$ is a class of functions (μ, ξ, η, π) on $\mathfrak{P}$ to $\mathfrak{A}$, having the properties L, C, D; and the operator J has the properties L and M.

3. *An Equation Equivalent to the Inequality.* The inequality (1) is obviously equivalent to an equation

$$\xi + J\kappa\xi = \pi, \tag{2}$$

where π is any function of $\mathfrak{M}$ which is everywhere positive on $\mathfrak{P}$. This situation suggests an additional postulate upon the class $\mathfrak{M}$, namely that it contain at least one positive function. We shall find it desirable presently to make a more comprehensive postulate.

Equation (2) is of the Fredholm type treated by Moore. Its solution depends upon the Fredholm determinant F_κ of the kernel κ. In case $F_\kappa \neq 0$, the equation admits the solution

$$\xi = \pi + J\lambda\pi, \tag{3}$$

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* Loc. cit., p. 349. We might have assumed the basis $\sum_6$ (p. 352) which differs from $\sum_5$ in the presence of two classes of functions $\mathfrak{M}'$ and $\mathfrak{M}''$ on two conceptually distinct ranges $\mathfrak{P}'$ and $\mathfrak{P}''$. The basis $\sum_6$ reduces to $\sum_5$ if $\mathfrak{P}' = \mathfrak{P}''$ and $\mathfrak{M}' = \mathfrak{M}''$. Our choice is in the interest of typographical simplicity. We were influenced also by the fact that we have discovered no instance involving distinct ranges $\mathfrak{P}'$ and $\mathfrak{P}''$ in which our postulate E (§4) is satisfied.

where λ is the reciprocal kernel (resolvent) for κ. The inequality (1) then admits, in this case, the solution (3) where π is any positive function of $\mathfrak{M}$.

Of more interest is the exceptional case in which $F_\kappa = 0$. The theory of equation (2) for this case (without the restriction that π be positive) has been treated in detail by Hildebrandt.* The facts are entirely analogous to those well known in the theory of ordinary integral equations. The equation (2) in general admits no solution. The corresponding homogeneous equation

$$\xi + J\kappa\xi = 0$$

admits a finite number of linearly independent solutions $\xi_1, \xi_2, \cdots, \xi_m$; and the adjoint homogeneous equation

$$\mu + J\mu\kappa = 0 \tag{4}$$

admits the same number of linearly independent solutions $\mu_1, \mu_2, \cdots, \mu_m$. A necessary and sufficient condition that the equation (2) admit a solution is that π be orthogonal to each of the functions μ_i. That is, that

$$J\mu_i\pi = 0, \qquad (i = 1, 2, \cdots, m). \tag{5}$$

The solution of (2) is then

$$\xi = \pi + J\lambda\pi + \sum_{i=1}^{m} c_i\xi_i$$

where λ is a pseudo-resolvent for κ, and the c_i are arbitrary constants.

To apply these results to the inequality (1), we note that the kernel κ is given; and if $F_\kappa = 0$, the set of fundamental solutions $\mu_1, \mu_2, \cdots, \mu_m$ of the adjoint equation (4) can be determined. The problem of the inequality then reduces to the determination of a positive function π in $\mathfrak{M}$ which satisfies conditions (5).

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* *On pseudo-resolvents of linear integral equations in general analysis*, Annals of Mathematics, vol. 21 (1920), pp. 323–330.

The machinery which suffices for the theory of the functional equation apparently affords no means of attack on this problem. We therefore introduce in the next section new postulates which are sufficient to secure our desired result.

4. *Two New Postulates.* The postulates to be introduced involve a certain property relative to functions, which has been used in an earlier paper,* and which may be defined as follows:

A real-valued function is said to be *M-definite* if it is somewhere positive and nowhere negative, or somewhere negative and nowhere positive on its range; in other words if it is not identically zero and does not change sign.

Postulate N: If μ is an M-definite function of $\mathfrak{M}$ and π is a positive function of $\mathfrak{M}$, then $J\mu\pi \neq 0$.

Postulate E: If $(\mu_1, \mu_2, \cdots, \mu_m)$ is a set of functions of $\mathfrak{M}$ such that no linear combination of them is M-definite, then there exists a positive function π in $\mathfrak{M}$ such that

$$J\mu_i\pi = 0, \qquad (i = 1,2, \cdots, m). \tag{5}$$

The postulate N is a natural generalization of a simple property possessed by the operator J in each of Moore's classical instances (I), (II), (III), and (IV).

The existential postulate E is not perhaps so natural, and it undoubtedly invites further analysis. It is however satisfied in each of the four classical instances, as has been shown elsewhere by the author.† This postulate (together

* See *On sets of functions of a general variable*, Transactions of this Society, vol. 29 (1927), pp. 463–470.

† For the instance (II), including (I), see *Note on certain associated systems of linear equalities and inequalities*, Annals of Mathematics, vol. 28 (1926), p. 41, Theorem II.

For the instance (III), the proof is contained in a paper entitled *A theorem on orthogonal sequences* which will probably be published at an early date.

For the instance (IV), see *A theorem on orthogonal functions with an application to integral inequalities*, which is to appear soon in the Transactions of this Society.

An equivalent (and in some respects preferable) formulation of the postulate may be given in terms of the notion M-rank defined in the paper

with the assumption that $\mathfrak{M}$ has the property L) implies the existence of at least one positive function in $\mathfrak{M}$.

5. *The Inequality and its Adjoint Equation.* By use of the postulates just introduced, we now obtain the following

THEOREM: *The inequality*

$$\xi + J_K\xi > 0 \tag{1}$$

admits a solution, if and only if the adjoint equation

$$\mu + J\mu K = 0 \tag{6}$$

admits no M-definite solution.

First, we note that the case $F_K \neq 0$ is consistent with the theorem, since in that case the inequality (1) admits a solution (3), and the equation (6) admits only the solution $\mu = 0$ which is not M-definite.

Suppose then that $F_K = 0$, and consider a fundamental set of solutions

$$\mu_1, \mu_2, \cdots, \mu_m \tag{7}$$

of the adjoint equation (6). That is, the linear combinations of this set comprise the totality of solutions of (6). There are then two possibilities.

If the set (7) admits no linear combination which is M-definite, then the equation (6) admits no M-definite solution, while the inequality (1) admits a solution, since by postulate E there is a positive function π satisfying (5).

If, on the other hand, the set (7) admits an M-definite linear combination, say $\mu^* = \sum c_i\mu_i$, then the equation (6) admits the M-definite solution $\mu = \mu^*$. And the inequality (1) admits no solution. For suppose it did admit a solution $\xi = \xi^*$, that is suppose $\xi^* + J_K\xi^* = \pi$, where π is a positive function. Then π must satisfy the conditions $J\mu_i\pi = 0$,

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referred to in the preceding footnote, namely: If $(\mu_1, \mu_2, \cdots, \mu_m)$ is a set of functions of M-rank zero, then there exists a positive function π in $\mathfrak{M}$ such that $J\mu_i\pi = 0$, $(i = 1, 2, \cdots, m)$.

and since J has the property L (linearity) it follows that $J\mu^*\pi = 0$, which contradicts postulate N.

We have shown that (1) admits a solution in those and only those cases in which (6) admits no M-definite solution. This establishes the theorem.

6. *The Non-Homogeneous Inequality.* The inequality

$$\xi + J_\kappa \xi > \eta \tag{8}$$

where η is a given function of $\mathfrak{M}$, may be replaced by the two simultaneous inequalities

$$\begin{aligned} \xi + J_\kappa \xi - \eta x &> 0, \\ x &> 0, \end{aligned} \tag{9}$$

where x is a real number to be determined. For any solution $\xi = \xi^*$ of (8) determines a solution ($\xi = \xi^*$, $x = 1$) of (9), and conversely any solution ($\xi = \xi^*$, $x = x^*$) of (9) determines a solution $\xi = \xi^*/x^*$ of (8).

By the method of adjunctional composition (see Moore, loc. cit., p. 355), the system (9) may be written in the homogeneous form

$$\xi' + J_\kappa' \xi' > 0$$

relative to a new basis $\sum_5'$:

$$(\mathfrak{A} \; ; \; \mathfrak{P}' \; ; \; \mathfrak{M}' \; ; \; \mathfrak{K}' \equiv (\mathfrak{M}'\mathfrak{M}')^* \; ; \; J' \text{ on } \mathfrak{K}' \text{ to } \mathfrak{A}).$$

This new basis is the adjunctional composite of our general basis $\sum_5$, and the particular and very simple basis

$$(\mathfrak{A} \; ; \; \mathfrak{P}^{\mathrm{I}} \; ; \; \mathfrak{M}^{\mathrm{I}} \equiv \mathfrak{A} \; ; \; \mathfrak{K}^{\mathrm{I}} \equiv \mathfrak{A}\mathfrak{A} \; ; \; J^{\mathrm{I}} \text{ on } \mathfrak{K}^{\mathrm{I}} \text{ to } \mathfrak{A}),$$

in which $\mathfrak{P}^{\mathrm{I}}$ is a class containing a single element, $\mathfrak{M}^{\mathrm{I}}$ is the class of real numbers, and J^{I} is the identity operation.

University of Saskatchewan

SUBSTITUTIONS WHICH TRANSFORM A REGULAR GROUP INTO ITS CONJOINT*

BY G. A. MILLER

In a recent article which appeared in this Bulletin† under a similar heading it was proved that a necessary and sufficient condition that there exists a substitution t of order 2 which is commutative with every substitution of the group of isomorphisms of a regular group R, transforms R into its conjoint, and involves only letters of R, is that R involves substitutions whose order exceeds 2. In what follows it will be assumed that R satisfies this condition, and hence R may represent any abstract group except the abelian group of order 2^m and of type $(1, 1, 1, \cdots)$. While in the previous article the main emphasis was laid on a method of determining t as a substitution on the letters of R, and comparatively little attention was then paid to the fact that t is commutative with every substitution of the group of isomorphisms of R, it is this fact which will receive the main attention in the present article, since it throws much light on the infinite system of substitution groups which are separately groups of isomorphisms of regular substitution groups.

To simplify the considerations which follow we note here the well known facts that the totality of the substitutions on the letters of R which transform R into itself, in the sense that they transform every substitution of R into such a substitution, constitutes a group H, known as the holomorphs of R, and that the subgroup H_1, composed of all the substitutions of H which omit a given letter, is simply isomorphic with the group of isomorphisms of R. There are always at least two such sub-groups in H, and t is not necessarily commutative with every substitution in each of

* Presented to the Society, September 9, 1927.

† Vol. 32 (1926), p. 631.

these subgroups. It must, however, be commutative with every substitution of such a subgroup whenever t transforms into their inverses all the cycles which involve a letter which is neither in t nor in the subgroup in question. In what follows such a subgroup will be implied when we speak of the group of isomorphisms of R in connection with t.

When R is abelian then t is one of the substitutions of H which transform every substitution of R into its inverse and omit at least one letter of R. In this case the degree of t cannot be less than $r/2$, where r is the order of R. On the other hand, the degree of t is $r/4$ whenever R is the direct product of the octic group and an abelian group of order 2^m and of type $(1, 1, 1, \cdots)$. Since this infinite category of groups is characterized by the fact that it is composed of all the groups which involve operators whose order exceeds 2 but the number of such operators is relatively less than in any other group,* it is clear that t is never of a smaller degree than $r/4$, and that it is of this degree only when R belongs to the category just noted. From the obvious fact that when R is the quaternion group it can be transformed into its conjoint by a transposition, while t is of degree 6 in this case, it results directly that it may sometimes be possible to transform R into its conjoint by means of a substitution which is of a lower degree than t. In what follows we shall use t_1 to represent a substitution of lowest degree which transforms R into its conjoint, except that t_1 cannot be the identity when R is abelian and therefore coincides with its conjoint.

It is easy to see that the degree of t_1 can never be less than $r/4$ since the product of t_1 and any of its conjugates under H must be found in H, in view of the fact that every substitution which transforms R into its conjoint must also transform this conjoint into R. Moreover, the degree of this product cannot be less than $r/2$, whenever this product is not the identity, since the number of letters omitted by any substitution of H_1 is equal to the number of substitutions

* G. A. Miller, Comptes Rendus, vol. 141 (1905), p. 591.

of R with which this substitution is commutative and hence H cannot involve any substitution besides the identity which involves less than $r/2$ letters. From this it results also that when t_1 is of degree $r/4$ then R must be non-abelian, and that a conjugate of t_1 cannot involve the same letters as t_1 involves unless it is identical with t_1. Hence, whenever t_1 is of degree $r/4$, it is of order 2 and has exactly four distinct conjugates under H. The continued product of these conjugates is invariant under H and hence is a characteristic substitution of R. In what follows we shall assume that t_1 is of degree $r/4$.

If H_1 omits a letter of t_1 all of its substitutions must be commutative with t_1 and hence t_1 is also commutative with every substitution of the group of isomorphisms of R. It therefore results that when R can be transformed into its conjoint by a substitution of degree $r/4$ and t is not of this degree then there are at least three different substitutions of order 2 which are commutative with every substitution in its group of isomorphisms. Since t_1 is commutative with one-fourth of the substitutions of R and transforms R into its conjoint it results that the central of R must be composed of one-fourth of its substitutions. Since the central quotient group is always non-cyclic it results that R involves three abelian subgroups of index 2. Hence the following theorem.

THEOREM I. *A regular group cannot be transformed into its conjoint by a substitution which is not the identity but involves less than one-fourth of the letters of this regular group, and when it is transformed into its conjoint by a substitution which involves exactly one-fourth of its letters its order must be divisible by 8 and it must be non-abelian but involve three abelian subgroups of index 2.*

We proceed to prove that the conditions expressed in this theorem are not only necessary but also sufficient in order that the regular group R can be transformed into its conjoint by t_1. When these conditions are satisfied R has a commutator subgroup of order 2. Let R_0 be one of its abelian subgroups

of index 2. This subgroup has two transitive constituents and involves the central of R, which has four such constituents. Let t_1 be the part of the commutator of R which appears in the last of these four constituents and let s be the substitution of order 2 which simply interchanges the corresponding letters of R_0, so that s is commutative with every substitution of R_0 and transforms t_1 into t_0. To obtain R it is obviously only necessary to extend R_0 by means of the substitution $s_0 t_0 t_1 s$, where s_0 is a substitution found both in the first transitive constituent of R_0 and also in the central of R.* The transform of R under t_1 is the conjoint of R since each of its substitutions is commutative with every substitution of R. In fact t_1 is commutative with one-half of the substitutions of R_0 and transforms each of the other substitutions of R_0 into itself multiplied by the substitution of order 2 in the second transitive constituent of R_0 which includes t_1. These transforms are obviously commutative with $s_0 t_0 t_1 s$. As the transform of $s_0 t_0 t_1 s$ under t_1 is also commutative with every substitution of R we have proved the following theorem.

THEOREM II. *A necessary and sufficient condition that a regular group can be transformed into its conjoint by a substitution which involves exactly one-fourth of its letters is that this group be non-abelian but involve more than one abelian subgroup of index* 2.

From this theorem it results that there are at least three substitutions of order 2 on the letters of the H of each of these groups which are commutative with every substitution of H_1, except possibly in the case when R is the direct product of the octic group and an abelian group of order 2^m and of type $(1, 1, 1, \cdots)$. In this special case it is obvious that there are always exactly three such substitutions in H. Hence it results that when the regular group R is non-abelian but involves more than one abelian subgroup of index 2 there

* G. A. Miller, American Journal of Mathematics, vol. 24 (1902), p. 395.

are always at least two distinct substitutions of order 2 which transform it into its conjoint and are commutative with every substitution of its group of isomorphisms. It also results directly from the same theorem that the only non-abelian groups which can be transformed into their conjoints by a transposition are the octic group and the quaternion group. It may be added that when R is any dihedral group of order $2n$, $n>2$, then t is a substitution of order 2 which transforms into its inverse a cycle of order n and involves $n-1$ or $n-2$ letters as n is odd or even. When R is the dicyclic group then t is the product of this substitution and the substitution of order 2 generated by the second cycle of order n.

If all the operators of order $k>2$ which appear in a group G are transformed according to a transitive substitution group under the group of isomorphisms of G then there are at least $\phi(k)$ substitutions on the letters of this transitive group which are commutative with each of its substitutions, $\phi(k)$ being the totient of k. This results from the fact that the subgroup composed of all the substitutions of this transitive group which omit one letter must omit at least the $\phi(k)$ letters corresponding to the operators of order k generated by a group operator of this order. In particular, there results the following theorem.

THEOREM III. *A necessary and sufficient condition that a constituent of t which involves the same letters as a transitive constituent of H involves appears in this transitive constituent is that each of the operators of G which corresponds to the letters of this constituent corresponds also to its inverse in some automorphism of G.*

This theorem explains the fact that the constituents of t appear in H when R is dihedral or dicyclic.

According to the general theorem under consideration there is always at least one substitution of order 2 on the letters of H which is commutative with every substitution

of H_1, and when there is only one such substitution it must be t. It may be of some interest to inquire what condition R must satisfy in order that there be only one such substitution. When R is abelian and $r>4$ the necessary and sufficient condition is obviously that the order of R be a power of any odd prime number and that its type be $(1, 1, 1, \cdots)$. If such an abelian group is extended by an operator of order 2 which transforms each of its operators into its inverse, there results a generalized dihedral group whose group of isomorphisms is the holomorph of this abelian group and hence t is again the only substitution of order 2 on the letters of its H which is commutative with every substitution of H_1. Another infinite system of groups in which t is the only substitution on the letters of R which is commutative with every substitution of H_1 may be constructed by extending the abelian group of order 2^m and of type $(1, 1, 1, \cdots)$ by means of an operator of prime order which is generated by any operator of order 2^m-1 in the group of isomorphisms of this abelian group. The tetrahedral group is the group of lowest order in this system. From what was noted above it results directly that whenever the order of a regular group is divisible by at least two distinct odd prime numbers there must be more than one substitution on the letters of R which is commutative with every substitution of its H_1.

The University of Illinois

THE FUNCTIONAL EQUATION DEFINING DIOPHANTINE AUTOMORPHISMS*

BY LOUIS WEISNER†

1. *Introduction.* A form $f(x_1, \cdots, x_m)$ is said to admit a *diophantine automorphism* if there exist m forms

$$\phi_1(x_1, \cdots, x_m), \cdots, \phi_m(x_1, \cdots, x_m),$$

such that

$$f[\phi_1(x_1, \cdots, x_m), \cdots, \phi_m(x_1, \cdots, x_m)] = f^n(x_1, \cdots, x_m), \tag{1}$$

where n is a positive integer greater than 1, the coefficients of the forms being integers or rational integral functions of one or more parameters with integral coefficients. In view of Bell's remarks (*A diophantine automorphism*, this Bulletin, vol. 28, p. 71) on the scarcity of what he calls *proper* diophantine automorphisms, it seems desirable to seek solutions of (1), regarded as a functional equation, no restrictions being placed on the algebraic nature of the solution or the coefficients.

Three methods of obtaining solutions of this functional equation are offered in the present paper. The first method gives a solution corresponding to any one-parameter continuous group having certain properties. The solution is generally a transcendental function, but I present this method here as the functional equation is of interest in analysis. In the second method f is assumed known, and the ϕ's are to be determined. It is shown that if there exists a Cremona transformation of which $x_i' = f(x_1, \cdots, x_m)$ is a constituent, then there exist *rational* functions $\phi_1, \cdots, \phi_m$ satisfying (1). This leads to rational solutions of the diophantine equation

$$f(x_1, \cdots, x_m) = u^n.$$

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† National Research Fellow.

The third method is more satisfactory for the purposes of diophantine analysis. It is shown that if the ϕ's are forms such that the equations

$$x_i' = \phi_i(x_1, \cdots, x_m), \qquad (i = 1, \cdots, m),$$

define a Cremona transformation of finite period, then there exists a form $f(x_1, \cdots, x_m)$ satisfying (1). Eisenstein's automorphism is obtained in this way. A diophantine automorphism may be obtained from every Cremona transformation of finite period, but not every diophantine automorphism may be obtained in this way. For example, the transformation

$$x_1' = x_1^n, \cdots, x_m' = x_m^n$$

is not a Cremona transformation, but it transforms $x_1 x_2 \cdots x_m$ into its nth power.

2. *First Method. One-Parameter Groups.* Let

$$x_i' = \phi_{it}(x_1, \cdots, x_m), \qquad (i = 1, \cdots, m) \tag{2}$$

be a one-parameter continuous group (t being the parameter) analytic in the neighborhood of $t=0$ and in some region of the x-space. The variables are non-homogeneous. We assume that the parameter of the product of two transformations of the group equals the sum of the parameters of the transformations, and that the identical transformation is given by $t=0$.*

Instead of (1) we consider

$$f(\phi_{1t}, \cdots, \phi_{mt}) = f^{nt}(x_1, \cdots, x_m), \tag{3}$$

which reduces to (1) when $t=1$. Taking the logarithms of both members of (3) we obtain

$$g(\phi_{1t}, \cdots, \phi_{nt}) = n^t g(x_1, \cdots, x_m), \tag{4}$$

where

$$g(x_1, \cdots, x_m) = \log f(x_1, \cdots, x_m). \tag{5}$$

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* See A. Cohen, *The Lie Theory of One-Parameter Groups*, §4.

Differentiating (4) with respect to t, and then putting $t=0$, we obtain

$$Ug(x_1, \cdots, x_m) = g(x_1, \cdots, x_m) \log n, \tag{6}$$

where U denotes the linear differential operator defining the infinitesimal transformation of the group (2). From (6) we deduce that

$$U^p g = g^p \log n.$$

Hence

$$\begin{aligned} g[\phi_{1t}, \cdots, \phi_{mt}] &= g + tUg + \frac{t^2}{2!}U^2 g + \cdots \\ &= g + tg \log n + \frac{t^2}{2!}\log^2 n + \cdots \\ &= gn^t. \end{aligned}$$

It follows that every solution of (6) is a solution of (4) which leads to a solution of (3) through the medium of (5). Now (6) is a linear partial differential equa ion of Lagrange's type which may be solved by the usual methods, the solution involving an arbitrary function. A solution of (1) may thus be obtained from any continuous group satisfying the stated conditions.

3. *Second Method. Arbitrary Transformations.* If

$$x_i' = f_i(x_1, \cdots, x_m), \qquad (i = 1, \cdots, m), \tag{7}$$

the variables being again non-homogeneous, is a reversible, but otherwise arbitrary transformation, the equations

$$f_i(x_1', \cdots, x_m') = F_i(x_1, \cdots, x_m), \quad (i = 1, \cdots, m), \tag{8}$$

where F_i is arbitrary, can be solved for $x_1', \cdots, x_m'$, giving, say,

$$x_i' = G_i(x_1, \cdots, x_m), \qquad (i = 1, \cdots, m). \tag{9}$$

Evidently

$$f_i(G_1, \cdots, G_m) = F_i(x_1, \cdots, x_m), \quad (i = 1, \cdots, m). \tag{10}$$

Regarding (10) as a system of functional equations in which $G_1, \cdots, G_m$ are the unknowns, a solution is provided by (9). If $h<m$ equations of the type (10) are given, we add $m-h$ other equations and obtain a solution involving $m-h$ arbitrary functions. Thus to find solutions of (1), f being given, select $m-1$ arbitrary functions $f_2, \cdots, f_m$, such that (7) is reversible ($f_1 \equiv f$), and let $F_2, \cdots, F_m$ be arbitrary functions. We obtain a solution involving $F_2, \cdots, F_m$.

Where f is a given *rational* function it is not always possible to find functions $f_2, \cdots, f_m$ such that $\phi_1, \cdots, \phi_m$ are rational. This will be the case, however, if (7) is a Cremona transformation, one of whose right members is f.

4. *An Example.* Applying the method of §3 to the equations

$$x' = \frac{x}{x^2 + y^2}, \qquad y' = \frac{y}{x^2 + y^2},$$

defining an inversion, we obtain the equations

$$\frac{x'}{x'^2 + y'^2} = \left(\frac{x}{x^2 + y^2}\right)^n, \qquad \frac{y'}{x'^2 + y'^2} = F,$$

where F is an arbitrary rational function of x and y. Solving for x' and y', we obtain

$$x' = \frac{x^n(x^2 + y^2)^n}{x^{2n} + (x^2 + y^2)^{2n}F^2},$$

$$y' = \frac{(x^2 + y^2)^n F}{x^{2n} + (x^2 + y^2)^{2n}F^2}.$$

We readily verify that this transformation transforms $x/(x^2+y^2)$ into its nth power.

5. *Third Method. Finite Cremona Transformations.* The variables in (1) are now assumed homogeneous and

$$x_i' = \phi_{i1} \equiv \phi_i(x_1, \cdots, x_m), \qquad (i = 1, \cdots, m),$$

the ϕ's being forms of order n, are assumed to define a Cremona transformation of finite period p. The rth iterate

of the ordered functions $\phi_{11}, \cdots, \phi_{m1}$ is denoted by $\phi_{1r}, \cdots, \phi_{mr}$, so that

$$\phi_{i0}(x_1, \cdots, x_m) \equiv x_i,$$
$$\phi_{ir}(x_1, \cdots, x_m) \equiv \phi_{i,r-1}(\phi_{11}, \cdots, \phi_{m1}).$$

Evidently

$$\phi_{1p} : \phi_{2p} : \cdots : \phi_{mp} = x_1 : x_2 : \cdots : x_m.$$

Since ϕ_{ip} is a form of order n^p, there exists a form $\psi(x_1, \cdots, x_m)$ of order $n^p - 1$ such that

$$x_i\psi(x_1, \cdots, x_m) = \phi_{ip}(x_1, \cdots, x_m).$$

Hence

$$\begin{aligned}\phi_{i1}(x_1, \cdots, x_m)\psi(\phi_{11}, \cdots, \phi_{m1}) &= \phi_{i,p+1}(x_1, \cdots, x_m) \\ &= \phi_{i1}(\phi_{1p}, \cdots, \phi_{mp}) \\ &= \phi_{i1}[x_1\psi(x_1, \cdots, x_m), \cdots, x_m\psi(x_1, \cdots, x_m)] \\ &= \phi_{i1}(x_1, \cdots, x_m)\psi^n(x_1, \cdots, x_m),\end{aligned}$$

since $\phi_{i1}(x_1, \cdots, x_m)$ is a form of order n. We conclude that

$$\psi(\phi_{11}, \cdots, \phi_{m1}) = \psi^n(x_1, \cdots, x_m).$$

Hence $\psi(x_1, \cdots, x_m)$ is a solution of (1). *There exists a solution of* (1) *corresponding to any Cremona transformation of order n and finite period.*

The function ψ thus determined need not be the simplest solution of the functional equation for a given Cremona transformation : it may be factorable, and one of its factors may be a solution. For example, for the Cremona transformation

$$\begin{aligned}x' &= 3xyz - x^2w - 2y^3, \\ y' &= 2xz^2 - xyw - y^2z, \\ z' &= xyz - 2y^2w + yz^2, \\ w' &= xw^2 - 3yzw + 2z^3,\end{aligned}$$

of period 2, the function ψ is found to be

$$(x^2w^2 - 3y^2z^2 + 4xz^3 + 4wy^3 - 6xyzw)^2.$$

The square root of this function is transformed into its cube by the Cremona transformation, and we have Eisenstein's automorphism.

6. *Automorphisms of Determinants.* If the elements of a determinant $D \equiv |a_{ij}|$ of order n are independent variables, and A_{ij} is the cofactor of a_{ij}, the transformation $a'_{ij} = A_{ij}$ is a Cremona transformation of period 2, which transforms D into D^{n-1}.

Now suppose each element is a function of m variables, say

$$a_{ij} \equiv a_{ij}(x_1, \cdots, x_m),$$

and let

$$A_{ij} \equiv A_{ij}(x_1, \cdots, x_m).$$

In general, the equations

$$a_{ij}(x_1', \cdots, x_m') = A_{ij}(x_1, \cdots, x_m)$$

are inconsistent, and, when consistent, do not define $x_1', \cdots, x_m'$ as rational integral functions of $x_1, \cdots, x_m$. In certain special cases we do obtain a transformation of the desired type.

Consider, for example, the symmetric determinant

$$\begin{vmatrix} a & d & e \\ d & b & f \\ e & f & c \end{vmatrix}.$$

The variables d, e, f occur twice, but their respective cofactors are the same in both cases. The determinant is transformed into its square by the transformation $a' = bc - f^2$, etc., each element being replaced by its cofactor. Eisenstein* credits this diophantine automorphism to Lagrange.

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* Journal für Mathematik, vol. 27, p. 105.

ALL POSITIVE INTEGERS ARE SUMS OF VALUES OF A QUADRATIC FUNCTION OF x*

BY L. E. DICKSON

1. *Introduction.* Fermat stated that he was the first to discover the beautiful theorem that every integer $A \geqq 0$ is a sum of $m+2$ *polygonal* numbers

$$p_{m+2}(x) = \tfrac{1}{2}m(x^2 - x) + x \tag{1}$$

of *order* $m+2$ (or $m+2$ sides), where x is an integer $\geqq 0$. The cases $m=1$ and $m=2$ state that every A is a sum of three *triangular* numbers $p_3(x) = \frac{1}{2}x(x+1)$, and also a sum of four squares $p_4(x) = x^2$.

Cauchy† was the first to publish a proof of Fermat's statement and showed that all but four of the polygonal numbers may be taken to be 0 or 1.

In this paper and its sequel we shall give a complete solution of the following more general question.

PROBLEM. Find every quadratic function $f(x)$ which takes integral values $\geqq 0$ for all integers $x \geqq 0$, such that every positive integer A is a sum of l of these values, where l depends on $f(x)$, but not on A.

2. LEMMA 1. *A quadratic function of x is an integer for every integer $x \geqq 0$ if and only if it is of the form*

$$f(x) = \tfrac{1}{2}mx^2 + \tfrac{1}{2}nx + c, \quad m + n \text{ even}, \tag{2}$$

where m, n, and c are integers.

Consider ux^2+vx+c. By its values for $x=0$, 1, and 2, c, $u+v$, and $4u+2v$ are integers. Subtract the double of the

* Presented to the Society, September 9, 1927.

† Oeuvres, (2), vol. 6, pp. 320–353. Pepin gave a modified proof, Atti dei Lincei, vol. 46 (1892–93), pp. 119–131. His proof requires a separate examination when $A < 110m$. For the simpler proof in §5, the limit is $A < 44m+32$.

second from the third. Hence $2u$ is an integer m. Since $\frac{1}{2}m+v$ is an integer, v is half an integer n.

3. *Positive Quadratic Functions Representing* 0 *and* 1. Let (2) be $\geqq 0$ for every integer $x \geqq 0$, whence $m>0$, $c \geqq 0$. Then

$$f(x+1) - f(x) = mx + \tfrac{1}{2}(m+n)$$

increases with x. Hence $f(x)$ does not represent every positive integer A. Thus $l>1$ in our problem, and a sum of two or more values of $f(x)$ must give $A=1$. Hence $f(u)=1$, $f(k)=0$ for certain integers $u \geqq 0$, $k \geqq 0$. We assume that k has its least value. Then

$$f(x) = f(x) - f(k) = \tfrac{1}{2}(x-k)[m(x+k)+n].$$

Since $f(u)=1$,

$$n = s - m(u+k), \quad s = 2/(u-k),$$

where s is an integer. Thus $u-k=\pm 1$ or ± 2, and

$$f(x) = \tfrac{1}{2}(x-k)[m(x-u)+s].$$

If $u-k = \pm 1$,

$$f(x) = \tfrac{1}{2}(x-k)[m(x-k \mp 1) \pm 2] = p_{m+2}(\pm x \mp k).$$

If $u-k=2$,

$f(k+1) = \frac{1}{2}(1-m) \geqq 0$ gives $m=1$, $f(x) = p_3(x-k-1)$.

Finally, if $u-k=-2$, then $k \geqq 2$ and $f(k-1)=\frac{1}{2}(1-m)$ is zero, since it is not negative. But this contradicts the definition of k as least.

THEOREM 1. *The functions derived from* (1) *by replacing* x *by* $x-k$ *or* $k-x$ *are the only quadratic functions of* x *which are integers* $\geqq 0$ *for every integer* $x \geqq 0$, *and which take the values* 0 *and* 1 *for certain integers* $x \geqq 0$.

The values of $p_3(-x)=\frac{1}{2}(x-1)x$ coincide with the triangular numbers. Hence if $m=1$, our problem is the same when $k \geqq 1$ as when $k=0$. This is evidently true also if $m=2$. Without loss of generality, we may then henceforth take $m>2$.

4. *Polynomials with an Excess.* Let $f(x)$ have an integral value $\geqq 0$ for every integer $x \geqq 0$, and let one value be zero. Let $M_s(A)$ denote the maximum sum $\leqq A$ of s values of $f(x)$, and write $E_s(A)$ for $A - M_s(A)$. In case $E_s(A)$ has a finite maximum E_s for all integers $A \geqq 0$, every integer $A \geqq 0$ is a sum of E_s numbers 0 or 1 and s values of $f(x)$. Then E_s is called the s-excess of $f(x)$. We shall drop the subscript 4 from E_4.

Let α, β, γ, δ denote the four integral values $\geqq 0$ of x and write

$$a = \Sigma\alpha^2, \qquad b = \Sigma\alpha. \tag{3}$$

Take (2) as $f(x)$ and insert the four values of x. Thus

$$A = \tfrac{1}{2}ma + \tfrac{1}{2}nb + 4c + r, \qquad 0 \leqq r \leqq E, \tag{4}$$

for a suitable integer r. Cauchy proved the following result.

LEMMA 2. *If a and b are positive odd integers such that $b^2 < 4a$ and*

$$b^2 + 2b + 4 > 3a, \tag{5}$$

equations (3) *have solutions α, β, γ, δ in integers $\geqq 0$.*

Multiply (5) by m and replace ma by its value from (4). The resulting inequality follows from that obtained by suppressing $6r$. Multiplication by $4m$ now yields the equivalent inequality

$$(2mb + \tau)^2 > U, \quad U = \tau^2 + 4m(6A - 24c - 4m), \tag{6}$$

where $\tau = 2m + 3n$. This inequality holds if

$$b > (U^{1/2} - \tau)/(2m), \quad U \geqq 0. \tag{7}$$

To satisfy $b^2 < 4a$, multiply by m^2 and replace ma by its value from (4). The resulting condition evidently follows from that with r replaced by E, and hence from

$$(mb + 2n)^2 < 4V, \quad V = n^2 + m(2A - 8c - 2E). \tag{8}$$

This inequality holds if

$$b < (2V^{1/2} - \quad)/m, \quad V \geqq 0, \tag{9}$$

and if $mb+2n \geqq 0$. For $n \geqq 0$, the latter evidently holds if $b>0$. For $n<0$, it holds by (7) if

$$4n - \tau + U^{1/2} \geqq 0 \tag{10}$$

and hence if

$$3A \geqq 12c + 2m - 2n - n^2/m \quad (\text{if } n < 0). \tag{11}$$

We desire that $b>0$. By (7), this will be true if $U^{1/2} \geqq \tau$. If $n<0$, this follows from (10). But if $n \geqq 0$, whence $\tau>0$, it holds if and only if $U \geqq \tau^2$, and hence if the quantity in the last parenthesis of (6) is $\geqq 0$:

$$A \geqq 4c + \tfrac{2}{3}m \quad (\text{if } n \geqq 0). \tag{12}$$

There will be at least d positive integers between the limits on b stated in (7) and (9) if

$$4V^{1/2} - U^{1/2} > P, \quad P = 2md - 2m + n. \tag{13}$$

The left member is $\geqq 0$ if

$$16V \geqq U, \tag{14}$$

and then (13) holds* if its square holds and hence if

$$F \equiv (2V + W)^2 - VU > 0, \quad 8W \equiv U - P^2, \quad P \geqq 0. \tag{15}$$

By the *minor conditions* we shall mean $U \geqq 0$, $V \geqq 0$, the inequality (14), and (11) or (12). Since

$$16V - U = 3(2m - n)^2 + 4n^2 + 8m(A - 4c - 4E), \tag{16}$$

it follows that (14) holds if

$$A \geqq 4c + 4E. \tag{17}$$

The latter implies $V \geqq 0$. Evidently $U \geqq 0$ if

$$A \geqq 4c + \tfrac{2}{3}m. \tag{18}$$

Hence the minor conditions all follow from (17) and (18) if $n \geqq 0$, but from these two and (11) if $n<0$. We shall speak of these as the *reduced* minor conditions.

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* Automatically if $P<0$.

5. *Polygonal Numbers.* We take (1) as the function (2). Thus $n=2-m$, $c=0$. By Table I, $E(2m+3)=m-2$. Hence E is not smaller than the value in

THEOREM 2. *For the function* (1), $E_4=m-2$ *if* $m\geqq 3$.

The reduced minor conditions are all satisfied if $A\geqq 4m$. Here (4) is $A=mg+b+r$, $g=\frac{1}{2}(a-b)$. If b takes the odd values β and $\beta+2$, while r takes the values $0, 1, \cdots, m-2$, the values of $b+r$ are $\beta+j(j=0, 1, \cdots, m)$. These, with $j=m$ omitted, form a complete set of residues modulo m. Hence for any A, the preceding equation yields an integral value of g and hence an odd integral value of a.

If there are at least $d=4$ integers between the limits for b, there will exist the desired two odd values for b. Then (6), (8), (13), and (15) give

$$U = 24mA - 15m^2 - 12m + 36, \quad V = 2mA - m^2 + 4,$$

$$P = 5m + 2, \quad W = 3mA - 5m^2 - 4m + 4,$$

$$F = m^2A^2 - 44m^3A - 32m^2A + 34m^4 + 44m^3 - 56m^2 - 48m > 0.$$

Evidently $F>0$ if $A\geqq 44m+32$.

Next, let $A<44m+32$. Then $A<M\equiv 48m+21$. In Table I the entries involving the same multiple of m, together with all intervening integers, will be said to form a *block*. We suppress $29m+10-12$ and $45m+10-13$. Down to M, the difference between any two consecutive numbers in any abridged block is now $\leqq 2$, whence $E(A)\leqq 1$ for every A within a block. For every $A<M$ not within a block, $E(A)\leqq m-2$. This will follow if proved when $A+1$ is the first number of any abridged block. Then A is the sum of $m-2$ and a number t occurring explicitly in the abridged table except as follows. If $A=10m+4$, $26m+12$, $27m+10$, or $28m+12$, then $A=m-3+t$. If $A=6m+3$, $21m+6$, $28m+7$, $30m+11$, $34m+11$, or $42m+12$, then $A=m-4+t$; while, if $m=3$, A is equal to the A for the next smaller m.

TABLE I.

Sums of Four Polygonal Numbers

0–4, m+2–5, $2m$+4–6, $3m$+3–7, $4m$+5–8, $5m$+7–8, $6m$+4–9, $7m$+6–9, $8m$+8–10, $9m$+7–10, $10m$+5–11, $11m$+7–9, 11, $12m$+8–12, $13m$+8–12, $14m$+10–12, $15m$+6–9, 11–13, $16m$+8–13, $17m$+10–13, $18m$+9–14, $19m$+11–14, $20m$+10–14, $21m$+7–13, 15, $22m$+9–15, $23m$+11–15, $24m$+10–16, $25m$+11–13, 15, 16, $26m$+13–16, $27m$+11–16, $28m$+8–11, 13–17, $29m$+10–12, 15–17, $30m$+12–17, $31m$+11–17, $32m$+13–18, $33m$+15–18, $34m$+12–18, $35m$+14–17, $36m$+9–19, $37m$+11–13, 15–19, $38m$+13–15, 17–19, $39m$+12–17, 19, $40m$+14–20, $41m$+16–20, $42m$+13–20, $43m$+14–17, 19, 20, $44m$+16–20, $45m$+10–13, 16–21, $46m$+12–21, $47m$+14–17, 19–21, $48m$+13–21, $49m$+15–21, $50m$+17–22, $51m$+14–17, 19–22, $52m$+16–22, $53m$+18–21, $54m$+17–19, 21, 22, $55m$+11–17, 19–23, $56m$+13–23, $57m$+15–21, 23, $58m$+14–23, $59m$+16, 17, 19–23, $60m$+16–24, $61m$+15–21, 23, 24, $62m$+17–24, $63m$+19–24, $64m$+17–24, $65m$+16–21, 23, 24, $66m$+12–15, 17–25, $67m$+14–16, 19–25, $68m$+16, 17, 19–25, $69m$+15–18, 20, 21, 23–25, $70m$+17–19, 21–25, $71m$+19–25, $72m$+16–26, $73m$+18–21, 23–26, $74m$+20–23, 25, 26, $75m$+19–25, $76m$+17–26, $77m$+19–21, 23–26, $78m$+13–16, 20–27, $79m$+15–17, 20–25, 27, $80m$+17, 18, 22–27, $81m$+16–21, 23–27, $82m$+18–27, $83m$+19–25, 27, $84m$+17–28, $85m$+19–21, 23–28, $86m$+21–28, $87m$+19–25, 27, 28, $88m$+18 22, 24 28, $89m$+20, 21, 23–28, $90m$+20 23, 25–28, $91m$+14–17, 20–25, 27–29, $92m$+16–18, 22–29, $93m$+18–21, 23–29, $94m$+17–29, $95m$+19, 20, 22–25 27–29, $96m$+21–29, $97m$+18–21, 23–29, $98m$+20–27, 29, 30, $99m$+20–25, 27–30, $100m$+21–30, $101m$+19–21, 23–29, $102m$+21–30, $103m$+22–25, 27–30, $104m$+22–30, $105m$+15–18, 24–29, 31, $106m$+17–23, 25–31, $107m$+19, 20, 22–25, 27–31, $108m$+18–21, 24–31, $109m$+20, 21, 23–29, 31, $110m$+22–31, $111m$+19–25, 27–31, $112m$+21–32, $113m$+23–29, 31, 32, $114m$+22–27, 29–32, $115m$+20–22, 24, 25, 27–32, $116m$+22, 23, 25–32, $117m$+23–29, 31, 32, $118m$+23–32, $119m$+22–25, 27–32, $120m$+16–19, 21–33, $121m$+18–20, 23–29, 31–33, $122m$+20, 21, 25–33, $123m$+19–25, 27–33, $124m$+21, 22, 25–33, $125m$+23, 25–29, 31–33, $126m$+20–31, 33, $127m$+22–25, 27–33, $128m$+24–26, 28–34, $129m$+23–29, 31–34, $130m$+21–23, 25–27, 29–34, $131m$+23, 24, 28–33, $132m$+24–34, $133m$+23–29, 31–34, $134m$+25–34, $135m$+22–24, 27–33, $136m$+17–20, 24–35, $137m$+19–21, 26–29, 31–35, $138m$+21, 22, 25, 26, 28–35, $139m$+20–23, 27–33, 35, $140m$+22. 23, 26, 27, 29–35, $141m$+23–29, 31–35, $142m$+21–31, 33–35, $143m$+23–25, 27–33, 35, $144m$+25–36, $145m$+24–29, 31–36, $146m$+22–27, 29–36, $147m$+24, 25, 27–33, 35, 36, $148m$+24–27, 29–36, $149m$+25–29, 31–36, $150m$+25–36, $151m$+23–25, 27–33, 35, 36, $152m$+25–36, $153m$+18–21, 27–29, 31–37, $154m$+20–22, 26–37, $155m$+22, 23, 28–33, 35–37, $156m$+21–37, $157m$+23–29, 31–37, $158m$+25–30, 33–35, 37, $159m$+22–25, 28–33, 35–37, $160m$+24–37, $161m$+26, 28, 29, 31–37, $162m$+25–27, 29–38, $163m$+23–25, 27–33, 35–38, $164m$+25 27, 30 38, $165m$+26–28, 31–37, $166m$+26–38, $167m$+28–33, 35–38, $168m$+24–26, 29–31, 33–38, $169m$+26–29, 31–37, $170m$+28–38, $171m$+19–22, 27–33, 35–39, $172m$+21–23, 26–39, $173m$+23, 24, 28, 29, 31–37, 39, $174m$+22 31, 33 35, 37–39, $175m$+24, 25, 27–33,

35–39, $176m+26$, 29–39, $177m+23$–26, 28, 29, 31–37, 39, $178m+25$–27, 29–39, $179m+27$, 31–33, 35–39, $180m+26$–40, $181m+24$–29, 31–37, 39, 40, $182m+26$–40, $183m+27$–33, 35–40, $184m+27$–40, $185m+29$, 31–37, 39, 40, $186m+25$–40, $187m+27$–33, 35–40, $188m+29$, 30, 32–40, $189m+27$–29, 31–37, 39, 40, $190m+20$–23, 29–31, 33–35, 37–41, $191m+22$–24, 28–33, 35–41, $192m+24$–41, $193m+23$–26, 28, 29, 31–37, 39–41, $194m+25$, 26, 30–32, 34–39, 41, $195m+27$, 29–33, 35–41, $196m+24$–27, 29–41, $197m+26$–28, 31–37, 39–41, $198m+28$–41, $199m+27$, 28, 30–32, 35, 36.

If $A=15m+5$ or $46m+11$, then $A=m-5+t$; while, if $m=3$ or 4, A is $\leqq$ the A for the next smaller m. Finally, if $A=36m+8$, then $A=m-6+t$; while, if $m=3$, 4, or 5, $A=34m+14$, 16, or 18, which belong to an earlier block. This completes the proof of Theorem 2.

By that theorem, every integer $A\geqq 0$ is a sum of $m+2$ polygonal numbers. Hence $E_s=0$ if $s\geqq m+2$. Next, let $4\leqq s<m+2$. If a sum by s of the polygonal numbers 0, 1, $m+2$, $3m+3$, $\cdots$ is $\leqq 2m+3$, at most one summand is $m+2$, whence the maximum such sum is $m+2+s-1$. Hence $E_s(2m+3)=m-s+2$. By Theorem 2, A is a sum of four polygonal numbers and $m-2$ numbers 0 or 1. Regard $s-4$ of the latter as polygonal numbers. Hence A is a sum of s polygonal numbers and $m-s+2$ numbers 0 or 1. All of these facts prove the following theorem.

THEOREM 3. *For the function* (1), $E_s=0$ *if* $s\geqq m+2$, *while* $E_s=m-s+2$ *if* $4\leqq s\leqq m+2$.

In the second case, $s+E_s=m+2$, so that the use of 5 or more polygonal numbers >1 yields no gain (but rather a loss) over the use of only four.

6. *Deductions from Table* I. We extended our table beyond the limit $48m+21$ required for the proof of Theorem 2 in order to deduce interesting facts concerning $E(A)$ for function (1), which are essential to the sequel.

LEMMA 3. *For* $54m+17\leqq A<74m+28$, $E(A)\leqq m-6$ *if* $m\geqq 7$, $E(A)\leqq 1$ *if* $m=5$ *or* 6.

From Table I we suppress $67m+14$-16. Since the difference between any two consecutive numbers in any abridged block is now $\leqq 2$, $E(A)\leqq 1$, for every A within a

block. Let f be the term free of m in the leader $qm+f$ of any abridged block. First, let $m=5$. For $q=56, \cdots, 74$, we find that $f+4$ is the term free of m in a number of the preceding abridged block. Hence $qm+f-1$ is the sum of $m-5$ and the number $(q-1)m+f+4$ in the abridged table. Finally, the E of $55m+10=53m+20$ is 1. Second, let $m\geqq 6$. Except for $q=55, 56, 62, 70$, $f+5$ is the term free of m in a number of the preceding abridged block, whence $E(A)\leqq m-6$. For the four q's, we use $f+6$ if $m>6$. If $m=6$, $56m+12=55m+18$, $62m+16=61m+22$, $70m+16=69m+22$, which fall within preceding blocks, while $55m+1<54m+17$.

LEMMA 4. *For* $74m+20\leqq A\leqq 199m+37$, $E(A)\leqq 1$ *if* $m=7$, $E(A)\leqq m-7$ *if* $m\geqq 8$, *except that* $E(80m+21)=m-6$ *if* $m=8$ *or* 9.

From each block we suppress all entries down to and including the last entry which differs by 3 or more from the next entry. As in Lemma 3, $f+6$ succeeds except for $q=80$, 106, 156, 158, 169, 195. For $m=7$, $a=80m+21$ equals $79m+28$, whose E is 1. For $m\geqq 10$, we restore the previously excluded $80m+17$-18 and then have the permissible value $E(a)=3$ and $80m+16=m-7+79m+23$. But for $m=8$ or 9, $80m+18\leqq 79m+27=a'$, and the full table includes no number numerically between a' and a, whence $E(a)=m-6$.

For $q=106$, 169, 195, we may use $f+6$ after suppressing $106m+17$-23, $169m+26$-29, and $195m+27$.

For $m\geqq 8$, $b=156m+20=m-8+155m+28$. But if $m=7$, $b=155m+27$, which was treated under $q=155$.

Finally, let $q=158$. If $m\geqq 9$ we restore the previously excluded $158m+25$-30, noting that $E(158m+32)=2$ is a permissible value. Then $158m+24=m-7+t$, where $t=157m+31$ is in the table. If $m=7$ or 8, the latter result is applicable since the missing $158m+31$-32 are equal to two of $159m+23$-25.

Assistance was provided by the Carnegie Institution for the construction of Table I and checking results by it.

THE UNIVERSITY OF CHICAGO

THE SUMMABILITY OF FOURIER SERIES

BY M. H. STONE

In an article of mine which appeared recently in the Transactions of this Society, I applied the general theorems deduced to the particular case of Fourier series. In these applications I omitted an essential step.* The object of this note is to extend some of the methods of that paper in the case of Fourier series, to prove some general theorems about the summability of infinite series, and to supply thereby the reasoning which I omitted in the earlier treatment.

M. Riesz has given a general method of summing an infinite series,† of which a special case may be described as follows: a series of functions of the real variable x,

$$u_0(x) + u_1(x) + u_2(x) + \cdots,$$

is said to be summable (α^Δ, δ) to the value $U(x)$ if

$$S_\Omega^{\Delta,\delta}(x) \to U(x), \quad \Omega \to +\infty,$$

where

$$S_\Omega^{\Delta,\delta} \equiv \sum_{\alpha=0}^{<\omega} \left(1 - \frac{\alpha^\Delta}{\Omega}\right)^\delta u_\alpha(x), \ \Omega = \omega^\Delta, \ \Delta > 0, \ \delta \geqq 0.$$

Here ω is a continuous variable. A more general method of summation is obtained if ω is restricted to a sequence of values $0 < \omega_1 < \omega_2 < \cdots$, $\omega_\nu \to +\infty$. If

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* Stone, Transactions of this Society, vol. 28 (1926), pp. 695–761. See Theorems 14, 33, 35, 36, 37, 38.

† M. Riesz, Comptes Rendus, vol. 149 (1909), pp. 909–912; Comptes Rendus, vol. 152 (1911), pp. 1651–1654; Proceedings of the London Society, (2), vol. 22 (1923–24), pp. 412–419; Hardy and Riesz, *The General Theory of Dirichlet's Series* (Cambridge Tracts, No. 18, 1915), Chapters IV and V; Hardy, Proceedings of the London Society, (2), vol. 15 (1915–16), pp. 72–88.

$$S_\Omega^{\Delta,\delta}(x) \to U(x),$$

the series may be referred to as summable $(\alpha^\Delta, \delta; \omega_\nu)$ to the value $U(x)$.* Clearly summability by the first method implies summability by the second; but the converse is not true in general.† The omission referred to above is due to the neglect of this fact. In order to remedy it, we may follow one of two courses: we may extend the method involving summability $(\alpha^\Delta, \delta; \omega_\nu)$ which was employed in that paper to yield a method involving summability (α^Δ, δ); or we may prove that with respect to Fourier series the two methods are interchangeable. We shall investigate both possibilities.

It is convenient to introduce the definition of a restricted kind of equivalence.

DEFINITION. *Two definitions of summability are said to be equivalent with respect to a class of series when every series of that class summable by one method is summable by the other to the same sum; and when, further, uniform summability on a range of the variable x in one case implies uniform summability on the same range in the other.*

In examining the two types of summability defined above and their equivalence with respect to the class of Fourier series and certain related classes of series, we shall restrict Δ to integral values. It is first necessary to find a compact analytical expression for the sums $S_\Omega^{n,\delta}$. Without loss of generality we may suppose that δ is greater than zero; and may assume that the Lebesgue-integrable function $f(x)$

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* This type of summability has been applied to the study of Fourier series by W. H. Young, Leipziger Berichte, vol. 63 (1911), pp. 369–387; by Stone, loc. cit.; and by Bailey, Annals of Mathematics, vol. 27 (1926), pp. 69–102.

† Riesz, Comptes Rendus, vol. 152 (1911), pp. 1651–1654; Proceedings of the London Society, vol. 22 (1923–24), pp. 412–419. Bailey, loc. cit., Lemmas III and IV, states two equivalence theorems which in view of Riesz's London paper do not seem to be correct; his reference to Young seems to be in error, as Young, loc. cit., does not give any discussion of equivalence.

whose Fourier series is under consideration satisfies the condition $\int_0^1 f\,dy=0$, since this restriction modifies only the constant term of the Fourier series on (0, 1). If we let $G(x, y; \lambda)$ be the Green's function for the differential system $u'+\lambda u=0,\ u(0)-u(1)=0$, so that*

$$G(x,y;\lambda) = \left\{e^{\lambda(y-x)};\ 0\right\} + e^{\lambda(y-1)-\lambda x}/(1-e^{-\lambda}),\quad \mathcal{R}(\lambda)\geqq 0,$$

$$G(x,y;\lambda) = \left\{0;\ -e^{\lambda(y-x)}\right\} + e^{\lambda y+\lambda(1-x)}/(1-e^{\lambda}),\ \mathcal{R}(\lambda)\leqq 0,$$

and if we choose appropriate contours S_1 and S_2 enclosing the poles of $\int_0^1 f(y)G(x, y; \lambda)dy$ on the upper and lower half of the λ-plane respectively, we may write

$$S_\Omega^{n,\delta} = \frac{1}{2\pi i}\int_{S_1}\int_0^1 f(y)\left[1-(-i\lambda/\Lambda)^n\right]^\delta\left\{\frac{\partial^k G}{\partial x^k};\ \frac{\partial^k G}{\partial x^k}\right\}dy\,d\lambda$$

$$+\frac{1}{2\pi i}\int_{S_2}\int_0^1 f(y)\left[1-(+i\lambda/\Lambda)^n\right]^\delta\left\{\frac{\partial^k G}{\partial x^k};\ \frac{\partial^k G}{\partial x^k}\right\}dy\,d\lambda,$$

$$\Lambda = 2\pi\omega, \qquad (k=0,1,2,\cdots),$$

for the sum obtained from the kth derived series of the Fourier series on the interval (0, 1) for the function $f(x)$.† The contours S_1 and S_2 which we shall employ may be described as the boundaries of the two semicircular regions into which the circle $|\lambda|\leqq\Lambda$ is divided by the axis of reals. It will be convenient to denote the circular arcs in S_1 and S_2 by C_1 and C_2 respectively. The formula given above can then be verified by the calculus of residues; it is valid even when C_1 and C_2 pass through poles of the Green's function, because of the presence of the factors $(1-(\pm i\lambda/\Lambda)^n)^\delta$, $\delta>0$.

We first study the contribution to $S_\Omega^{n,\delta}$ from the bracketed terms in the Green's function. The method followed is sufficiently exemplified by the treatment of the portion of this contribution arising from that part of S_1 for which

* Stone, loc. cit., p. 717.

† Stone, loc. cit., §2 and Lemma 13.

$\mathcal{R}(\lambda) \geqq 0$. The integral over this path may be replaced by an integral from 0 to Λi along the axis of imaginaries, and therefore becomes

$$\frac{1}{2\pi i}\int_0^{\Lambda i}\int_0^x f(y)[1-(-i\lambda/\Lambda)^n]^\delta(-\lambda)^{\cdot}e^{\lambda(y-x)}dy\,d\lambda$$

$$=\frac{1}{2\pi}\int_0^x\int_0^1 f(-z+x)(1-\phi^n)^\delta(-i\Lambda\phi)^k\Lambda e^{-\Lambda i\phi z}d\phi\,dz$$

by the change of variables $z=x-y$, $\phi=-\lambda i/\Lambda$. The total contribution from the bracket terms is found in this way to be

$$\frac{1}{2\pi}\int_0^{1-x} f(z+x)\Lambda^{k+1}\Phi_{n,k,\delta}(\Lambda z)dz$$

$$+\frac{(-1)^k}{2\pi}\int_0^x f(-z+x)\Lambda^{k+1}\Phi_{n,k,\delta}(\Lambda z)dz,$$

where

$$\Phi_{n,k,\delta}(\alpha)=\int_0^1(1-\phi^n)^\delta\phi^k[(-i)^k e^{\alpha i\phi}+i^k e^{-\alpha i\phi}]d\phi.$$

By integrating $k+1$ times by parts in the integral defining $\Phi_{n,\,k,\,\delta}(\alpha)$ we can obtain a more convenient form for this function when $\delta \geqq k$. We denote by $h_m(\phi)$ the mth derivative of the function $(1-\phi^n)^\delta\,\phi^k$. If δ is greater than or equal to k the functions $h_0, \cdots, h_k$ are continuous functions of ϕ on the interval (0, 1); and h_{k+1} is continuous except at $\phi=1$ and is integrable in the sense of Lebesgue on (0, 1). Furthermore, the equations $h_m(0)=h_m(1)=0$ are satisfied when $m=0, \cdots, k-1$; and $h_k(1)=0$ if $\delta>k$. Thus the result of integrating $k+1$ times by parts is seen to be*

* In a similar integration by parts which occurs in the proof of Theorem 35 of my paper cited above one term has been overlooked; the necessary correction is easily supplied.

$$\Phi_{n,k,\delta}(\alpha) = (-1)^k \alpha^{-k} \int_0^1 h_k(\phi)\left[(-1)^k e^{\alpha i\phi} + (-1)^k e^{-\alpha i\phi}\right] d\phi$$

$$= 2\alpha^{-k} \int_0^1 h_k(\phi) \cos \alpha\phi \, d\phi$$

$$= 2\alpha^{-k-1} h_k(1) \sin\alpha - 2\alpha^{-k-1} \int_0^1 h_{k+1}(\phi) \sin \alpha\phi \, d\phi.$$

By the theorem of Riemann-Lebesgue, the integral in the last expression is $o(1)$.

We now apply this result to the further simplification of the contribution from the bracket terms investigated above. We restrict the variable x to lie on a fixed interval (a, b) completely interior to $(0, 1)$. We can then choose a positive number A less than both a and $1-b$. Now

$$\left| \frac{1}{2\pi} \int_A^{1-x} f(z+x) \Lambda^{k+1} \Phi_{n,k,\delta}(\Lambda z) dz - \frac{h_k(1)}{\pi} \int_A^{1-x} f(z+x) z^{-k-1} \sin \Lambda z \, dz \right|$$

is less than or equal to

$$\frac{1}{\pi} \int_A^{1-x} |f(z+x)| z^{-k-1} \left| \int_0^1 h_{k+1}(\phi) \sin \Lambda z\phi \, d\phi \right| dz$$

$$\leqq A^{-k-1}\eta \int_A^{1-x} |f(z+x)| \, dz \leqq A^{-k-1}\eta \int_0^1 |f(y)| \, dy \leqq \epsilon$$

for $\Lambda z \geqq \Lambda A$ sufficiently large. We can treat the second part of the contribution in the same way. Thus we can write the whole expression as

$$\frac{1}{2\pi} \int_0^A \left[f(z+x) + (-1)^k f(-z+x)\right] \Lambda^{k+1} \Phi_{n,k,\delta}(\Lambda z) dz + \frac{h_k(1)}{\pi} \left[\int_A^{1-x} f(z+x) z^{-k-1} \sin \Lambda z \, dz + \int_A^x (-1)^k f(-z+x) z^{-k-1} \sin \Lambda z \, dz \right] + o(1).$$

If we change the variables of integration in the second term, it may be written

$$\frac{h_k(1)}{\pi}\int_0^1 f(y)K(x,y\,;\,\Lambda)dy$$

where

$$K(x,y\,;\,\Lambda) = \sin\Lambda(y-x)/(y-x)^{k+1}, \qquad 0 \leqq y \leqq x-A,$$
$$K(x,y\,;\,\Lambda) = 0 \qquad\qquad x-A < y < x+A,$$
$$K(x,y\,;\,\Lambda) = \sin\Lambda(y-x)/(y-x)^{k+1}, \qquad x+A \leqq y \leqq 1,$$

for $a \leqq x \leqq b$; and this integral can easily be shown to be $o(1)$ on (a, b) by the theory of singular integrals.* The final form of the contribution from the bracket terms in G is therefore

$$\frac{1}{2\pi}\int_0^A [f(z+x)+(-1)^k f(-z+x)]\Lambda^{k+1}\,\Phi_{n,k,\delta}(\Lambda z)dz + o(1),$$

uniformly on (a, b), when δ is a positive number greater than or equal to k.

We now turn to the study of the contribution to $S_\Omega^{n,\delta}$ from the remainder of the Green's function, under the assumption that x lies on (a, b) and that δ is a positive number greater than or equal to k. First we show that the integrals taken along the real axis are negligible, and then discuss the integrals over C_1 and C_2.

The contribution from the positive real axis is seen to be

$$\frac{1}{2\pi i}\int_0^\Lambda \{[1-(-i\lambda/\Lambda)^n]^\delta - [1-(i\lambda/\Lambda)^n]^\delta\}$$
$$\times \int_0^1 [f(y)(-\lambda)^k e^{\lambda(y-1)-\lambda x}/(1-e^{-\lambda})]dy\,d\lambda.$$

Under our assumption that $\int_0^1 f\,dy$ vanishes the integrand is analytic in λ at the origin. Furthermore

* Lebesgue, Annales de Toulouse, (3), vol. 1 (1909), pp. 52–55.

$$\int_0^1 [f(y)(-\lambda)^k e^{\lambda(y-1)-\lambda x}/(1-e^{-\lambda})]\,dy = O(\lambda^k e^{-\lambda x}) = O(\lambda^k e^{-\lambda a}).$$

If we split the integral over $(0, \Lambda)$ into two others, one over $(0, \Lambda^{1/2})$, the other over $(\Lambda^{1/2}, \Lambda)$, we find the first to be

$$O\left(\int_0^{\Lambda^{1/2}} O(\Lambda^{-n/2}) O(\lambda^k e^{-\lambda a}) d\lambda\right) = o(1)$$

and the second to be

$$O\left(\int_{\Lambda^{1/2}}^{\Lambda} O(1) O(\lambda^k e^{-\lambda a}) d\lambda\right) = o(1).$$

In exactly the same way we can show that the integral over the negative axis of reals is also $o(1)$ uniformly on (a, b).

In discussing the integrals over the arcs C_1 and C_2 we shall give the details of the method used only for the integral taken over the arc c of C_1 lying on the first quadrant. This may be written

$$\int_0^1 f(y) k(x, y\,;\,\Lambda) dy,$$

where

$$k(x, y\,;\,\Lambda) = \frac{1}{2\pi i}\int_c \frac{[1-(-i\lambda/\Lambda)^n]^\delta (-\lambda)^k e^{\lambda(y-1)-\lambda x}}{(1-e^{-\lambda})} d\lambda.$$

It is our purpose to show that

$$k(x, y\,;\,\Lambda) = O(1),$$

$$\int_\alpha^\beta k(x, y\,;\,\Lambda)\,dy = o(1), \quad 0 \leqq \alpha < \beta \leqq 1,$$

uniformly on (a, b); the theory of singular integrals then shows that $\int_0^1 f(y)\ k(x, y; \Lambda) dy$ is $o(1)$ uniformly on (a, b).* We introduce the variable θ defined by the equation $e^{-i\theta} = -i\lambda/\Lambda$. When λ is on c, θ is a real variable on the interval $(0, \pi/2)$, so that

* Lebesgue, loc. cit.

$$[1 - (-i\lambda/\Lambda)^n]^\delta = \theta^\delta O(1),$$

$$(-\lambda)^k e^{\lambda(y-1)-\lambda x} = O(\Lambda^k e^{-\Lambda A\theta/2}),$$

$$\int_\alpha^\beta (-\lambda)^k e^{\lambda(y-1)-\lambda x} dy = O(\Lambda^{k-1} e^{-\Lambda A\theta/2}),$$

$$d\lambda = \Lambda O(1) d\theta.$$

For a given value of Λ, we determine an integer l such that $(2l-1)\pi \leqq \Lambda \leqq (2l+1)\pi$. If we then set $\lambda = \mu + 2l\pi i$ we find

$$1/(1 - e^{-\lambda}) = 1/(1 - e^{-\mu}) = O(1 + 1/|\mu|)$$

in the region composed of the half-plane $\mathcal{R}(\mu) \geqq m > 0$ and the rectangle $0 \leqq \mathcal{R}(\mu) \leqq m$, $-3\pi/2 \leqq \mathcal{I}(\mu) \leqq +3\pi/2$. If Λ and l are large c lies on the corresponding region for the variable λ. In terms of the variable θ we have

$$1/|\mu| = (1/\Lambda)|e^{-i\theta} - 2l\pi/\Lambda| = (1/\Lambda)[4l\pi(1 - \cos\theta)/\Lambda + (1 - 2l\pi/\Lambda)^2]^{1/2}$$

$$= O\{(1/\Lambda)[4l\pi(1 - \cos\theta)/\Lambda]^{1/2}\} = O\{1/(\Lambda\theta)\},$$

when λ is on c. Consequently

$$1/(1 - e^{-\lambda}) = O\{1/(\Lambda\theta)\} + O(1).$$

Thus

$$k(x, y; \Lambda) = O\left(\int_0^{\pi/2} \Lambda^k \theta^{\delta-1} e^{-\Lambda A\theta/2} d\theta\right) + O\left(\int_0^{\pi/2} \Lambda^{k+1} \theta^\delta e^{-\Lambda A\theta/2} d\theta\right)$$

$$= O(1),$$

if δ is a positive number not less than k; and

$$\int_\alpha^\beta k(x, y; \Lambda) dy = O\left(\int_0^{\pi/2} \Lambda^{k-1} \theta^{\delta-1} e^{-\Lambda A\theta/2} d\theta\right) + O\left(\int_0^{\pi/2} \Lambda^k \theta^\delta e^{-A\Lambda\theta/2} d\theta\right)$$

$$= O(1/\Lambda)$$

uniformly on (a, b) under the same conditions.* This establishes the desired result. It is evident now that the integrals over the other parts of C_1 and C_2 will yield to similar treatment. Thus we can state the first theorem.

THEOREM I. *The sum $S_\Omega^{n,\delta}$ formed for the kth derived series of the Fourier series for a Lebesgue-integrable function $f(x)$ on the interval $(0, 1)$ can be expressed as*

$$S_\Omega^{n,\delta} = \frac{1}{2\pi}\int_0^A [f(z+x) + (-1)^k f(-z+x)]\Lambda^{k+1} \cdot \Phi_{\delta,k,n}(\Lambda z)dz + o(1),$$

uniformly $0 < a \leqq x \leqq b < 1$, where δ is a positive number not less than k and A is a positive number less than both a and $1-b$. Here $\Lambda = 2\pi\omega$ is a continuous variable.

This theorem amounts to an extension of the methods of our paper cited above to include a treatment of summability (α^n, δ); it serves to bridge the gap which was left in the proofs of those theorems enumerated in the first footnote. By using this theorem we can arrive at further theorems, concerning the relation of summability (α^n, δ) to summability $(\alpha^n, \delta; \omega_\nu)$ with respect to the class of Fourier series and the class of derived series of Fourier series.

If $f(x)$ is absolutely continuous—that is, if it is an integral in the sense of Lebesgue—we can integrate by parts in the integral appearing in Theorem I. In view of the fact that

$$\int \Phi_{n,k,\delta}(\Lambda z)dz = -\Phi_{n,k-1,\delta}(\Lambda z)/\Lambda, \qquad k \geqq 1,$$

we obtain for $k \geqq 1$

$$S_\Omega^{n,\delta} = -\Lambda^k[f(A+x) + (-1)^k f(-A+x)]\Phi_{n,k-1,\delta}(\Lambda A) + \frac{1}{2\pi}\int_0^A [f'(z+x) + (-1)^{k-1} f(-z+x)]\Lambda^k \Phi_{n,k-1,\delta}(\Lambda z)dz + o(1).$$

* See Stone, loc. cit., Lemma V, for a more detailed discussion of the final steps here.

To evaluate the first term here we treat $\Phi_{n,k-1,\delta}(\Lambda.4)$ by the method of integration by parts outlined above. Since $\delta \geqq k$, we have

$$\Phi_{n,k-1,\delta}(\alpha) = -2\alpha^{-k}\int_0^1 \frac{d^k}{d\phi^k}[(1-\phi^n)^\delta \phi^{k-1}]\sin \alpha\phi\, d\phi.$$

In consequence the first term is $o(1)$ uniformly $a \leqq x \leqq b$. By repeating this process we obtain the second theorem.

THEOREM II. *If $f^{(l-1)}(x)$ is absolutely continuous,*

$$S_\Omega^{n,\delta} = \frac{1}{2\pi}\int_0^A [f^{(l)}(z+x) + (-1)^{(k-l)} f^{(l)}(-z+x)]\Lambda^{k-l+1} \cdot \Phi_{n,k-l,\delta}(\Lambda z)dz + o(1),$$

uniformly on (a, b), if $\delta \geqq k \geqq 1$ and $l \leqq k$. In other words, the sum formed for the kth derived series for $f(x)$ and the sum formed for the $(k-l)$th derived series for $f^{(l)}(x)$ have a difference which is uniformly $o(1)$ on (a, b), $l = 0, \cdots, k$.

We now compare summability (α^n, δ) and $(\alpha^n, \delta; \omega_\nu)$ for a Fourier series, $\delta > 0$, under the hypothesis that $\omega_{\nu+1} - \omega_\nu = o(1)$. A necessary and sufficient condition that the series be summable $(\alpha^n, \delta; \omega_\nu)$ to the sum $S(x)$ is that

$$I_\nu = \frac{1}{2\pi}\int_0^A F(z)\Lambda_\nu \Phi_{n,0,\delta}(\Lambda_\nu z)dz = o(1)$$

$$F(z) = f(z+x) + f(-z+x) - 2S(x), \quad \Lambda_\nu = 2\pi\omega_\nu,$$

the limit being approached uniformly on any closed set $\mathcal{E}$ on which the summability is uniform.* We consider the quantity

$$J_\nu = \frac{1}{2\pi}\int_0^A F(z)[\Lambda\Phi_{n,0,\delta}(\Lambda z) - \Lambda_\nu \Phi_{n,0,\delta}(\Lambda_\nu z)]dz$$

* For amplification of the reasoning here, see Stone, loc. cit., Theorem 14.

where $\Lambda_\nu \leqq \Lambda \leqq \Lambda_{\nu+1}$. By the process of integration by parts used above

$$\Lambda\Phi_{n,0,\delta}(\Lambda z) - \Lambda_\nu\Phi_{n,0,\delta}(\Lambda_\nu z) =$$
$$\int_0^1 \left\{ \frac{d}{d\phi}(1-\phi^n)^\delta \cdot [\sin \Lambda_\nu \phi z - \sin \Lambda \phi z]/z \right\} dz = O(1),$$

since

$$(\sin \Lambda_\nu \phi z - \sin \Lambda \phi z)/z = \phi \cos \Lambda'_\nu \phi z (\Lambda_\nu - \Lambda),$$

$\Lambda_\nu < \Lambda'_\nu < \Lambda$ by the law of the mean. Consequently,

$$|J_\nu| \leqq K \int_0^A |F(z)|dz \leqq K\Big(\int_0^A |f(z+x)|dz$$
$$+ \int_0^A |f(-z+x)|dz + A|S(x)| \Big),$$

where K is a suitably chosen positive constant. The last expression is $o(1)$ for each value of x when $A \to 0$. The first two terms are uniformly $o(1)$ on (a, b) by a fundamental property of the Lebesgue integral. On any closed set $\mathcal{E}$ on which the given Fourier series is uniformly summable $(\alpha^n, \delta; \omega_\nu)$, the function $S(x)$ is continuous and bounded; on this set $\mathcal{E}$, therefore, $A|S(x)|$ is uniformly $o(1)$ when $A \to 0$. Thus by choosing A sufficiently small we can make $|J_\nu| < \epsilon/2$, where ϵ is a preassigned positive constant. The value assigned to A may be taken independent of x, when x lies on a closed set $\mathcal{E}$ on which the summability $(\alpha^n, \delta; \omega_\nu)$ is uniform. Now

$$S_\Omega^{n,\delta} - S(x) = I_\nu + J_\nu + o(1),$$

where $I_\nu = o(1)$, for any value of x at which $S(x)$ exists. When, for such a value of x, a value of A has been determined so that $|J_\nu| < \epsilon/2$, a number L can be found so that for $\Lambda_{\nu+1} \geqq \Lambda \geqq \Lambda_\nu \geqq L$ we have

$$|I_\nu + o(1)| < \epsilon/2.$$

Thus we find

$$S_\Omega^{n,\delta} - S(x) = o(1),$$

the limit being uniform on any closed set $\mathcal{E}$ on which the summability $(\alpha^n, \delta; \omega_\nu)$ is uniform. Thus we can assert that summability $(\alpha^n, \delta; \omega_\nu)$ and summability (α^n, δ) are equivalent on (a, b) with respect to the class of Fourier series.

In order to extend the result about equivalence to the whole interval (0,1), we need only appeal to an elementary property of Fourier series. If $f(x)$ is defined outside the interval (0, 1) by the periodic relation $f(x+1)=f(x)$, it is found that the Fourier series for $f(x)$ and $f(x-1/2)$ have a simple and important relationship: when the variable x in the second series is replaced by $x+1/2$, the first series results. Thus the behavior of the Fourier series for $f(x)$ at $x=0$ or $x=1$ can be completely discussed by considering the behavior of the Fourier series for $f(x-1/2)$ at $x=1/2$. It is thus seen that summability $(\alpha^n, \delta; \omega_\nu)$ and summability (α^n, δ) are equivalent with respect to the class of Fourier series, without restriction as to the range considered.

THEOREM III. *If* $\omega_{\nu+1}-\omega_\nu=o(1)$, *then summability* $(\alpha^n, \delta; \omega_\nu)$ *and summability* (α^n, δ) *are equivalent with respect to the class of Fourier series.*

By combining Theorems II and III we find without difficulty a more general result.

THEOREM IV. *If* $\omega_{\nu+1}-\omega_\nu=o(1)$, *then summability* (α^n, δ) *and summability* $(\alpha^n, \delta; \omega_\nu)$ *are equivalent with respect to the class of kth derived series of the Fourier series of functions whose* $(k-1)$*th derivatives are absolutely continuous,* $\delta \geqq k \geqq 1$.

COLUMBIA UNIVERSITY

ON THE SEPARATION OF THE PLANE BY IRREDUCIBLE CONTINUA†

BY W. A. WILSON

1. *Introduction.* This question was investigated by A. Rosenthal‡ in 1919. His principal results may be stated as follows. If F is the union of two bounded continua, C_1 and C_2, which are irreducible between the points a and b and have no other common points, then the complement of F with respect to its plane consists of two principal component regions (*Hauptgebiete*), each of which has F as its frontier, and possibly of a number of secondary component regions (*Nebengebiete*), each of which has its frontier wholly in either C_1 or C_2.

It is a simple matter to construct two irreducible continua which together constitute the frontier of precisely two of the complementary regions, but which have more than two points in common. For example, let bmc and bnc be two complementary arcs of a circumference, and let am^* and an^* be two wavy lines intersecting only at a and approaching asymptotically bmc and bnc, respectively. Then $C_1=am^*+bmc$ and $C_2=an^*+bnc$ are both irreducible between a and b and divide the plane into three regions, of which two only have C_1+C_2 as their frontier. But C_1 and C_2 have in common three points, not two.

In this particular case, to be sure, Rosenthal's theorem can be used, for C_1+C_2 can be expressed as the union of two continua, irreducible between a pair of points and intersecting only in these points. This is possible here because C_1 and C_2 can each be separated into two irreducible continua having only one common point. Such a decomposability,

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† Presented to the Society, October 30, 1926.

‡ (A) A. Rosenthal, *Teilung der Ebene durch irreduzibele Kontinua*, Sitzungsberichte der Münchener Akademie, 1919, pp. 91–109.

however, is not a general property of irreducible continua, even when no indecomposable continua are involved. Therefore a removal of the requirement that the two irreducible continua have but two points in common is a real extension of the theorem quoted above.

It is the purpose of the present article to give some general conditions under which this extension is possible and also to investigate the frontiers of the secondary regions. The principal results will be found in §§ 5 and 8.

In certain parts of the work free use is made of the oscillatory set of a continuum about a point and its properties, which have been developed by the author in the papers listed below.* The notation of these papers is also used.

2. *Some Generalizations.* For what follows it is convenient to extend the notion of a continuum irreducible between two points in two ways which must be kept distinct.

DEFINITION I. *If A is a closed sub-set of the continuum C, then C is irreducible about A if no proper sub-continuum of C contains A.*

DEFINITION II. *If α and β are closed sets without common points and the continuum C contains one or more points of both sets, then C is irreducible between α and β if no proper sub-continuum of C contains points of both sets.*

The second definition is due to Miss Anna M. Mullikin.† It is readily seen that if A or $\alpha+\beta$ consists of two points, both definitions yield the ordinary continuum irreducible between two points. The following properties will be used.

LEMMA I. *If C is a bounded continuum irreducible about the sum of the two continua A and B, which have no common points, then $C-(A+B)$ is connected.*

* (B) *On the oscillation of a continuum,* Transactions of this Society, vol. 27.

(C) *Some properties of the irreducible continuum,* ibid., vol. 28.

(D) *On the structure of the irreducible continuum,* American Journal, vol. 48.

† (E) Anna M. Mullikin, *Certain theorems relating to plane connected sets,* Transactions of this Society, vol. 24.

The proof given by Rosenthal (loc. cit., p. 104) for the case where A and B are points is immediately applicable.

LEMMA II. *Let C be bounded and closed and contain points of both the closed sets α and β. Let $\alpha \cdot \beta = 0$. Then C can be separated into two closed sets C_1 and C_2, such that $C_1 \cdot C_2 = \alpha \cdot C_2 = \beta \cdot C_1 = 0$, or else C contains a sub-continuum irreducible between α and β.*

This theorem is stated and proved in an equivalent form by Miss Mullikin (loc. cit., p. 147).

LEMMA III. *Let C be a bounded continuum irreducible between the closed subsets α and β, which have no common points. Let A be the oscillatory set of C about some point a of α. Then A contains α and, if D is a sub-continuum of C containing a point x of $C-A$ and a point a' of α, D contains A.*

PROOF. We omit the trivial case that C is indecomposable. If A is complete, $C-A$ is a semi-continuum by a theorem proved elsewhere.* Obviously it contains β and hence can contain no point of α. Let E be a sub-continuum of $C-A$ joining x and β. Then $D+E$ joins α and β, and hence equals C. As $A \cdot E = 0$, D contains A.

If A is not complete, it is indecomposable and is not a continuum of condensation.† Then $\overline{C-A}$ is a proper sub-continuum‡ of C and contains β. Hence it contains no point of α. Thus A contains α in this case too. As x is a point of $C-A$, $D+\overline{C-A} = C$; hence D contains $C-\overline{C-A}$. Since $\overline{C-\overline{C-A}}$ is a sub-continuum of A and contains all points of A whose distance from a is less than some positive δ, it is identical with A. Thus D contains A.

3. *A Special Type of Frontier Set.* Let $F = H_1 + H_2$, where H_1 and H_2 are bounded continua and $F - H_2 = H_1$

* See reference (C), p. 543.

† See reference (D), p. 155.

‡ (F) C. Kuratowski, *Théorie des continus irréductibles*, Fundamenta Mathematicae, vol. 3, Theorem 3.

$-H_1 \cdot H_2$ and $F-H_1=H_2-H_1 \cdot H_2$ are connected sets. Certain properties of this type of set and its complementary regions are easily deduced.

If Z denotes the plane, it follows from the fact that $F-H_1$ is connected that $F-H_1$ lies in some one component G_1 of $Z-H_1$. Likewise, $F-H_2$ lies in one component G_2 of $Z-H_2$.

Now let G' be any other component of $Z-H_1$ and G'' any other component of $Z-H_2$. We first show that, if $G' \cdot G_2 \neq 0$, then G_2 contains G'. For otherwise G' would contain points of H_2, the frontier of G_2, since G' is connected. This is impossible, for G_1 is the only component of $Z-H_1$ containing points of H_2. Likewise, either $G'' \cdot G_1=0$ or G_1 contains G''.

A second property regarding these components is that either $G' \cdot G''=0$ or else $G'=G''$ and the frontier of G' is a part of some component of $H_1 \cdot H_2$. To prove this let us assume that $G' \cdot G'' \neq 0$. Then, if G'' contained points not in G', it would contain points of the frontier of G', which is a part of H_1. This is impossible by the second paragraph above. Likewise, G' contains no point not in G'' and hence $G'=G''$. Let g' denote the frontier of G'. Then $g' \subseteq H_1H_2$. As g' is a continuum, it is a part of some component of H_1H_2.

With these preliminaries, we are in a position to prove the following lemma.

4. Lemma IV. *Let F be the union of two bounded continua H_1 and H_2 having these properties: $F-H_1$ and $F-H_2$ are connected;** $H_1 \cdot H_2=A+B$, where A and B are continua and $A \cdot B=0$; H_1 and H_2 contain sub-continua C_1 and C_2, respectively, such that $\alpha=C_1 \cdot C_2 \cdot A \neq 0$, $\beta=C_1 \cdot C_2 \cdot B$, C_1 and C_2 are irreducible between α and β, and $F=C_1+C_2$. Then F cuts the plane and is the frontier of exactly two components of its complement.*

* In view of §2, Lemma I, this requirement may be replaced by the hypothesis that H_1 and H_2 are irreducible about A and B.

PROOF.* It was shown in §3 that, if Z denotes the plane, one component G_1 of $Z-H_1$ contains $H_2-(A+B)$ and one component G_2 of $Z-H_2$ contains $H_1-(A+B)$. Let $K_1=Z-G_1$ and $K_2=Z-G_2$. Then K_1 and K_2 are continua which do not cut the plane.

Now consider $K_1 \cdot K_2$. As K_1 consists of H_1 and the components G' of $Z-H_1$ differing from G_1, and K_2 is similarly constituted, it is evident from §3 that $K_1 \cdot K_2$ consists of $A+B$ and such components G' of $Z-H_1$ as coincide with components G'' of $Z-H_2$. In §3 it was shown that, if $G'=G''$, then g', the frontier of G', is a part of either A or B. In consequence of these facts $K_1 \cdot K_2$ is the sum of two continua, which may be denoted by A' and B', where A' contains A and may contain one or more regions whose frontiers form a part of A, B' contains B and may contain one or more regions whose frontiers form a part of B, and $A' \cdot B'=0$.

Two cases must now be considered: (a) one of the regions G_1 and G_2 is unbounded; (b) neither G_1 nor G_2 is unbounded. We shall complete the proof for the first case and then return to the second.

If G_1 is unbounded, K_1 is bounded. For otherwise a point in K_1 could be joined to one in G_1 by a broken line not cutting the bounded set H_1, which is the frontier of G_1. Likewise K_2 is bounded, if G_2 is unbounded. Thus either K_1 or K_2 is bounded. We have seen that neither of these continua separates the plane and that their divisor is two continua without common points. We can therefore apply a theorem of Miss Mullikin,† which shows that K_1+K_2 cuts the plane into exactly two regions, R_1 and R_2. Let F_1 and F_2 be the frontiers of these regions. If we can show that $F_1=F_2=F$, the theorem is proved. To this end we first observe that, since no point of F_1 or F_2 can be an inner point of K_1 or K_2, then F_1 and F_2 are parts of $H_1+H_2=F=C_1+C_2$.

* (G) This proof is a modification of the proof of Rosenthal's theorem given by S. Straszewicz in his paper *Über die Zerschneidung der Ebene durch abgeschlossene Mengen*, Fundamenta Mathematicae, vol. 7, p. 187.

† See reference (E), p. 160; also reference (G), §§18, 19.

It is easily seen that $R_1+R_2=G_1 \cdot G_2$. Now if m is a point of R_1 and n is one of R_2, there is a broken line in G_1 joining m and n and not cutting H_1, for $G_1 \cdot H_1=0$. Thus H_1 (and in like fashion H_2) is not an $S(m, n)$.*

There are two sub-cases: either F_1 contains C_1 or C_2, or it does not. To fix the ideas, let $C_1 \subset F_1$. Then $F_1 \cdot C_2$ is not void by the previous paragraph. Since C_2 is irreducible between α and β, it follows from §2, Lemma II, that, if $F_1 \cdot C_2 \neq C_2$, then $F_1 \cdot C_2$ is the sum of two closed sets M and N, such that $M \cdot N=M \cdot \beta=N \cdot \alpha=0$. (One of these, but not both, may be void.) Then $F_1=M+C_1+N$. Since $M+A \subseteq H_2$ and $A+C_1 \subseteq H_1$, neither $M+A$ nor $A+C_1$ is an $S(m, n)$. Hence by a theorem of Janiszewski† $M+A+C_1$, and consequently $M+C_1$, is not an $S(m, n)$. Likewise, C_1+N is not an $S(m, n)$ and a second application of the theorem referred to gives the contradiction that F_1 is not an $S(m, n)$. Hence $F_1 \cdot C_2=C_2$, and $F_1=C_1+C_2=F$.

Now suppose that neither C_1 nor C_2 is a part of F_1. Since C_1 is irreducible between α and β, $F_1 \cdot C_1$ is the sum of two closed sets M and N, such that $M \cdot N=N \cdot \alpha=M \cdot \beta=0$. Either M or N, but not both, may be void. For the same reason $F_1 \cdot C_2=P+Q$, where P and Q are closed and $P \cdot Q=P \cdot \beta=Q \cdot \alpha=0$. Then $F_1=(M+P)+(N+Q)$ and $(M+P)(N+Q)=0$, which is a contradiction, as F_1 is a continuum, unless both M and P, or N and Q, are void. If $M=P=0$, $F_1=N+Q \subseteq N+B+Q$. As neither $N+B$ nor $B+Q$ is an $S(m, n)$, this is a contradiction.

Thus in both sub-cases $F_1=C_1+C_2=F$, and in like fashion $F_2=F$.

It now remains to take care of the situation that arises when neither G_1 nor G_2 is unbounded. If v is a point of G_1

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* A set H is an $S(m, n)$ if every continuum joining m and n cuts H.

† (H) Z. Janiszewski, *Sur les coupures du plan faites par des continus*, Prace Matematyczno-Fizyczne, vol. 26, Theorem A: "If P and Q are bounded closed sets, if $P \cdot Q$ is connected, and neither P nor Q is an $S(m, n)$, then $P+Q$ is not an $S(m, n)$." See also Straszewicz, Fundamenta Mathematicae, vol. 4, p. 129.

not on H_2 and the plane is inverted with respect to v as a center, the image F^* of F will satisfy the conditions of the first case. This follows from the facts that the correspondence between F and F^* is homeomorphic, that the image G_1^* of G_1 is unbounded, that the images of the frontiers of the component regions of $Z-F$ are the frontiers of the images of these regions, and that the property of irreducibility is an invariant of analysis situs. Then by the proof given above there are precisely two components, R_1^* and R_2^*, of $Z-F^*$ which have F^* as their frontier. Inverting again, we have the inverse images R_1 and R_2 of R_1^* and R_2^*, respectively, as the only components of $Z-F$ which have F as a frontier.

5. *A General Theorem.* It is obvious that Rosenthal's theorem is the special case of the above lemma obtained when A and B are points, in which case $H_1=C_1$ and $H_2=C_2$. We can, however, obtain from this lemma the following theorem, which is, aside from one exceptional case, more general than the theorem in question.

THEOREM I. *Let F be the union of two bounded continua C_1 and C_2 having these properties: $C_1 \cdot C_2=\alpha+\beta$, where α and β are closed and $\alpha \cdot \beta=0$; both C_1 and C_2 are irreducible between α and β; and either C_1 and C_2 are both decomposable, or C_1 is indecomposable and C_2 is decomposable and not the union of two indecomposable continua. Then F cuts the plane and is the frontier of exactly two components of its complement.*

PROOF. This theorem is a corollary of §4. If we let A_1 and A_2 denote the oscillatory sets of C_1 and C_2, respectively, about some point a of α, and let B_1 and B_2 have the same meaning for a point b of β, it follows from §2, Lemma III, that $\alpha=A_1 \cdot A_2$ and $\beta=B_1 \cdot B_2$.

If both continua are decomposable, take $A=A_2$ and $B=B_1$, and set $H_1=A+C_1+B=A_2+C_1$ and $H_2=A+C_2+B=C_2+B_1$. Then $H_1 \cdot H_2=A+B=A_2+B_1$. If C_1 is indecomposable, take $A=A_2$ and $B=B_2$, and set $H_1=A+C_1+B=A_2+C_1+B_2$ and $H_2=A+C_2+B=C_2$. Again $H_1 \cdot H_2=A+B$.

In both cases $A \cdot B = 0$, H_1 and H_2 are irreducible about $A+B$, and $F = H_1+H_2 = C_1+C_2$. Hence §4 gives the theorem.

6. *The Secondary Regions.* Let R_1 and R_2 be the two principal regions, which have the frontier $F = C_1+C_2$, and let R' denote any one of the other components of $Z-F$, that is, one of the secondary regions defined by F if any exist.

Since R' is a component of $Z-F$, it is a part of a component of $Z-H_1$ and of a component of $Z-H_2$. It is not a part of both G_1 and G_2, for, as seen in §4, $G_1 \cdot G_2 = R_1+R_2$. To fix the ideas let $R' \subseteq G'$, where G' is a component of $Z-H_1$ different from G_1. By definition of G' and G_1, $G' \cdot H_2 = 0$; hence $\overline{G}' \cdot F \subseteq H_1$. Then, if F' is the frontier of R', $F' \subseteq H_1$. Likewise, if R' is a part of some component of $Z-H_2$ different from G_2, $F' \subseteq H_2$.

Since $F' \subset F$ and $R_1 \cdot R' = 0$ by hypothesis, it follows that, if m is a point of R' and n one of R_1, then F' is an $S(m, n)$. It is also an irreducible $S(m, n)$.* For, if K is a closed sub-set of F' and x is a point of $F'-K$, then for a sufficiently small positive δ there is a circular region U_δ containing x and points of both R' and R_1, but no point of K. Then m and n can be joined by a broken line not cutting K. Thus no proper closed part of F' is an $S(m, n)$.

If $F' \subseteq H_1$, but F' is not a part of C_1, there are two cases to consider. If C_1 is decomposable, it follows from the definition of H_1 (§5) that F' contains points of A_2 not in C_1. If $F' \cdot (C_1-A_1) \neq 0$, we have a contradiction. For then by §2, Lemmas II and III, $F' \supset A_1$. Then $F' \cdot C_1$ and $F' \cdot (A_1+A_2)$ have in common the continuum A_1. As both of these sets are proper closed parts of F', neither is an $S(m, n)$, which gives the contradiction that F' is not an $S(m, n)$. Thus we have $F' \cdot (C_1-A_1) = 0$ and in this case $F' \subseteq A_1+A_2$.

If C_1 is indecomposable, then C_2 is decomposable and not the sum of two indecomposable continua. Then $A_2 \cdot B_2 = 0$ and the argument used in the previous paragraph shows

* A set P is called an irreducible $S(m, n)$ if no proper closed part of P sian $S(m, n)$.

that we arrive at a contradiction if we assume that F' contains points of both A_2 and B_2 not in C_1. As C_1 is indecomposable, $A_1=B_1=C_1$; hence either $F'\subseteq A_1+A_2$, or $F'\subseteq B_1+B_2$.

The same sort of discussion is employed if $F'\subseteq H_2$. Thus we have found that, *if R' is a secondary region defined by the continuum F of §5, then the frontier of R' is a part either of C_1, or of C_2, or of A_1+A_2, or of B_1+B_2.* For a closer determination of this frontier we need the following theorem.

7. THEOREM II. *Let C be a bounded plane continuum irreducible between the points a and b. Let m and n be two points not on C and let K be a closed sub-set of C which is an irreducible $S(m, n)$. Then K is a part or the whole of the oscillatory set of C about one of its points**.

PROOF. CASE I. *K is a continuum of condensation.* Let K_a and K_b be the saturated semi-continua of $C-K$ containing a and b, respectively, and let neither be void. Since K is a continuum of condensation, $C=\overline{K}_a+\overline{K}_b$. Then $K\cdot\overline{K}_a$ and $K\cdot\overline{K}_b$ have a common point c, which is a limiting point of both K_a and K_b. Then the reasoning in §13 of the paper mentioned in reference (C) shows that K is a part of the oscillatory set about c. Similar reasoning establishes the theorem for the case that either K_a or K_b is void.

CASE II. *K is not a continuum of condensation.* Let us assume that K contains neither a nor b. Then there are subcontinua A_k and B_k, irreducible between $\overline{K}$ and a and b, respectively, and $A_k\cdot B_k=0$.* Then $L=\overline{C-(A_k+B_k)}$ is irreducible‡ between A_k and B_k, and $L\subseteq K$. We first show that L is indecomposable or is the union of two indecomposable continua. For, if not, there is an irreducible de-

* If every oscillatory set of C is complete, the theorem can be deduced easily from Theorem 9 of a paper by R. L. Moore, *Concerning upper semi-continuous collections of continua*, Transactions of this Society, vol. 27 (1925), pp. 416–428.

† See reference (F), Theorems II and IV.

‡ See reference (C), §4.

composition* of L into two proper sub-continua M and N such that

$$M \cdot B_k = N \cdot A_k = 0$$

and either M or N, say N, is decomposable. Obviously N is irreducible between $A_k + M$ and B_k and there is an irreducible decomposition of N into two proper sub-continua P and Q such that

$$P \cdot B_k = (A_k + M) \cdot Q = 0.$$

Then we have

$$C = (A_k + M + P) + (P + Q + B_k),$$
$$(A_k + M + P) \cdot (P + Q + B_k) = P.$$

Since $K \supseteq L$,

$$K = K(A_k + M + P) + K(P + Q + B_k)$$

is a decomposition of K into two closed proper sub-sets whose common part is a continuum. This is a contradiction, by reference (H). There are, then, two sub-cases to discuss: (*a*) when L is the union of two indecomposable continua; (*b*) when L is indecomposable.

(*a*) Let $L = M + N$, where M and N are indecomposable and $M \cdot B_k = N \cdot A_k = 0$. Reasoning similar to the above shows that in this case K contains no points not on L. Let x be a point of $M \cdot N$; then it is not a point of A_k or B_k. It is easily seen that $A_k + M$ is irreducible between a and x, and $N + B_k$ between x and b. Then, since M and N are indecomposable, M and N are the oscillatory sets of $A_k + M$ and $N + B_k$ respectively about x. Hence

$$K = L = M + N$$

is the oscillatory set of C about x.†

(*b*) Let L be indecomposable. From the definition of oscillatory sets it is evident that L is the oscillatory set of C about any point of $C - (A_k + B_k)$; hence the theorem holds if $K = L$. If $K \neq L$, it follows by reasoning similar to

* See reference (D), p. 156.

† See reference (D), p. 153.

that used above that either $(K-L)\cdot A_k$ or $(K-L)\cdot B_k$ is void, say the latter. Then $K \subseteq A_k+L$. Let y be a point of $A_k \cdot L$; then A_k is irreducible between a and y, and $L+B_k$ is irreducible between y and b. Let Y' be the oscillatory set of A_k about y. Since y does not lie on B_k, the oscillatory set of $L+B_k$ about y is L. Then $Y=Y'+L$ is the oscillatory set of C about y.*

If $K \subseteq Y$, the theorem is proved; if not, we arrive at a contradiction. For K will contain a point z on A_k-Y'. Then, if Y_a' is the saturated semi-continuum of A_k-Y' containing a, $\overline{Y}_a' \supset A_k-Y' \supset z$.† Then $\overline{Y}_a'=A_k$, since A_k is irreducible between a and K. Hence Y' is complete‡ and z is a point of Y_a'. As K joins Y_a' and B_k, this shows that $K \supset Y'$. Then $K \cdot A_k$ and $Y'+L$ are closed proper parts of K and their divisor is the continuum Y'. This is a contradiction, as their union is K, which is an irreducible $S(m, n)$.

Thus the proof is complete, except for the special cases where K contains a or b, or both. The above demonstration holds for these cases, if we merely replace A_k by a if K contains a and B_k by b if K contains b.

8. THEOREM III. *Let F be the union of two bounded continua C_1 and C_2 having these properties: $C_1 \cdot C_2=\alpha+\beta$, where α and β are closed and $\alpha \cdot \beta=0$; both C_1 and C_2 are irreducible between α and β; and either C_1 and C_2 are both decomposable, or C_1 is indecomposable and C_2 is decomposable and is not the union of two indecomposable continua. Then the frontier of each secondary region determined by F is a part or the whole of some oscillatory set of C_1 or C_2, or it is a part or the whole of the union of the oscillatory sets of C_1 and C_2 about some point of α or β.*

PROOF. Let R' be any secondary region and let F' be its frontier. Let A_1, A_2, B_1, and B_2 be the oscillatory sets of C_1

* See referrence (D), p. 153.
† See reference (C), §10.
‡ See reference (C), §15.

and C_2 about a point a of α and a point b of β, respectively.

It was shown in §6 that, if F' is not a part of A_1+A_2 or B_1+B_2, then F' is a part of C_1 or C_2. Since C_1 and C_2 are each irreducible between α and β, each of them is irreducible between a point a of α and a point b of β. Hence, if $F' \subseteq C_1$, the theorem of §7 shows that F' is a part of some oscillatory set of C_1. Thus the theorem is proved.

9. *Conclusion.* A consequence of the previous theorem is that under the hypotheses there stated the frontier of a secondary region is a part of either a continuum of condensation, an indecomposable continuum, a pair of indecomposable continua, or the union of a continuum of condensation and an indecomposable continuum. This follows at once from the proof of §7 and the fact that the oscillatory set of a bounded irreducible continuum about one of its end points is either a continuum of condensation or an indecomposable continuum.* Obviously there are no secondary regions unless A_1+A_2, or B_1+B_2, or some oscillatory set of C_1 or C_2 cuts the plane.

The conditions imposed on the character of the continua C_1 and C_2 in §5 take care of all possible cases with two exceptions. These are that both continua are indecomposable or that one is indecomposable and the other is the union of two indecomposable continua. With regard to the first it has been shown by Kuratowski† and Knaster‡ that the extended theorem of Rosenthal (§5) need not hold in this case. Whether or not it holds in the second case remains to be proved.

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* See reference (D), p. 155.

† C. Kuratowski, *Sur les coupures irréductibles du plan*, Fundamenta Mathematicae, vol. 6, p. 138.

‡ B. Knaster, *Quelques coupures singulières du plan*, Fundamenta Mathematicae, vol. 7, p. 281.

ON THE MAPPING OF THE SEXTUPLES OF THE SYMMETRIC SUBSTITUTION GROUP G_6 IN A PLANE UPON A QUADRIC*

BY ARNOLD EMCH

1. *Introduction.* The six permutations of three elements x_1, x_2, x_3 considered as projective coordinates in a plane determine an involution of sextuples of points which may be mapped on a rational surface.† I shall show that in case of the involution thus defined the map is a quadric whose relation with the plane, established with sufficient details, will lead to some interesting geometric applications. The map of every configuration on the quadric will be a configuration in the plane, invariant under the G_6, whose geometric properties have been investigated before.‡

2. *Mapping of the G_6.* Let ϕ_1, ϕ_2, ϕ_3 represent the elementary symmetric functions $\phi_1 = x_1 + x_2 + x_3$, $\phi_2 = x_2x_3 + x_3x_1 + x_1x_2$, $\phi_3 = x_1x_2x_3$, so that the general symmetric function of degree four has the form

$$y_i = a_i\phi_1^4 + b_i\phi_1^2\phi_2 + c_i\phi_1\phi_3 + d_i\phi_2^2, \tag{1}$$

depending upon three effective constants. Hence, there are four linearly independent functions y_i; $i = 1, 2, 3, 4$, which we may set proportional to the four projective coordinates of a point in a space S_3. Thus to every point in (x), and consequently to every sextuple $I_6\{(x_1x_2x_3), (x_1x_3x_2), (x_2x_1x_3), (x_2x_3x_1), (x_3x_1x_2), (x_3x_2x_1)\}$, corresponds in S_3 a point (y), and as the system of sextuples is a continuous ∞^2 manifold, the locus of such points (y) must be a surface, which will be proved to be a quadric cone Q. For the y_i's we may

* Presented to the Society, September 9, 1927.

† Castelnuovo, G., *Sulla razionalità delle involuzioni piane*, Mathematische Annalen, vol. 44 (1894), pp. 125–155.

‡ Emch, A., *Some geometric applications of symmetric substitution groups*, American Journal, vol. 45 (1923), pp. 192–207.

evidently choose any four linearly independent symmetric quartics. For every choice of four such functions we obtain a certain surface Q. But all these surfaces are obviously collinearly related. The simplest choice is

$$(2)\quad \begin{cases} \rho y_1 = x_1^4 + x_2^4 + x_3^4 = \phi_1^4 - 4\phi_1^2\phi_2 + 4\phi_1\phi_3 + 2\phi_2^2, \\ \rho y_2 = x_2^2 x_3^2 + x_3^2 x_1^2 + x_1^2 x_2^2 = \phi_2^2 - 2\phi_1\phi_3, \\ \rho y_3 = x_1^3(x_2 + x_3) + x_2^3(x_3 + x_1) + x_3^3(x_1 + x_2) \\ \qquad = \phi_1^2\phi_2 - 2\phi_2^2 - \phi_1\phi_3, \\ \rho y_4 = x_1^2 x_2 x_3 + x_2^2 x_3 x_1 + x_3^2 x_1 x_2 = \phi_1\phi_3. \end{cases}$$

A simple elimination process of ϕ_1, ϕ_2, ϕ_3 leads to the required relations between the y's:

$$(3)\quad y_1(y_2 + 2y_4) + 2y_2^2 - y_3^2 - y_4^2 + 4y_2y_4 - 2y_3y_4 = 0,$$

which is a quadric cone Q with the vertex $V(4, -2, -1, -1)$, as can easily be verified.

To a plane section of Q corresponds in (x) a quartic which may be any of the reducible or irreducible types (1). Of particular importance are, of course, the exceptional elements of the (1, 6) correspondence between Q and (x). In the first place for the intersections $I(1, \omega, \omega^2)$, $J(1, \omega^2, \omega)$ of $\phi_1 = 0$ and $\phi_2 = 0$ there is $y_1 = y_2 = y_3 = y_4 = 0$, so that I and J are fundamental points in (x). To the first neighborhoods of I and J, $(1+\alpha_1, \omega+\alpha_2, \omega^2+\alpha_3)$ and $(1+\alpha_1, \omega^2+\alpha_2, \omega+\alpha_3)$, correspond on Q for $\alpha_1+\alpha_2+\alpha_3 \neq 0$, $\rho y_1 = 4(\alpha_1+\alpha_2+\alpha_3)$, $\rho y_2 = -2(\alpha_1+\alpha_2+\alpha_3)$, $\rho y_3 = -(\alpha_1+\alpha_2+\alpha_3)$, $\rho y_4 = \alpha_1+\alpha_2+\alpha_3$, or the point $V(4, -2, -1, 1)$. To a sextuple on $\phi_1 = 0$, distinct from I and J, corresponds on Q the point $T(2, 1, -2, 0)$. A plane $p = y_1 + \lambda y_2 + \mu y_3 + \nu y_4 = 0$ cuts Q in a conic K and the join VT in a point R on K, to which corresponds in (x) the quartic

$$(4)\quad \phi_1^4 + (\mu - 4)\phi_1^2\phi_2 + (4 - 2\lambda - \mu + \nu)\phi_1\phi_3 + (2 + \lambda - 2\mu)\phi_2^2 = 0,$$

which has $\phi_1 = 0$ as a double tangent. To a generic point R on VT correspond thus the first neighborhoods of I and J on $\phi_1 = 0$. To a plane

(5) $$y_1 + \lambda y_2 + \mu y_3 + (2\lambda + \mu - 4)y_4 = 0$$

through V, corresponds the quartic

(6) $$\phi_1^4 + (\mu - 4)\phi_1^2\phi_2 + (2 + \lambda - 2\mu)\ \phi_2^2 = 0,$$

which clearly reduces to the product of two conics of the type $\phi_1^2 + k\phi_2 = 0$. Thus a generic plane of the bundle through V cuts Q in two generatrices to which correspond in (x) two conics of the symmetric pencil $\phi_1^2 + k\phi_2 = 0$. To a tangent plane of Q corresponds a double conic $(\phi_1^2 + k\phi_2)^2 = 0$. The tangent plane at R

$$y_1 + 6y_2 + 4y_3 + 12y_4 = 0$$

touches Q along VT, and to this intersection of the tangent plane, VT counted twice, corresponds in (x) the quadruple line $\phi_1^4 = 0$.

3. *Mapping of Intersections of the Quadric Cone.* A generic surface F_n cuts Q in a space curve C_{2n} to which corresponds in (x) a symmetric curve C'_{4n} (curve in which the coordinates enter symmetrically). As F_n cuts VT in n points, $\phi_1 = 0$ is, in general, an M-fold double tangent of C'_{4n}. This also appears directly from the fact that in (1) ϕ_2^2 appears in y_1, y_2, y_3, and ϕ_1 is a factor of all other terms in which ϕ_2^2 is not contained.

Conversely to a symmetric n-ic C'_n in (x) corresponds on Q a curve, whose order can easily be determined in every case. For instance, when C'_n does not pass through I and J, which is the case when C'_n contains the term ϕ_3^m, $3m = n$, then a generic quartic C'_4 cuts C'_n in $12m$ points which form $2m$ sextuples. To these correspond on Q, $2m$ points which lie on a plane of the corresponding conic K of C_4, and which are the intersections of the curve C on Q, corresponding to C'_4. The order of C is therefore $2m$. The curve C_{2m} on Q is cut out by a surface F_m, which may possibly pass through generatrices of Q, or through the point T. For example when C'_n is a sextic, then F_m is a quadric which passes through a generatrix of Q, or is a quadric cone with its

vertex at T. This is in agreement with the counting of constants. The number $N+1$ of terms of a symmetric n-ic is equal to the number of positive integral solutions of the diophantine equation $\alpha+2\beta+3\gamma=n$ resulting from the general term $\phi_1^{\alpha}\phi_2^{\beta}\phi_3^{\gamma}$ of the n-ic, and is equal to the nearest positive integer contained in $(n+3)^2/12$. For $n=6$, $N+1=7$. The number of constants in F_2 is 10 which is reduced to 7 by the condition to pass through a generatrix of Q. To the intersection of F_2 with Q corresponds in (x) an octavic. But to the generatrix of Q, common to F_2, corresponds a symmetric conic as a factor of the octavic, so that a sextic remains as a residual curve.

More generally F_m cuts Q in a curve to which corresponds in (x) a curve of order $4m$. In order that this reduce to $3m$ it is necessary that a factor of order m split off. These factors are of the form $\phi_1^{\alpha}\phi_2^{\beta}$, with $\alpha+\beta=m$, and F_m must pass α times through R and contains β generatrices of Q. Thus in case of $n=9$, $m=3$, F_m must be either a cubic cone with vertex at T, or a cubic surface through T and a generatrix of Q. The condition to pass through T and a generatrix of Q absorbs 5 constants of F_3, and leaves 15 (14 effective) disposable constants. But through the intersection of a quadric and a cubic surface there are ∞^3 other cubics, so that there are ∞^{12} linearly independent residual quintics on Q to which corresponds in (x) the same manifold of conics, which is in agreement with the number of solutions of the diophantine equation, in case of $n=9$.

4. *Symmetric Quartics.* To a pencil of planes through a line s cutting Q in A and B corresponds in (x) a pencil of quartics with the same double tangent $\phi_1=0$ and with two sextuples A' and B', corresponding to A and B, as base-points outside of the double tangencies at I and J. When s is tangent to Q, the quartics of the pencil all touch each other in the points of a sextuple. Now consider any two conics K' and K'' on Q. The common tangent-planes of K' and K'' envelope two cones. Through a generic

point of Q there are two tangent-planes to each of these cones. To K' and K'' correspond in (x) two quartics C_4' and C_4''. Every tangent plane of one of these cones cuts Q in a conic which touches K' and K''. To this conic corresponds a quartic which touches each C_4' and C_4'' in points of a sextuple, one for each C_4' and C_4''. Hence we may state the following theorem.

THEOREM. *Given two symmetric quartics C_4' and C_4'', then there exist two ∞^1 systems of symmetric quartics of index 2, such that every quartic of each system has a sextuple contact (contact in points of a sextuple) with each C_4' and C_4''.*

To the intersection of a plane through T with Q corresponds in (x) a symmetric cubic $\phi_1^3+\lambda\phi_1\phi_2+\mu\phi_3=0$. Let K' again be a conic not through I. Now I is the vertex of a quadric cone through K', whose tangent planes cut Q in conics tangent to K'. To these correspond in (x) cubics and a quartic respectively. This leads to the following theorem.

THEOREM. *For a symmetric quartic corresponding to a generic conic on Q there exists a system of cubics of index 2 with the property of sextuple contacts with the quartic.**

5. *A Problem in Closure.* Let K' and K'' be again two conics on Q not intersecting in real points and O a generic point in space. O as a vertex determines with K' and K'' two cones C' and C'', which intersect Q in two other conics L' and L''. Assume O such that C' and C'' have no real generatrix in common, moreover so that it is possible to construct a closed pyramid of n faces inscribed to one cone, say C', and circumscribed to C''. To the four conics K', K'', L', L'' correspond in (x) four quartics C_4', C_4'', D_4', D_4''. To a conic cut out on Q by a face of P corresponds in (x) a quartic which cuts each C_4' and D_4' in a sextuple and touches each C_4'' and D_4'' in points of a sextuple. As there

* Such systems of sextuple tact cubics for the general quartic were established by A. Clebsch, Mathematische Annalen, vol. 3 (1871), pp. 45–75.

are ∞^1 such inscribed and circumscribed pyramids P the mapping upon (x) gives the following theorem.

THEOREM. *Let C_4' and C_4'' be two fixed symmetric quartics in (x). Construct a quartic $C_4^{(1)}$ with a sextuple contact with C_4'', cutting C_4' in two sextuples S_1 and S_2. Through S_2 draw another quartic $C_4^{(2)}$ with a sextuple contact with C_4'', which cuts C_4' in another sextuple S_3. Through S_3 draw similarly a third quartic $C_4^{(3)}$, cutting C_4' in a sextuple S_4, and so forth. Suppose that after drawing n such quartics, the last $C_4^{(n)}$ through S_n cuts C_4' in a sextuple S_{n+1} which coincides with S_1. If this happens once then there exists an infinite number of such series of quartics with the closure property.*

Moreover there exist two other fixed quartics D_4' and D_4'' which are related to these series in precisely the same manner as C_4' and C_4''.

THE UNIVERSITY OF ILLINOIS

CONGRUENCES OF LINES OF SPECIAL ORIENTATION RELATIVE TO A SURFACE OF REFERENCE*

BY M. C. FOSTER

1. *Introduction.* With each line l of a rectilinear congruence let us associate the point M in which l intersects a surface of reference S. We refer S to any orthogonal system. Let α, β, γ be the direction-cosines of l relative to the moving trihedral of S at M, the x-axis being chosen tangent to the curve $v = \text{const.}$ By congruences of special orientation relative to S, we shall mean those congruences for which the functions α, β, γ are of a special form. The present paper is concerned primarily with the case when α, β, γ are constant.

2. *Normal Congruences.* Relative to the moving trihedral the coordinates of any point P on l are

* Presented to the Society, December 28, 1926.

$$x = \alpha t, \quad y = \beta t, \quad z = \gamma t, \tag{1}$$

where t is the distance along l from M to P. The condition that there exist a surface Σ normal to l is that the displacements of some point on l satisfy the relation

$$\sum \alpha \delta x = 0, \tag{2}$$

for all values of dv/du. This becomes on using the well known formulas*

$$\frac{\partial t}{\partial u} du + \frac{\partial t}{\partial v} dv + \alpha\xi \, du + \beta\eta_1 \, dv \equiv 0. \tag{3}$$

Hence

$$\frac{\partial t}{\partial u} + \alpha\xi = 0, \qquad \frac{\partial t}{\partial v} + \beta\eta_1 = 0. \tag{4}$$

The condition of integrability,

$$\frac{\partial}{\partial v}(\alpha\xi) = \frac{\partial}{\partial u}(\beta\eta_1), \tag{5}$$

is therefore a necessary condition that the congruence be normal. It is also sufficient. For if (5) is satisfied, the function t as given by (4) will satisfy (3), and consequently there exists a one-parameter family of surfaces Σ normal to the congruence.

Let us now assume that

$$\alpha = U, \qquad \beta = V, \tag{6}$$

where U and V are functions of u and v alone respectively. The relation (5) becomes

$$U\frac{\partial \xi}{\partial v} = V\frac{\partial \eta_1}{\partial u},$$

which may be written†

$$U\eta_1 r = -V\xi r_1, \tag{7}$$

* Eisenhart, *Differential Geometry of Curves and Surfaces*, p. 170.

† Eisenhart, p. 170.

or

$$-\frac{U}{V}=\frac{\xi r_1}{\eta_1 r}=\frac{\rho_{gu}}{\rho_{gv}}, \tag{8}$$

where ρ_{gu} and ρ_{gv} are the radii of geodesic curvature of the curves $v=$const. and $u=$const., respectively. From (8) we have the following theorem.

THEOREM 1. *If the lines l be of fixed orientation relative to the trihedral of S, a necessary and sufficient condition that the congruence be normal is that the corresponding radii of geodesic curvature of the curves $v=$const. and $u=$const., respectively, be in the constant ratio $\alpha:-\beta$.*

The condition that the curves $v=$const., and $u=$const., be geodesics is that $r=0$, and $r_1=0$, respectively. Hence from (7) we have the following theorems.*

THEOREM 2. *If α, (β), be zero, a necessary and sufficient condition that the congruence be normal is that the curves $u=$const., $(v=$const.$)$, be geodesics.*

THEOREM 3. *If the curves $u=$const., $(v=$const.$)$, be geodesics, a necessary and sufficient condition that the congruence be normal is that either the curves $v=$const., $(u=$const.$)$, be geodesics and S be developable, or α, (β), be zero.*

We note that Theorem 2 includes as a special case the well known theorem that a necessary and sufficient condition that the tangents to a family of curves on a surface constitute a normal congruence is that these curves be geodesics.

3. *Equation Defining the Developables.* We shall assume for the moment that α, β, γ are any functions whatever of u and v. For l to generate a developable surface the displacement of some point on l must satisfy the relations

$$\frac{\delta x}{\alpha}=\frac{\delta y}{\beta}=\frac{\delta z}{\gamma},$$

or

$$\alpha\delta z-\gamma\delta x=0, \qquad \beta\delta z-\gamma\delta y=0. \tag{9}$$

* We exclude the case where the lines are normal to S.

When t is eliminated between these equations we get the following equation for the curves on S defining the developables,

$$(10) \qquad \begin{vmatrix} \alpha & \xi du & A_1 du + A_2 dv \\ \beta & \eta_1 dv & B_1 du + B_2 dv \\ \gamma & 0 & C_1 du + C_2 dv \end{vmatrix} = 0,$$

where we have set

$$(11) \qquad \begin{cases} A_1 = \dfrac{\partial \alpha}{\partial u} + q\gamma - r\beta, & A_2 = \dfrac{\partial \alpha}{\partial v} + q_1\gamma - r_1\beta, \\ B_1 = \dfrac{\partial \beta}{\partial u} + r\alpha - p\gamma, & B_2 = \dfrac{\partial \beta}{\partial v} + r_1\alpha - p_1\gamma, \\ C_1 = \dfrac{\partial \gamma}{\partial u} + p\beta - q\alpha, & C_2 = \dfrac{\partial \gamma}{\partial v} + p_1\beta - q_1\alpha. \end{cases}$$

When dv/du is eliminated between equations (9) we get the following equation in t for the distances to the focal points:

$$(12) \qquad \begin{vmatrix} \alpha & A_1 t + \xi & A_2 t \\ \beta & B_1 t & B_2 t + \eta_1 \\ \gamma & C_1 t & C_2 t \end{vmatrix} = 0.$$

4. *Distance to a Line of Striction.* The direction-cosines

$$\alpha + \delta\alpha, \quad \beta + \delta\beta, \quad \gamma + \delta\gamma$$

of a neighboring line l' of the congruence are found by considering the displacements $\delta\alpha$, $\delta\beta$, $\delta\gamma$ of a point (α, β, γ) on the unit sphere, relative to a trihedral of fixed vertex at the center of the unit sphere, whose axes are parallel to the axes of the moving trihedral. If a point P on l generate the line of striction of the ruled surface defined by a value of dv/du, the displacement of P must be orthogonal to both l and l'. Hence

$$\sum \alpha \delta x = 0, \qquad \sum (\alpha + \delta\alpha)\delta x = 0,$$

or

$$\sum \delta\alpha\delta x = 0.$$

This becomes, by means of well known formulas,

$$
\begin{aligned}
&\left\{\frac{\partial\alpha}{\partial u}du + \frac{\partial\alpha}{\partial v}dv + \gamma(q\,du + q_1 dv) - \beta(r\,du + r_1\,dv)\right\} \\
&\cdot\left\{\left(\frac{\partial\alpha}{\partial u}du + \frac{\partial\alpha}{\partial v}dv\right)t + \xi\,du + \gamma t(q\,du + q_1 dv)\right. \\
&\left. - \beta t(r\,du + r_1\,dv)\right\} \\
&+\left\{\frac{\partial\beta}{\partial u}du + \frac{\partial\beta}{\partial v}dv + \alpha(r\,du + r_1\,dv) - \gamma(p\,du + p_1\,dv)\right\} \\
&\cdot\left\{\left(\frac{\partial\beta}{\partial u}du + \frac{\partial\beta}{\partial v}dv\right)t + \eta_1 dv + \alpha t(r\,du + r_1\,dv)\right. \\
&\left. - \gamma t(p\,du + p_1\,dv)\right\} \\
&+\left\{\frac{\partial\gamma}{\partial u}du + \frac{\partial\gamma}{\partial v}dv + \beta(p\,du + p_1 dv) - \alpha(q du + q_1 dv)\right\} \\
&\cdot\left\{\left(\frac{\partial\gamma}{\partial u}du + \frac{\partial\gamma}{\partial v}\right)t + \beta t(p\,du + p_1\,dv)\right. \\
&\left. - \alpha t(q\,du + q_1\,dv)\right\} = 0.
\end{aligned}
\tag{13}
$$

If a value be assigned to t this equation gives us the equation of the curves on S defining the ruled surfaces for which t is the distance to their lines of striction; and if a value be assigned to dv/du we have an equation in t which gives us the distance to the line of striction of the ruled surface so determined.

5. *The Direction Cosines α, β, γ, constant.* We now assume the lines of the congruence to be of fixed orientation relative to the moving trihedral. The curves defining the ruled surfaces whose lines of striction lie on S will be found by setting $t = 0$ in (13). We get, for the equation of these curves,

$$\xi(\beta r - \gamma q)du^2 + \{\xi(\beta r_1 - \gamma q_1) + \eta_1(\gamma p - \alpha r)\}du\,dv + \eta_1(\gamma p_1 - \alpha r_1)dv^2 = 0. \tag{14}$$

The curves $v=\text{const.}$, or $u=\text{const.}$, will be the lines of striction of the ruled surfaces which they define if

$$\beta r - \gamma q = 0,$$

or

$$\gamma p_1 - \alpha r_1 = 0,$$

respectively. If

$$\beta r - \gamma q = 0,$$

we have*

$$\frac{\beta}{\gamma} = \text{const.} = \frac{q}{r} = -\operatorname{ctn}\omega_u, \tag{15}$$

where ω_u is the angle between the positive principal normal of the curve $v=\text{const.}$, and the positive normal to S, the angle being measured toward the positive binormal. Hence we have the following theorem.

THEOREM 4. *If l be of fixed orientation relative to the trihedral, a necessary and sufficient condition that the curves $v=\text{const.}$ be the lines of striction of the ruled surfaces which they define is that their osculating planes meet the normal under the constant angle defined by* (15).

Let us here recall a theorem of ruled surfaces due to Bonnet:

If a curve upon a ruled surface have two of the following properties, it has the third also:

1. that it cut the rulings under constant angle,
2. that it be the line of striction,
3. that it be a geodesic.

Hence Theorem 4 may be extended to state that the curves $v=\text{const.}$ are geodesics on the ruled surfaces which they

* Eisenhart, p. 167.

define; and the principal normals meet the rulings orthogonally. A similar theorem holds for the curves $u=\text{const.}$

The condition that S be the middle surface is that the coefficient of t in (12) shall vanish. We get as this condition

$$\xi p_1(\alpha^2-1)+\eta_1 q(1-\beta^2)+\alpha\beta(\xi q_1-\eta_1 p)+\gamma(\alpha\xi r_1-\beta\eta_1 r)=0. \tag{16}$$

If S be minimal we have $\xi p_1-\eta_1 q=0$; consequently, if $\alpha=\beta$, and S be referred to its lines of curvature, (16) reduces to

$$\xi r_1=\eta_1 r,$$

or

$$\rho_{gu}=\rho_{gv}.$$

Hence we have the following theorem.

THEOREM 5. *If S be a minimal surface referred to its lines of curvature, a necessary and sufficient condition that the congruence of fixed orientation for which $\alpha=\beta$, have S as its middle surface, is that the corresponding radii of geodesic curvature of the parametric curves be equal.*

We note from (7) that such congruences are not normal.

Let the lines of curvature be parametric. The condition that the developables of the congruence be represented on S by the parametric system is that the coefficients of du^2 and dv^2 in (9) shall vanish. This condition is

$$\alpha C_2-\gamma A_2=0,\quad \gamma B_1-\beta C_1=0. \tag{17}$$

Hence

$$\begin{cases} -\dfrac{\alpha}{\gamma}=\text{const.}=\dfrac{r_1}{p_1}=\tan\omega_v, \\ \dfrac{\beta}{\gamma}=\text{const.}=-\dfrac{r}{q}=\tan\omega_u. \end{cases} \tag{18}$$

Consequently, the osculating planes of the lines of curvature meet the tangent planes under constant angle, and they are

therefore plane curves.* We have in consequence the following theorem.

THEOREM 6. *If S be referred to its lines of curvature, a necessary and sufficient condition that the developables of the congruence of lines of fixed orientation be defined by the parametric curves is that these curves be plane in both systems, and that their osculating planes meet the normals under the constant angles defined by* (18).

When relations (18) are substituted in (16), this condition that S be the middle surface reduces to

$$\xi p_1 - \eta_1 q = 0.$$

Hence if we further assume S to be the middle surface, it must also be minimal. We note that in this case the congruence is of the Ribaucour type; for the developables intersect the middle surface in a conjugate system.

6. *Isotropic Congruences.* We shall again assume for the moment that α, β, γ are any functions whatever of u and v. The necessary and sufficient condition that a congruence be isotropic is that all the lines of striction lie on the middle surface. Evidently if all lines of striction lie on some surface, that surface is the middle surface. We suppose that all lines of striction lie on S. Then from (13) we must have

$$t \equiv 0,$$

or

$$A_1 = 0, \quad \xi A_2 + \eta_1 B_1 = 0, \quad B_2 = 0. \tag{19}$$

Hence when S is the middle surface (19) represents the necessary and sufficient condition that a congruence be isotropic.

We again assume α, β, γ constant. The relations (19) become

$$\begin{cases} q\gamma - r\beta = 0, \\ \xi(q_1\gamma - r_1\beta) + \eta_1(r\alpha - p\gamma) = 0, \\ r_1\alpha - p_1\gamma = 0, \end{cases} \tag{20}$$

* Eisenhart, p. 150.

from which we have

$$(21)\qquad \begin{cases} \dfrac{\gamma}{\beta} = \text{const.} = \dfrac{r}{q} = -\tan\omega_u, \\ \dfrac{\gamma}{\alpha} = \text{const.} = \dfrac{r_1}{p_1} = \tan\omega_v. \end{cases}$$

Using the first and third equations of (20) the second member reduces to

$$(22)\qquad \eta_1 p_1 r^2 - \xi q r_1^2 + 2\xi q_1 r r_1 = 0.$$

From (21) we see that the osculating planes of the parametric curves meet the normals under constant angles. We have therefore the following theorem.

THEOREM 7. *A condition that the congruence of lines of fixed orientation relative to its middle surface be an isotropic congruence, is that the osculating planes of the parametric curves meet the normals under the constant angles defined by* (21), *and that the fundamental quantities of the middle surface satisfy* (22).

7. *Congruences of Ribaucour.* The condition that a congruence be of the Ribaucour type is that its developables meet the middle surface in a conjugate system. We assume S to be the middle surface referred to its lines of curvature. From (10) we readily find the condition that the curves defining the developables form a conjugate system; this reduces to

$$(23)\qquad p_1 r\alpha - q r_1 \beta = 0.$$

Hence when S is taken as the middle surface referred to its lines of curvature, the relation (23) represents the necessary and sufficient condition that a congruence of fixed orientation relative to S be of the Ribaucour type. This condition is readily reducible to

$$\frac{\alpha}{\beta} = \text{const.} = -\tan\bar{\omega}_v \operatorname{ctn}\bar{\omega}_u.$$

8. *Condition that the Lines of Curvature on S and on the Normal Surfaces be in Correspondence.* We have seen in §5 that when the lines of curvature on S are parametric, a condition that the developables be represented by the parametric curves is

$$\frac{\alpha}{\gamma} = -\frac{r_1}{p_1}, \qquad \frac{\gamma}{\beta} = -\frac{q}{r}.$$

From these we get

$$\frac{\alpha}{\beta} = \frac{r_1 q}{p_1 r}. \tag{24}$$

When the congruence is normal we have, from (7),

$$\frac{\alpha}{\beta} = -\frac{\xi r_1}{\eta_1 r}; \tag{25}$$

and from (24) and (25) we have

$$\xi p_1 + \eta_1 q = 0. \tag{26}$$

Hence, when S is referred to its lines of curvature, the relation (26) represents a condition that the parametric curves on S correspond to the lines of curvature on the surfaces normal to the congruence of fixed orientation.

WILLIAMS COLLEGE

A NEW TABLE OF THE ZEROS OF THE BESSEL FUNCTIONS $J_0(x)$ AND $J_1(x)$ WITH CORRESPONDING VALUES OF $J_1(x)$ AND $J_0(x)$*

BY H. T. DAVIS AND W. J. KIRKHAM

1. *Introduction.* In connection with a problem involving the calculation of the distribution of energy in a diffraction pattern of sixty-six rings, it was found necessary to extend existing tables of the zeros of the Bessel functions $J_0(x)$ and $J_1(x)$ with corresponding values of $J_1(x)$ and $J_0(x)$. In the hope that these computations may be of use to other investigators they are reproduced here.

The first ten zeros in Table I and the first fifty zeros with the associated values of $J_0(x)$ in Table II are incorporated from tables by E. Meissel published in 1888 and 1889–90 respectively.† The values of the zeros of $J_0(x)$ and their logarithms from $s=11$ to $s=40$ have been taken from a paper by R. W. Willson and B. O. Peirce.‡ The associated values of $J_1(x)$ have been recomputed to ten decimal places and one correction in the Willson-Peirce table is noted, the value corresponding to $s=35$ being in error.

2. *Formulas Used in the Calculation of the Zeros of $J_0(x)$ and $J_1(x)$.* In computing the values of $x_0^{(s)}$ and $x_1^{(s)}$, the roots of $J_0(x)$ and $J_1(x)$ respectively, the formulas of G. C. Stokes were used.§ Thus we have

$$x_0^{(s)} = \frac{1}{4}\pi a\left(1+\frac{\alpha_2}{a^2}+\frac{\alpha_4}{a^4}+\frac{\alpha_6}{a^6}+\frac{\alpha_8}{a^8}\right), \qquad a = 4s-1,$$

* These tables were made possible by a grant of funds from the Waterman Institute, Indiana University.

† Mathematische Abhandlungen der Akademie der Wissenschaften zu Berlin, 1888; Annual report of the Ober-Realschule at Kiel, 1889–90.

‡ This Bulletin, vol. 3 (1897), pp. 153–155.

§ Cambridge Philosophical Transactions, vol. 9, or Mathematical and Physics Papers, vol. II, pp. 353, 355. See also J. McMahon, Annals of Mathematics, vol. 9 (1894–95), p. 25.

where

$$\alpha_2 = 2/\pi^2, \quad \alpha_4 = -62/(3\pi^4), \quad \alpha_6 = 15116/(15\pi^6),$$
$$\alpha_8 = -12554474/(105\pi^8);$$

$$x_0^{(s)} = \pi s - .7853981634 + \frac{k_1}{a} - \frac{k_2}{a^3} + \frac{k_3}{a^5} - \frac{k_4}{a^7},$$

where

$$\log k_1 = 9.2018201316, \qquad \log k_2 = 9.2217608253,$$
$$\log k_3 = 9.9155362692, \qquad \log k_4 = 0.9955001222.$$

$$x_1^{(s)} = \frac{1}{4}\pi b\left(1 + \frac{\beta_2}{b^2} + \frac{\beta_4}{b^4} + \frac{\beta_6}{b^6} + \frac{\beta_8}{b^8}\right), \qquad b = 4s + 1,$$

where

$$\beta_2 = -6/\pi^2, \quad \beta_4 = 6/\pi^4, \quad \beta_6 = -4716/(5\pi^6),$$
$$\beta_8 = 3902418/(35\pi^8);$$

$$x_1^{(s)} = \pi s + .7853981634 - \frac{m_1}{b} + \frac{m_2}{b^3} - \frac{m_3}{b^5} + \frac{m_4}{b^7},$$

where

$$\log m_1 = 9.6789413863, \qquad \log m_2 = 8.6846416409,$$
$$\log m_3 = 9.8867944373, \qquad \log m_4 = 0.9651566416.$$

The values of $x_0^{(s)}$ and $x_1^{(s)}$ were calculated from these formulas by means of Vega's ten-place table of logarithms. In checking the zeros, the differences between successive values were computed on a ten-bank Monroe calculator with the aid of Barlow's table of reciprocals and cubes from the following formulas:

$$x_0^{(s+1)} - x_0^{(s)} = \pi - .1591549431[1/(4s-1) - 1/(4s+3)] + .1666329280[1/(4s-1)^3 - 1/(4s+3)^3],$$

$$x_1^{(s+1)} - x_1^{(s)} = \pi + .4774648293[1/(4s+1) - 1/(4s+5)] - .0483773016[1/(4s+1)^3 - 1/(4s+5)^3].$$

3. *The Calculation of* $J_1[x_0^{(s)}]$ *and* $J_0[x_1^{(s)}]$. For the calculation of $J_1[x_0^{(s)}]$ and $J_0[x_1^{(s)}]$ new asymptotic formulas were used. It is well known* that $J_0(x)$ and $J_1(x)$ have the following asymptotic expansions:

$$(1)\quad \begin{cases} J_0(x) = \left(\dfrac{2}{\pi x}\right)^{1/2}[P_0 \cos(x - \pi/4) + Q_0 \sin(x - \pi/4)], \\ J_1(x) = \left(\dfrac{2}{\pi x}\right)^{1/2}[P_1 \cos(x - \pi/4) + Q_1 \sin(x - \pi/4)], \end{cases}$$

where

$$P_0 = 1 - \frac{1^2 3^2}{2!(8x)^2} + \frac{1^2 3^2 5^2 7^2}{4!(8x)^4} - \cdots,$$

$$Q_0 = \frac{1^2}{1!8x} - \frac{1^2 3^2 5^2}{3!(8x)^3} + \frac{1^2 3^2 5^2 7^2 9^2}{5!(8x)^5} - \cdots,$$

$$P_1 = \frac{3}{1!8x} + \frac{3 \cdot 5 \cdot 21}{3!(8x)^3} + \frac{3 \cdot 5 \cdot 21 \cdot 45 \cdot 77}{5!(8x)^5} - \cdots,$$

$$Q_1 = 1 + \frac{3 \cdot 5}{2!(8x)^2} - \frac{3 \cdot 5 \cdot 21 \cdot 45}{4!(8x)^4} + \cdots.$$

If in the formulas (1) we let $x = x_1^{(s)}$ and use the identity $\sin^2(x-\pi/4) + \cos^2(x-\pi/4) = 1$ to eliminate the sine and cosine terms, we get the following asymptotic formula for the calculation of $J_0(x_1^{(s)})$:

$$(2)\quad \begin{aligned} J_0(x_1^{(s)}) &= \pm\left(\frac{2}{\pi x_1}\right)^{1/2}\left[1 + \frac{1 \cdot 3}{2!(2x_1)^2} - \frac{1^4 3^3 5}{4!(2x_1)^4} + \frac{1^4 3^4 5^3 7}{6!(2x_1)^6} - \cdots\right]^{-1/2}, \\ &= \pm\left(\frac{2}{\pi x_1}\right)^{1/2}\left[1 - \frac{3}{2^4 x_1^2} + \frac{3^2 13}{2^9 x_1^4} - \frac{3^2 \cdot 5 \cdot 7 \cdot 23}{2^{13} x_1^6} + \frac{3^4 \cdot 5 \cdot 10487}{2^{19} x_1^8} + \cdots\right]. \end{aligned}$$

* See Gray, Mathews and MacRobert, *A Treatise on Bessel Functions*, London, 1922, pp. 57–59.

The sign is plus when s is even, and is minus when s is odd.

Similarly, replacing x by $x_0^{(s)}$ and eliminating the sine and cosine terms from (1) we obtain the following formula for $J_1(x_0^{(s)})$:

$$J_1(x_0^{(s)}) = \pm\left(\frac{2}{\pi x_0}\right)^{1/2}\left[1 - \frac{1^4}{2!(2x_0)^2} + \frac{1^4 3^4}{4!(2x_0)^4} - \frac{1^4 3^4 5^4}{6!(2x_0)^6} + \cdots\right]^{-1/2},$$

$$= \pm\left(\frac{2}{\pi x_0}\right)^{1/2}\left[1 + \frac{1}{2^4 x_0^2} - \frac{3\cdot 17}{2^9 x_0^4} + \frac{43\cdot 101}{2^{13} x_0^6} - \frac{3025837}{2^{19} x_0^8} + \cdots\right]. \tag{3}$$

The sign is plus when s is odd, and is minus when s is even.

If in (2) we substitute the value of $x_1^{(s)}$ from Stokes' formula, and in (3) the value of $x_0^{(s)}$, we obtain new formulas which are remarkable in that terms involving the reciprocals of $(4s+1)^2$ and $(4s-1)^2$ are absent:

$$J_0(x_1^{(s)}) = \pm\frac{2\cdot 2^{1/2}}{\pi b^{1/2}}\left[1 + \frac{24}{\pi^4 b^4} - \frac{19584}{10\pi^6 b^6} + \frac{2466720}{7\pi^8 b^8}\right], \qquad b = 4s+1,$$

$$J_1(x_0^{(s)}) = \pm\frac{2\cdot 2^{1/2}}{\pi a^{1/2}}\left[1 - \frac{56}{3\pi^4 a^4} + \frac{9664}{5\pi^6 a^6} - \frac{7381280}{21\pi^8 a^8}\right], \qquad a = 4s-1.$$

From these formulas we see that to ten decimal places

$$J_0(x_1^{(s)}) = \frac{2\cdot 2^{1/2}}{\pi}\,\frac{1}{(4s+1)^{1/2}} \text{ and } J_1(x_0^{(s)}) = \frac{2\cdot 2^{1/2}}{\pi}\,\frac{1}{(4s-1)^{1/2}}$$

for values of s greater than 10. In the calculation of $J_0(x_1^{(s)})$ and $J_1(x_0^{(s)})$ from these formulas Vega's ten-place table of logarithms was used. Each value was checked by direct computation on the Monroe calculator. The first five values of $J_1(x_0^{(s)})$ were reduced from twelve-place values obtained by interpolation from Meissel's table.

4. *Formulas for the Calculation of $J_n(x)$ from Tables of $J_0(x)$ and $J_1(x)$.* It should be noted that corresponding values of $J_n(x)$ can be obtained from tables of $J_0(x)$ and $J_1(x)$ by the well known recurrence formula:

$$\frac{2nJ_n(x)}{x} = J_{n-1}(x) + J_{n+1}(x).$$

However, when n is large, the labor of calculation becomes very great. The following formulas, which, as far as the authors are aware, are not found in standard treatises on the Bessel functions, are very useful in the calculation of $J_n(x)$ from tables of $J_0(x)$ and $J_1(x)$:

$$J_{2n}(x) = (-1)^n\left[1 + \sum_{r=1}^{n-1}(-1)^r \frac{2^{2r}n^2(n^2-1)^2\cdots}{(2r)!x^{2r}} \cdot \frac{[n^2-(r-1)^2]^2(n^2-r^2)}{1}\right]J_0(x)$$

$$+ (-1)^{n+1}\left[\sum_{r=0}^{n-1}(-1)^r \frac{2^{2r+1}n^2(n^2-1)^2(n^2-4)^2}{(2r+1)!x^{2r+1}} \cdot \frac{\cdots(n^2-r^2)^2}{1}\right]J_1(x).$$

$$J_{2n-1}(x) = (-1)^{n+1}\left[\sum_{r=0}^{n-2}(-1)^r \frac{2^{2r+1}n^2(n^2-1)^2\cdots}{(2r+1)!x^{2r+1}} \cdot \frac{[n^2-(r-1)^2]^2(n^2-r^2)(n-r)(n-r-1)}{1}\right]J_0(x)$$

$$+ (-1)^{n+1}\left[1 + \sum_{r=1}^{n-1}(-1)^r \frac{2^{2r}n^2(n^2-1)^2\cdots}{(2r)!x^{2r}} \cdot \frac{[n^2-(r-1)^2]^2(n-r)^2}{1}\right]J_1(x).$$

These formulas are easily proved by induction.

TABLE I.

THE FIRST 150 ROOTS OF $J_0(x)$ WITH CORRESPONDING VALUES OF $J_1(x)$

s	$x_0^{(s)}$	$\log x_0^{(s)}$	$J_1[x_0^{(s)}]$	$\log \pm J_1[x_0^{(s)}]$
1	2.4048255577	0.3810835788	+0.5191474973	9.715290765
2	5.5200781103	0.7419452231	−0.3402648065	9.531817033
3	8.6537279129	0.9372032361	+0.2714522999	9.433693526
4	11.7915344391	1.0715703238	−0.2324598214	9.366347900
5	14.9309177086	1.1740865018	+0.2065464331	9.315017699
6	18.0710639679	1.2569837232	−0.1877288030	9.273530911
7	21.2116366299	1.3265741787	+0.1732658943	9.238713084
8	24.3524715308	1.3865430443	−0.1617015504	9.208714184
9	27.4934791320	1.4392297006	+0.1521812139	9.182361044
10	30.6346064684	1.4862123057	−0.1441659779	9.158862782
11	33.7758202136	1.5286059043	+0.1372969435	9.137660869
12	36.9170983537	1.5672275586	−0.1313246267	9.118346175
13	40.0584257646	1.6026938781	+0.1260694971	9.100610020
14	43.1997917132	1.6354816528	−0.1213986249	9.084213767
15	46.3411883717	1.6659671666	+0.1172111989	9.068969108
16	49.4826098974	1.6944525978	−0.1134291926	9.054724841
17	52.6240518411	1.7211842839	+0.1099911432	9.041357716
18	55.7655107550	1.7463656842	−0.1068478884	9.028765943
19	58.9069839261	1.7701667872	+0.1039595729	9.016864487
20	62.0484691902	1.7927310714	−0.1012934991	9.005581573
21	65.1899648002	1.8141807465	+0.0988225538	8.994856073
22	68.3314693299	1.8346207594	−0.0965240405	8.984635493
23	71.4729816036	1.8541418997	+0.0943787940	8.974874423
24	74.6145006437	1.8728232368	−0.0923705049	8.965533317
25	77.7560256304	1.8907340543	+0.0904851942	8.956577523
26	80.8975558711	1.9079354006	−0.0887108024	8.947976507
27	84.0390907769	1.9244813451	+0.0870368634	8.939703232
28	87.1806298436	1.9404200023	−0.0854542429	8.931733631
29	90.3221726372	1.9557943757	+0.0839549293	8.924046200
30	93.4637187819	1.9706430570	−0.0825318613	8.916621640
31	96.6052679510	1.9850008094	+0.0811787883	8.909442565
32	99.7468198587	1.9988990584	−0.0798901543	8.902493260
33	102.8883742542	2.0123663047	+0.0786610017	8.895759473
34	106.0299309165	2.0254284784	−0.0774868911	8.889228237
35	109.1714896498	2.0381092361	+0.0763638333	8.882887721
36	112.3130502805	2.0504302219	−0.0752882326	8.876727102
37	115.4546126537	2.0624112882	+0.0742568382	8.870736454
38	118.5961766309	2.0740706879	−0.0732667027	8.864906647
39	121.7377420880	2.0854252422	+0.0723151467	8.859229272
40	124.8793089132	2.0964904866	−0.0713997282	8.853696559

TABLE I. (Continued)

s	$x_0^{(s)}$	$\log\ x_0^{(s)}$	$J_1[x_0^{(s)}]$	$\log \pm J_1[x_0^{(s)}]$
41	128.0208770059	2.1072807979	+0.0705182163	8.848301319
42	131.1624462752	2.1178095078	−0.0696685682	8.843036885
43	134.3040166383	2.1280890011	+0.0688489095	8.837897066
44	137.4455880203	2.1381308034	−0.0680575164	8.832876097
45	140.5871603528	2.1479456587	+0.0672928009	8.827968605
46	143.7287335737	2.1575435988	−0.0665532972	8.823169576
47	146.8703076258	2.1669340045	+0.0658376495	8.818474318
48	150.0118824570	2.1761256608	−0.0651446023	8.813878437
49	153.1534580192	2.1851268069	+0.0644729905	8.809377815
50	156.2950342685	2.1939451798	−0.0638217315	8.804968583
51	159.4366111643	2.2025880546	+0.0631898176	8.800647102
52	162.5781886689	2.2110622805	−0.0625763097	8.796409948
53	165.7197667480	2.2193743133	+0.0619803313	8.792253893
54	168.8613453692	2.2275302450	−0.0614010632	8.788175891
55	172.0029245031	2.2355358310	+0.0608377387	8.784173063
56	175.1445041219	2.2433965140	−0.0602896398	8.780242689
57	178.2860842001	2.2511174463	+0.0597560927	8.776382192
58	181.4276647137	2.2587035104	−0.0592364646	8.772589131
59	184.5692456406	2.2661593371	+0.0587301608	8.768861190
60	187.7108269600	2.2734893229	−0.0582366213	8.765196170
61	190.8524086526	2.2806976452	+0.0577553186	8.761591984
62	193.9939907001	2.2877882769	−0.0572857554	8.758046644
63	197.1355730857	2.2947649996	+0.0568274620	8.754558260
64	200.2771557933	2.3016314151	−0.0563799947	8.751125031
65	203.4187388082	2.3083909573	+0.0559429339	8.747745239
66	206.5603221162	2.3150469021	−0.0555158823	8.744417247
67	209.7019057043	2.3216023771	+0.0550984638	8.741139490
68	212.8434895599	2.3280603705	−0.0546903214	8.737910475
69	215.9850736715	2.3344237388	+0.0542911166	8.734728774
70	219.1266580280	2.3406952151	−0.0539005280	8.731593019
71	222.2682426191	2.3468774157	+0.0535182499	8.728501903
72	225.4098274349	2.3529728465	−0.0531439918	8.725454173
73	228.5514124661	2.3589839095	+0.0527774772	8.722448626
74	231.6929977040	2.3649129085	−0.0524184425	8.719484113
75	234.8345831404	2.3707620540	+0.0520666369	8.716559527
76	237.9761687673	2.3765334684	−0.0517218210	8.713673807
77	241.1177545773	2.3822291905	+0.0513837662	8.710825933
78	244.2593405633	2.3878511803	−0.0510522546	8.708014926
79	247.4009267187	2.3934013221	+0.0507270777	8.705239844
80	250.5425130370	2.3988814292	−0.0504080363	8.702499779

TABLE I. (Continued)

s	$x_0^{(s)}$	$\log x_0^{(s)}$	$J_1[x_0^{(s)}]$	$\log \pm J_1[x_0^{(s)}]$
81	253.6840995122	2.4042932472	+0.0500949399	8.699793860
82	256.8256861386	2.4096384570	−0.0497876061	8.697121244
83	259.9672729106	2.4149186784	+0.0494858602	8.694481124
84	263.1088598231	2.4201354726	−0.0491895350	8.691872717
85	266.2504468710	2.4252903453	+0.0488984701	8.689295272
86	269.3920340498	2.4303847494	−0.0486125117	8.686748061
87	272.5336213547	2.4354200870	+0.0483315122	8.684230384
88	275.6752087815	2.4403977121	−0.0480553299	8.681741563
89	278.8167963262	2.4453189327	+0.0477838286	8.679280944
90	281.9583839846	2.4501850127	−0.0475168778	8.676847897
91	285.0999717532	2.4549971743	+0.0472543516	8.674441808
92	288.2415596282	2.4597565990	−0.0469961292	8.672062089
93	291.3831476063	2.4644644304	+0.0467420942	8.669708166
94	294.5247356841	2.4691217748	−0.0464921346	8.667379487
95	297.6663238584	2.4737297040	+0.0462461428	8.665075516
96	300.8079121264	2.4782892553	−0.0460040147	8.662795734
97	303.9495004850	2.4828014340	+0.0457656504	8.660539638
98	307.0910889315	2.4872672144	−0.0455309532	8.658306742
99	310.2326774632	2.4916875409	+0.0452998301	8.656096573
100	313.3742660776	2.4960633298	−0.0450721913	8.653908673
101	316.5158547720	2.5003954693	+0.0448479502	8.651742598
102	319.6574435444	2.5046848218	−0.0446270230	8.649597916
103	322.7990323923	2.5089322243	+0.0444093289	8.647474210
104	325.9406213135	2.5131384890	−0.0441947898	8.645371073
105	329.0822103059	2.5173044056	+0.0439833303	8.643288109
106	332.2237993677	2.5214307406	−0.0437748773	8.641224937
107	335.3653884968	2.5255182391	+0.0435693603	8.639181183
108	338.5069776912	2.5295676252	−0.0433667110	8.637156486
109	341.6485669493	2.5335796034	+0.0431668633	8.635150492
110	344.7901562692	2.5375548582	−0.0429697534	8.633162861
111	347.9317456494	2.5414940560	+0.0427753191	8.631193258
112	351.0733350881	2.5453978449	−0.0425835005	8.629241359
113	354.2149245839	2.5492668559	+0.0423942396	8.627306850
114	357.3565141352	2.5531017031	−0.0422074799	8.625389422
115	360.4981037404	2.5569029846	+0.0420231669	8.623488778
116	363.6396933985	2.5606712829	−0.0418412476	8.621604625
117	366.7812831078	2.5644071655	+0.0416616706	8.619736681
118	369.9228728672	2.5681111852	−0.0414843861	8.617884667
119	373.0644626753	2.5717838810	+0.0413093457	8.616048316
120	376.2060525309	2.5754257783	−0.0411365025	8.614227364

TABLE I. (Continued)

s	$x_0^{(s)}$	$\log x_0^{(s)}$	$J_1[x_0^{(s)}]$	$\log \pm J_1[x_0^{(s)}]$
121	379.3476424328	2.5790373893	+0.0409658109	8.612421556
122	382.4892323800	2.5826192136	−0.0407972266	8.610630640
123	385.6308223712	2.5861717385	+0.0406307066	8.608854375
124	388.7724124055	2.5896954395	−0.0404662091	8.607092521
125	391.9140024818	2.5931907804	+0.0403036936	8.605344848
126	395.0555925990	2.5966582142	−0.0401431204	8.603611128
127	398.1971827563	2.6000981830	+0.0399844514	8.601891141
128	401.3387729527	2.6035111185	−0.0398276490	8.600184671
129	404.4803631872	2.6068974423	+0.0396726770	8.598491506
130	407.6219534590	2.6102575659	−0.0395195001	8.596811442
131	410.7635437673	2.6135918920	+0.0393680838	8.595144276
132	413.9051341111	2.6169008134	−0.0392183947	8.593489813
133	417.0467244898	2.6201847145	+0.0390704003	8.591847860
134	420.1883149024	2.6234439708	−0.0389240687	8.590218230
135	423.3299053483	2.6266789495	+0.0387793690	8.588600738
136	426.4714958268	2.6298900094	−0.0386362712	8.586995206
137	429.6130863371	2.6330775018	+0.0384947459	8.585401458
138	432.7546768784	2.6362417700	−0.0383547646	8.583819321
139	435.8962674503	2.6393831501	+0.0382162993	8.582248629
140	439.0378580519	2.6425019709	−0.0380793229	8.580689217
141	442.1794486827	2.6455985539	+0.0379438088	8.579140923
142	445.3210393420	2.6486732140	−0.0378097313	8.577603591
143	448.4626300292	2.6517262595	+0.0376770652	8.576077067
144	451.6042207439	2.6547579922	−0.0375457858	8.574561198
145	454.7458114852	2.6577687076	+0.0374158692	8.573055839
146	457.8874022529	2.6607586950	−0.0372872920	8.571560843
147	461.0289930463	2.6637282380	+0.0371600312	8.570076070
148	464.1705838648	2.6666776142	−0.0370340647	8.568601380
149	467.3121747080	2.6696070956	+0.0369093705	8.567136638
150	470.4537655754	2.6725169490	−0.0367859274	8.565681710

TABLE II.

THE FIRST 150 ROOTS OF $J_1(x)$ WITH CORRESPONDING VALUES OF $J_0(x)$

s	$x_1^{(s)}$	$\log x_1^{(s)}$	$J_0[x_1^{(s)}]$	$\log \pm J_0[x_1^{(s)}]$
1	3.8317059702	0.5833921757	−0.4027593957	9.605045681
2	7.0155866698	0.8460639942	+0.3001157525	9.477288791
3	10.1734681351	1.0074690286	−0.2497048771	9.397427025
4	13.3236919363	1.1246245823	+0.2183594072	9.339171907
5	16.4706300509	1.2167102124	−0.1964653715	9.293286014
6	19.6158585105	1.2926073202	+0.1800633753	9.255425387
7	22.7600843806	1.3571738678	−0.1671846005	9.223196271
8	25.9036720876	1.4133613337	+0.1567249863	9.195138240
9	29.0468285340	1.4630987210	−0.1480111100	9.170294315
10	32.1896799110	1.5077166581	+0.1406057982	9.148003230
11	35.3323075501	1.5481720021	−0.1342112403	9.127788890
12	38.4747662348	1.5851759898	+0.1286166221	9.109297099
13	41.6170942128	1.6192717536	−0.1236679608	9.092257199
14	44.7593189977	1.6508834702	+0.1192498120	9.076457703
15	47.9014608872	1.6803487586	−0.1152736941	9.061730211
16	51.0435351836	1.7079407452	+0.1116704969	9.047938448
17	54.1855536411	1.7338835151	−0.1083853489	9.034970580
18	57.3275254379	1.7583631957	+0.1053740554	9.022733694
19	60.4694578453	1.7815360748	−0.1026005671	9.011149761
20	63.6113566985	1.8035346583	+0.1000351468	9.000152613
21	66.7532267341	1.8244722636	−0.0976530158	8.989685660
22	69.8950718375	1.8444465556	+0.0954333390	8.979700119
23	73.0368952256	1.8635423032	−0.0933584533	8.970153648
24	76.1786995846	1.8818335547	+0.0914132722	8.961009255
25	79.3204871755	1.8993853729	−0.0895848220	8.952234435
26	82.4622599144	1.9162552327	+0.0878618760	8.943800472
27	85.6040194364	1.9324941569	−0.0862346634	8.935681873
28	88.7457671449	1.9481476480	+0.0846946348	8.927855900
29	91.8875042517	1.9632564558	−0.0832342730	8.920302190
30	95.0292318080	1.9778572186	+0.0818469379	8.913002436
31	98.1709507308	1.9919829969	−0.0805267394	8.905940115
32	101.3126618230	2.0056637255	+0.0792684317	8.899100266
33	104.4543657913	2.0189265960	−0.0780673254	8.892469301
34	107.5960632595	2.0317963811	+0.0769192140	8.886034837
35	110.7377547809	2.0442957136	−0.0758203116	8.879785565
36	113.8794408476	2.0564453260	+0.0747672005	8.873711120
37	117.0211218989	2.0682642573	−0.0737567865	8.867801987
38	120.1627983281	2.0797700333	+0.0727862602	8.862049406
39	123.3044704886	2.0909788222	−0.0718530644	8.856445295
40	126.4461386985	2.1019055715	+0.0709548658	8.850982183

TABLE II. (Continued)

s	$x_1^{(s)}$	$\log x_1^{(s)}$	$J_0[x_1^{(s)}]$	$\log \pm J_0[x_1^{(s)}]$
41	129.5878032451	2.1125641276	−0.0700895302	8.845653149
42	132.7294643885	2.1229673418	+0.0692551013	8.840451769
43	135.8711223648	2.1331271628	−0.0684497820	8.835372069
44	139.0127773887	2.1430547201	+0.0676719183	8.830408488
45	142.1544296559	2.1527603970	−0.0669199848	8.825555833
46	145.2960793452	2.1622538953	+0.0661925720	8.820809257
47	148.4377266203	2.1715442942	−0.0654883757	8.816164219
48	151.5793716314	2.1806401023	+0.0648061865	8.811616466
49	154.7210145163	2.1895493043	−0.0641448816	8.807162008
50	157.8626554019	2.1982794036	+0.0635034167	8.802797092
51	161.0042944054	2.2068374597	−0.0628808191	8.798518190
52	164.1459316346	2.2152301230	+0.0622761818	8.794321978
53	167.2875671898	2.2234636653	−0.0616886575	8.790205319
54	170.4292011631	2.2315440083	+0.0611174540	8.786165254
55	173.5708336411	2.2394767493	−0.0605618292	8.782198984
56	176.7124647027	2.2472671842	+0.0600210877	8.778303862
57	179.8540944227	2.2549203290	−0.0594945768	8.774477380
58	182.9957228701	2.2624409390	+0.0589816830	8.770717160
59	186.1373501093	2.2698335268	−0.0584818292	8.767020948
60	189.2789762004	2.2771023782	+0.0579944721	8.763386600
61	192.4206011997	2.2842515672	−0.0575190996	8.759812079
62	195.5622251597	2.2912849701	+0.0570552283	8.756295447
63	198.7038481298	2.2982062777	−0.0566024018	8.752834860
64	201.8454701561	2.3050190072	+0.0561601888	8.749428559
65	204.9870912823	2.3117265128	−0.0557281809	8.746074867
66	208.1287115488	2.3183319956	+0.0553059917	8.742772184
67	211.2703309942	2.3248385127	-0.0548932546	8.739518981
68	214.4119496545	2.3312489858	+0.0544896223	8.736313798
69	217.5535675637	2.3375662093	−0.0540947647	8.733155236
70	220.6951847537	2.3437928575	+0.0537083686	8.730041961
71	223.8368012552	2.3499314908	−0.0533301360	8.726972691
72	226.9784170965	2.3559845630	+0.0529597833	8.723946199
73	230.1200323046	2.3619544264	−0.0525970408	8.720961311
74	233.2616469051	2.3678433375	+0.0522416513	8.718016896
75	236.4032609225	2.3736534628	−0.0518933698	8.715111873
76	239.5448743794	2.3793868825	+0.0515519623	8.712245201
77	242.6864872978	2.3850455956	−0.0512172058	8.709415881
78	245.8280996983	2.3906315240	+0.0508888870	8.706622952
79	248.9697116003	2.3961465162	−0.0505668022	8.703865490
80	252.1113230227	2.4015923515	+0.0502507566	8.701142605

TABLE II. (Continued)

s	$x_1^{(s)}$	$\log x_1^{(s)}$	$J_0[x_1^{(s)}]$	$\log \pm J_0[x_1^{(s)}]$
81	255.2529339830	2.4069707427	−0.0499405637	8.698453440
82	258.3945444983	2.4122833400	+0.0496360453	8.695797172
83	261.5361545844	2.4175317339	−0.0493370302	8.693173004
84	264.6777642566	2.4227174575	+0.0490433548	8.690580170
85	267.8193735296	2.4278419898	−0.0487548620	8.688017931
86	270.9609824173	2.4329067582	+0.0484714011	8.685485573
87	274.1025909327	2.4379131406	−0.0481928275	8.682982407
88	277.2441990888	2.4428624679	+0.0479190024	8.680507768
89	280.3858068975	2.4477560258	−0.0476497924	8.678061013
90	283.5274143702	2.4525950573	+0.0473850692	8.675641520
91	286.6690215183	2.4573807640	−0.0471247098	8.673248689
92	289.8106283521	2.4621143085	+0.0468685953	8.670881938
93	292.9522348815	2.4667968154	−0.0466166118	8.668540705
94	296.0938411167	2.4714293739	+0.0463686494	8.666224446
95	299.2354470668	2.4760130382	−0.0461246021	8.663932633
96	302.3770527404	2.4805488296	+0.0458843682	8.661664756
97	305.5186581464	2.4850377379	−0.0456478493	8.659420320
98	308.6602632927	2.4894807224	+0.0454149506	8.657198846
99	311.8018681874	2.4938787130	−0.0451855807	8.654999867
100	314.9434728378	2.4982326121	+0.0449596513	8.652822935
101	318.0850772513	2.5025432949	−0.0447370774	8.650667609
102	321.2266814346	2.5068116110	+0.0445177767	8.648533467
103	324.3682853946	2.5110383851	−0.0443016697	8.646420095
104	327.5098891378	2.5152244180	+0.0440886797	8.644327093
105	330.6514926702	2.5193704875	−0.0438787324	8.642254073
106	333.7930959978	2.5234773496	+0.0436717561	8.640200656
107	336.9346991265	2.5275457389	−0.0434676814	8.638166475
108	340.0763020616	2.5315763695	+0.0432664410	8.636151173
109	343.2179048084	2.5355699358	−0.0430679701	8.634154402
110	346.3595073722	2.5395271133	+0.0428722055	8.632175826
111	349.5011097577	2.5434485591	−0.0426790865	8.630215115
112	352.6427119699	2.5473349127	+0.0424885539	8.628271950
113	355.7843140132	2.5511867967	−0.0423005506	8.626346020
114	358.9259158923	2.5550048172	+0.0421150209	8.624437021
115	362.0675176112	2.5587895644	−0.0419319113	8.622544658
116	365.2091191742	2.5625416132	+0.0417511694	8.620668644
117	368.3507205852	2.5662615238	−0.0415727448	8.618808700
118	371.4923218481	2.5699498420	+0.0413965883	8.616964550
119	374.6339229664	2.5736071000	−0.0412226523	8.615135931
120	377.7755239442	2.5772338166	+0.0410508905	8.613322583

TABLE II. (Continued)

s	$x_1^{(s)}$	$\log x_1^{(s)}$	$J_0[x_1^{(s)}]$	$\log \pm J_0[x_1^{(s)}]$
121	380.9171247846	2.5808304976	−0.0408812580	8.611524252
122	384.0587254911	2.5843976364	+0.0407137112	8.609740691
123	387.2003260668	2.5879357143	−0.0405482076	8.607971661
124	390.3419265151	2.5914452012	+0.0403847061	8.606216926
125	393.4835268389	2.5949265553	−0.0402231666	8.604476258
126	396.6251270412	2.5983802242	+0.0400635503	8.602749432
127	399.7667271248	2.6018066446	−0.0399058191	8.601036230
128	402.9083270927	2.6052062433	+0.0397499364	8.599336438
129	406.0499269472	2.6085794367	−0.0395958663	8.597649849
130	409.1915266912	2.6119266320	+0.0394435740	8.595976259
131	412.3331263272	2.6152482268	−0.0392930254	8.594315469
132	415.4747258576	2.6185446099	+0.0391441877	8.592667285
133	418.6163252847	2.6218161610	−0.0389970285	8.591031516
134	421.7579246109	2.6250632515	+0.0388515167	8.589407978
135	424.8995238385	2.6282862444	−0.0387076217	8.587796488
136	428.0411229696	2.6314854948	+0.0385653138	8.586196870
137	431.1827220064	2.6346613498	−0.0384245640	8.584608949
138	434.3243209507	2.6378141492	+0.0382853441	8.583032555
139	437.4659198047	2.6409442255	−0.0381476266	8.581467523
140	440.6075185704	2.6440519037	+0.0380113847	8.579913690
141	443.7491172494	2.6471375021	−0.0378765921	8.578370897
142	446.8907158436	2.6502013323	+0.0377432234	8.576838988
143	450.0323143550	2.6532436992	−0.0376112537	8.575317810
144	453.1739127852	2.6562649015	+0.0374806587	8.573807214
145	456.3155111358	2.6592652316	−0.0373514146	8.572307055
146	459.4571094085	2.6622449759	+0.0372234984	8.570817188
147	462.5987076049	2.6652044151	−0.0370968875	8.569337473
148	465.7403057266	2.6681438239	+0.0369715599	8.567867774
149	468.8819037749	2.6710634717	−0.0368474939	8.566407955
150	472.0235017515	2.6739636224	+0.0367246686	8.564957885

Waterman Institute,
Indiana University

FRANK NELSON COLE

BY T. S. FISKE

Frank Nelson Cole, son of Otis Cole and Frances Maria Pond Cole, was born at Ashland, Massachusetts, September 20, 1861. Prepared for college at the Marlborough (Massachusetts) High School, he was admitted to Harvard in September 1878 and at graduation in 1882 stood second in his class. After studying mathematics for a year in the Graduate School of Harvard University, he was appointed to a traveling fellowship and spent the following two years (1883–1885) at the University of Leipsic, where he was a member of the mathematical seminar of Professor Felix Klein.

Returning to Harvard in 1885, he was for two years lecturer in the Graduate School. In 1886 he received from Harvard the degrees of A.M. and Ph.D. In the summer of 1888 he revisited Germany and married Miss Martha Marie Streiff of Hamburg.

In the autumn of 1888 he became instructor in mathematics at the University of Michigan, and a year later was promoted to an assistant professorship. In 1895 he was called to Columbia University as professor of mathematics with assignments to the University Faculty of Pure Science and to the Faculty of Barnard College. For fourteen years (1901–1914) he served on the Committee on Admissions of Barnard College. During three years (1904–1906) he was an examiner in mathematics for the College Entrance Examination Board.

He was elected Secretary of the American Mathematical Society in 1895, became a member of the editorial staff of the Bulletin in 1898, and succeeded the author of this sketch as editor-in-chief of the Bulletin in 1899. In 1920 he relinquished his administrative and editorial labors for the American Mathematical Society. However, he continued his work in the Columbia University Faculty of Pure Science and at Barnard College.

In the spring of 1926 he applied to Columbia University to be released from active service on September 20, 1926, the sixty-fifth anniversary of his birth. Suddenly taken seriously ill he died of heart failure May 26, 1926. He was survived by his wife, two sons, and a daughter.

A picture of Frank Nelson Cole in 1885 after his return from Leipsic is sketched in the following excerpt from an address "Professor Bôcher's Scientific Start in Life" delivered by Professor William Fogg Osgood at a meeting of the Harvard Mathematical Club held February 12, 1925:

> "The other course, Mathematics 13, was given by Mr. (at that time not yet Dr.) Frank Nelson Cole, who had just returned from Germany and was aglow with the enthusiasm which Felix Klein inspired in his students. Cole was not the first to give a formal course of lectures at Harvard on the theory of functions of a complex variable, Professor James Mills Peirce having lectured on this subject in the seventies. That presentation was, however, solely from the Cauchy

standpoint, being founded on the treatise of Briot and Bouquet, *Fonctions Elliptiques*. Cole brought home with him the geometric treatment which Klein had given in his noted Leipsic lectures of the winter of 1881–1882. Cole also gave a course in Modern Higher Algebra, with its applications to geometry. The enthusiasm which he felt for his subject was contagious. Interesting as were the other courses I have mentioned, they stood as the Old over against the New and of the latter Cole was the apostle. The students felt that he had seen a great light. Nearly all the members of the Department,—Professor J. M. Peirce, Dr. B. O. Peirce, and I think, Professor Byerly,—attended his lectures. It was the beginning of a new era in in graduate instruction at Harvard, and mathematics has been taught here in that spirit ever since."

In the publication of scientific papers the most productive period of Professor Cole's life was the seven years (1888–1895) during which he was connected with the University of Michigan. He had previously published three papers, one of them a long critique of Klein's *Ikosaeder*, in the American Journal of Mathematics. While at Michigan he published eight papers and a translation of the work of Eugen Netto on *The Theory of Substitutions*. In the period of thirty-one years during which he was connected with Columbia University he published nine scientific papers, over one hundred reports of meetings of the American Mathematical Society, and innumerable unsigned notes containing items of interest to the mathematical community.

From the time of his arrival in New York in 1895, Cole devoted the larger part of his time, thought, and efforts to the administrative and editorial activities of the American Mathematical Society. His greatest and most absorbing interest in life seemed to be that the affairs of the American Mathematical Society should be conducted wisely, efficiently, and with good taste. He was tireless in his opposition to anything that seemed to him slovenly, unscholarly, or born of an excess of zeal.

In 1920, upon the conclusion of a quarter of a century of service to the Society, some hundreds of his fellow members joined in presenting him with a purse as a token of their admiration and gratitude. This he turned back to the Society, and upon the recommendation of the Council it was used for the purpose of founding a prize to be known as the Frank Nelson Cole prize in Algebra. At the same time in recognition of his distinguished services volume 27 of the Bulletin (the thirtieth volume counting from the establishment of the Bulletin in 1891) was dedicated to him. His portrait appears as the frontispiece of that volume.

At the meeting of the Mathematical Society held September 9, 1926, the following minute was adopted:

"On the recommendation of its Council, the American Mathematical Society, meeting in Columbus, Ohio, on September 8, 1926, expresses its deep sense of loss in the death of Frank Nelson Cole on May 26, 1926. He was for many years the Society's most active executive officer. From an early date in its history until 1920 when he passed his duties on to others, he ably guided the development of the

Society. As Secretary from 1895, and as Editor-in-Chief of the Bulletin from 1899, he led the Society from its modest beginnings to a state of solid accomplishment. He exercised his functions with a skill which excited admiration and which gave the American Mathematical Society an established place in the scientific world. When he retired he turned over to his successors a healthy structure that was able to withstand the stresses of the very difficult post-war period.

The Society wishes now to place on record its grateful recognition of his devoted service to the ideals of American Mathematical Science. His memory will remain an inspiration to all who may in the future serve the interests of the Society and the cause of mathematics in America."

In order to realize how valuable and timely was the service Cole rendered to mathematicians by publishing his translation of Netto's *Theory of Substitution Groups*, one must notice that it was practically the first textbook on the subject in English. European students had easy access to Groups in Serret's *Algèbre Superieure* from the 3rd edition (1866), and in the German edition of Netto's book (1882). Yet no English author had at that date taken up the task of presenting the subject for beginners; if we except seven pages in Chrystal's *Algebra*, and journal papers by Cayley and Bolza. An additional value attaches to Cole's book, in that it gave Netto opportunity to revise, omit, and recast considerable portions of the German text.

A very definite contribution to group theory is his work on simple groups, in two papers, one with Glover's cooperation. These showed only three simple groups of orders between 200 and 661. Burnside and Dickson later carried the upper boundary to 1000.

The papers cited below on triad systems and their groups mark an interesting outgrowth of the Netto translation. Debate arose over a matter left indefinite in that book, whether every triad system must have a transitive group. A survey of the literature showed examples to the contrary, one of the two possible triad systems on thirteen elements having a group of order six. The possibility of enumerating all the triad systems on fifteen elements was then discussed, whereupon Miss Cummings developed the earliest examples of incongruent systems exhibiting congruent groups, and extended the number of known systems to twenty-four. Mr. White followed this with a complete synthesis of all having actual groups bringing the number up to forty-four. Cole meantime had begun, but deferred the completion of, an exhaustive discussion of the structure of such systems, based on "interlacings." When he was shown some novel systems, which admit no group of substitutions save the identity, a race began for results. The two others found a way of deriving new systems from old, and had produced thirty-three of this new sort, before Cole finished his investigation showing about fifty such, in addition to those admitting groups. Miss Cummings, applying to all these her speedy method of comparison by "indices," discovered many duplicates. Cole had not overlooked any that were already known, but had found thirty-six "groupless" systems, the last three being entirely new. This brought the whole list up to eighty; and

Cole's deductive method furnished the climax, the proof that now the list is complete. This was probably the only investigation in which he collaborated with others; and even in this his methods were entirely independent, only the results permitting comparison and control.

The short paper on Kirkman Parades was essentially an extract from this larger work, and served to settle conclusively the "fifteen schoolgirl problem."

List of Professor Cole's Publications

Reports of Meetings of the American Mathematical Society: this Bulletin, vol. 3, p. 1; vol. 4, pp .1, 87, 175, 291, 415; vol. 5, pp. 1, 113, 217, 325, 423; vol. 6, pp. 95, 117, 267, 365; vol. 7, pp. 1, 113, 199, 289; vol. 8, pp. 1, 183, 367; vol. 9, pp. 183, 281, 393, 525; vol. 10, pp. 53, 119, 171, 221, 373, 485; vol. 11, pp. 111, 231, 347, 461; vol. 12, pp. 53, 107, 223, 325, 423; vol. 13, pp. 55, 153, 261, 315, 423; vol. 14, pp. 53, 157, 247, 351, 407; vol. 15, pp. 157, 275, 323, 419; vol. 16, pp. 53, 169, 281, 395, 451; vol. 17, pp. 61, 167, 277, 389, 505; vol. 18, pp. 51, 159, 213, 327, 485; vol. 19, pp. 51, 163, 275, 387, 497; vol. 20, pp. 169, 283, 395, 507; vol. 21, pp. 57, 161, 269, 373, 481; vol. 22, pp. 161, 263, 373, 481; vol. 23, pp. 57, 157, 257, 299, 435; vol. 24, pp. 57, 169, 265, 369, 465; vol. 25, pp. 49, 145, 433; vol. 26, pp. 145, 241, 337, 433; vol. 27, p. 97.

The potential of a shell bounded by confocal ellipsoidal surfaces, Proceedings of the American Academy of Arts and Sciences, vol. 9 (1883), pp. 226–231.

A contribution to the theory of the general equation of the sixth degree, American Journal of Mathematics, vol. 8 (1885–86), pp. 265–286.

Klein's Ikosaeder, American Journal of Mathematics, vol. 9 (1886–87), pp. 45–61.

On rotations in space of four dimensions, American Journal of Mathematics, vol. 12 (1889–1890), pp. 181–210.

The linear functions of a complex variable, Annals of Mathematics, vol. 5, (1889–90), pp. 121–176.

Simple groups from order 201 *to order* 500, American Journal of Mathematics, vol. 14 (1891), pp. 378–388.

Review of Felix Klein, *Vorlesungen über die Theorie der elliptischen Modulfunctionen*, Bulletin of the New York Mathematical Society, vol. 1 (1891–92), p. 105.

Translation of Eugen Netto, *The Theory of Substitutions and its Applications to Algebra*. Ann Arbor, 1892.

Note on the substitution groups of six, seven, and eight letters, Bulletin of the New York Mathematical Society, vol. 2 (1892–93), pp. 184–190.

The transitive substitution groups of nine letters, Bulletin of the New York Mathematical Society, vol. 2 (1892–93), pp. 250–258.

On groups whose orders are products of three prime factors (with J. W. Glover), American Journal of Mathematics, vol. 15 (1893), pp. 191–220.

Simple groups as far as order 600, American Journal of Mathematics, vol. 15 (1893), pp. 303–315.

Review of Charlotte Angas Scott, *An Introductory Account of Certain Modern Ideas in Plane Analytical Geometry*, Bulletin of this Society, vol. 2 (1895–96), pp. 265–271.

On the factoring of large numbers, Bulletin of this Society, vol. 10 (1903–04), pp. 134–137.

The groups of order p^3q^β, Transactions of this Society, vol. 5 (1904), pp. 214–219.

The triad system of thirteen letters, Transactions of this Society, vol. 14 (1913), pp. 1–5.

Note on solvable quintics, This Bulletin, vol. 21 (1914–15), pp. 462–464.

The complete enumeration of triad systems in fifteen elements (with L. D. Cummings and H. S. White), Proceedings of the National Academy, vol. 3 (1917), pp. 197–199.

Complete classification of the triad systems on fifteen elements (with L. D. Cummings and H. S. White), Memoirs of the National Academy, vol. 14, No. 2 (1919); 89 pp.

Kirkman Parades, Bulletin of this Society, vol. 28 (1922), pp. 435–437.

On simple groups of low order. Bulletin of this Society, vol. 30 (1924), 489–492.

HAUSDORFF'S REVISED MENGENLEHRE

Mengenlehre. By Felix Hausdorff. Second edition. Berlin and Leipzig, Walter de Gruyter, 1927. 285 pp.

There are second editions and second editions. Some are merely second printings with misprints corrected and a few pages added to bring the subject matter up to date; others are complete revisions. Hausdorff's second edition is an extreme example of the latter type; here even the title has been revised the first edition having appeared in 1914 under the title *Grundzüge der Mengenlehre.**

Not only has the title of the book been abbreviated in this new edition, but the book itself has been reduced in size so that the number of pages is considerably less than two-thirds that of the first edition. This is due principally to a difference in the literary style of the two editions, the *Grundzüge* seeming extremely verbose when compared with the conciseness of the second edition. It is questionable whether the new edition will be as "teachable" as the old. Certainly it cannot be read with as much ease by a student approaching the subject for the first time, as so much more is left for him to supply for himself. Probably the best approach to the subject will be found to lie in a judicious selection of topics from both editions. We regret that we find in the second edition such a conscious effort on the part of the author to save space.

As in the first edition, there are two main topics considered, the first five chapters being concerned with the general theory of aggregates and the remaining chapters with the more special theory of point sets. In discussing the various chapters we shall give in general only the omissions from and the additions to the material contained in the first edition.

The *Vorbemerkungen* and Chapter 1 (pp. 9–24) contain the essential parts of the first two chapters of the *Grundzüge* in one-third the space required there. Among the topics omitted are symmetric sets and the principle of duality, and much less space is devoted to the "algebra" of sets. Among the topics retained the order is often changed; for example, a section on functions defined over a set is placed before the section on the fundamental ideas of the sum and common part† of two sets. Here the author has replaced the cumbersome $\mathfrak{S}$, $\mathfrak{D}$ notation of the *Grundzüge* by a more common notation: the sum S of two sets A, B is indicated by $S=A+B$, the common part D by $D=AB$. As in the *Grundzüge*, the notation $S=A+B$ is used if and only if $D=0$, thus differing from the usual notation which indicates the sum by $S=A+B$ in all cases.

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* Reviewed by Professor Henry Blumberg in this BULLETIN, vol. 27 (1920–21), pp. 116–129.

† *Durchschnitt*, translated by Blumberg as *section*. The most common expression is *product*, but Hausdorff uses this in a different sense.

Chapter 2 (pp. 25–41) on cardinal numbers and Chapter 3 (pp. 41–55) on order types correspond almost exactly to Chapters 3 and 4 respectively of the *Grundzüge*, although the number of pages has been reduced by half. This is partly accounted for by the omission of certain sections on ordered sets, the topic of ordered sets being given very brief consideration in this edition.

The fourth chapter (pp. 55–77) contains all the material on well-ordered sets and ordinal numbers that has been retained from Chapters 5 and 6 of the *Grundzüge*, where these topics covered five times the space. Practically all of Chapter 6 (partially-ordered sets, etc.) has been omitted, the only section that remains essentially as a unit being the section on general products and powers of ordered sets. The other topics considered are: the well-ordering theorem, the comparability of ordinal numbers, combination of ordinal numbers, and the Alephs.

In the fifth chapter (pp. 77–93) the author begins a comprehensive treatment of Borel and Suslin (or Souslin) sets, to which portions of Chapters 7 and 8 are also devoted. Suslin sets were first studied in 1917 by Michael Suslin (1894–1919), a Russian. These sets are a generalization of the well known Borel sets.

The remainder of the book (see however §40) is devoted to a discussion of sets of points considered as subsets of a metric space, that is, a space in which there exists a definition of the distance between each pair of points, subject to certain distance axioms. This is a departure from the viewpoint of the *Grundzüge*, where Hausdorff begins with a topological space, i.e., a space satisfying certain neighborhood axioms (*Umgebungsaxiome*). Having proved a number of theorems on the basis of this set of axioms, he then specializes the space step by step by adding additional restrictions, obtaining in turn a metric space, a euclidean space of n dimensions, and finally a euclidean plane, with appropriate groups of theorems in each case.

We do not quarrel with the author's decision to omit the discussion of euclidean spaces from the second edition, although we feel that the student loses something if he does not have brought to his attention the special methods applicable in euclidean spaces.

We do regret exceedingly, however, that he has treated only metric spaces when he might so easily have followed the outline of the *Grundzüge*. Instead of doing so, he proceeds as follows: having assumed that the space is metric, he defines (§22) a *neighborhood* of radius ρ of a point x, as the set of points whose distance from x is less than ρ. A system of neighborhoods defined thus will satisfy the *Umgebungsaxiome*. In proving many of the theorems that follow, the author makes use of this special system of neighborhoods, and in his proofs does not use all the properties of a metric space, but only those properties which are necessary to insure that the space contain a system of neighborhoods satisfying the *Umgebungsaxiome*. In other words, a number of theorems which are true in a topological space are proved by Hausdorff only for a metric space, when with a slight rearrangement of material he could have proved these theorems in all their generality. Since the author does not hesitate to assume whenever

necessary that his given metric space is complete, compact, or separable, if one of these hypotheses is needed to prove a particular theorem, it seems strange that he does not take the most general viewpoint throughout this part of the book, and assume merely that the point sets considered are subsets of a *topological* space, assuming it to be also metrical when and only when it is necessary. In this connection, compare §22 and §23 of the second edition with §2 and §3 of Chapter 7 of the first edition.

Chapter 6 (pp. 94–164) on point sets and Chapter 7 (pp. 164–193) on point sets and ordinal numbers correspond to Chapters 7 and 8 of the *Grundzüge*. Dyadic sets and sets of the first and second categories of Baire are given more prominence than in the first edition. A few pages are devoted to the idea of a locally connected (=connected *im kleinen*) set, an idea which Hahn and Mazurkiewicz were developing at the time of the appearance of the first edition. The results of the sixth chapter are obtained without any reference to ordinal numbers, while Chapter 7 is devoted to theorems in which this concept occurs or which are more easily proved by making use of this idea. Here the author shows that the totality of Suslin sets is not identical with the totality of Borel sets, but that there exist Suslin sets which are not Borel sets. In the concluding section, necessary and sufficient conditions are given in order that a Suslin set be a Borel set. In an appendix are given alternative proofs by Lusin of the theorems of this section, which proofs were communicated to the author by letter too late to be put in the proper place in the text. This is a striking proof of the current interest in this topic.

Chapter 8 (pp. 193–232) on correspondences between two spaces and Chapter 9 (pp. 232–275) on real functions comprise some material from Chapter 9 of the *Grundzüge*, but much of the material is new. The author first considers the subject of continuous curves: it is shown that the class of dyadic continua is identical with that of continuous curves; the Sierpinski and the Hahn-Mazurkiewicz characterizations of continuous curves are given. Next the subject of correspondences is taken up, under which are Lavrientieff's theorem on the extension of a continuous (1–1) correspondence, and a convenient table (p. 219) giving the image of various classes of point sets under correspondences of various types. Another section is devoted to the work of Hahn and R. L. Moore on the prime parts of a continuum. In the final section of Chapter 8 (§40), Hausdorff discusses briefly topological spaces, giving various sets of axioms used to distinguish the class of metric spaces from the more general class of topological spaces.

The ninth and concluding chapter gives a much more complete treatment of Baire's functions than was the case in the *Grundzüge*, and concludes with the theorems of Hahn and Sierpinski on the nature of the convergence set of a sequence of real continuous functions.

In addition to the omissions which we have mentioned above, the second edition omits the entire tenth chapter of the *Grundzüge* which treated content and measure of point sets and the applications of this theory to Lebesgue integrals.

The book closes with a list of the literature of the subject, a list of sources, and an index. Unlike the first edition, the list of sources gives

merely the original place of publication of the theorems cited; there are no supplementary remarks. The enlarged list of literature over that given in the first edition testifies to the growing interest in the subject of aggregate theory—an interest which the appearance of the *Grundzüge* did much to arouse and foster. The index is remarkably complete, and is an improvement over that of the first edition in that it contains also references to authors. Another improvement is the consecutive numbering of sections. The work of the publishers is excellent. We have discovered only a few unimportant misprints.

It is not within our province to criticize the author's choice of material. The book aims to be a textbook and not a report on the subject of aggregate theory, and therefore many topics have had to be omitted altogether, and in the case of those topics which have been included a choice of theorems has had to be made. Under such circumstances, the only criterion for the goodness of the author's choice is the taste of the individual reader.

The *Grundzüge* has been out of print for the past four years and we therefore heartily welcome the appearance of this new edition. We wish to state without qualification that this is an indispensable book for all those interested in the theory of aggregates and the allied branches of real variable theory and analysis situs. If we have seemed critical of some phases of the book, it is only because this excellent book is not better.

H. M. Gehman

GENERAL MATHEMATICS IN EUROPE

Leçons de Mathématiques générales. By L. Zoretti. With preface by P. Appell. Second edition. Paris, Gauthier-Villars, 1925. 16+788 pp.

Nouveau Traité de Mathématiques générales. By C. Fabry. Fourth edition. Vol. 2. Paris, Hermann, 1925. 276 pp.

Einführung in die Elemente der höheren Mathematik. By H. Hahn and H. Tietze. Leipzig, Hirzel, 1925. 12+331 pp.

Cours Complet de Mathématiques Spéciales. By J. Haag. Vol. 1, *Algèbre et Analyse.* 1914. 6+402 pp. Vol. 2, *Géométrie.* 1921. 7+662 pp. Vol. 3, *Mécanique.* 1922. 8+192 pp. Paris, Gauthier-Villars.

In the United States the term "general mathematics" has been used recently to denote a course for high school students. Such a course may have many advantages but it presents a troublesome problem to educational authorities when a pupil seeks entrance to college. It is difficult to determine whether certain definite entrance requirements have been met and it is sometimes equally puzzling to know whether the student is qualified to take certain courses in the college curriculum. General mathematics in a high school may consist of a mixture of arithmetic, algebra, geometry, and trigonometry in all sorts of proportions. The purpose usually is to prepare the pupils for practical living and college preparation is not the chief aim of the course.

On the continent of Europe the term "general mathematics" has been used to denote a course corresponding more nearly to those given in

American colleges. The course consists of a mixture of analysis, geometry, and mechanics, the emphasis on different topics varying considerably with the authors. The purpose is to furnish the mathematical equipment required by physicists, chemists, engineers, and other scientists and the training of mathematicians is not the chief aim of the course.

The first edition of the book by Zoretti was well reviewed by Professor J. B. Shaw in this Bulletin (April, 1918, p. 355). The second edition differs from the first only in a few details, notably in an increase of the use of graphical methods and in the omission of the chapter on elliptic functions.

The book by Fabry, of which only the second volume is at hand, is now in its fourth edition, from which we may infer that the book has been successful in meeting the need for which it was designed. This volume includes analysis, mechanics, and theory of errors.

The book by Hahn and Tietze covers much less ground than the others and the point of view is that of the pure mathematician. The four parts into which the book is divided are: 1. Numbers; 2. Functions; 3. Differential Calculus; 4. Integral Calculus. The first two parts give an exposition of the elements of the theory of functions of real variables, furnishing a firm foundation for the introduction to the calculus which follows.

The work by Haag is more extensive than any of the others. It is based on a course given by the author preparing students for entrance to the École Polytechnique and the École Normale Supérieure, but has been extended considerably beyond this course with the idea of giving adequate preparation for advanced work in mathematics. Apparently this accounts for the title "special mathematics." A student of the so-called general mathematics would be expected to omit certain parts.

The complete work consists of four volumes. In addition to the three listed above, volume 4 has been announced with the title "Géométrie descriptive. Trigonométrie." Each volume of text is accompanied by a separate volume of exercises, some of which are worked out while others are left to the student. The author lays great stress upon the exercises as necessary for a complete understanding of the subject matter and has provided a very generous collection. Any student who would follow the author's advice and work through the volumes of exercises after studying the corresponding text, could not fail to be well grounded in the mathematics usually covered in America through the first year of the graduate school.

W. R. Longley

REYMOND ON SCIENCE IN ANTIQUITY

History of the Sciences in Greco-Roman Antiquity. By Arnold Reymond. Translated from the French by Ruth Gheury de Bray, with preface by Léon Brunschvicg. New York, E. P. Dutton. x+246 pp. $2.50.

In his preface to this work M. Brunschvicg calls attention to the fact that Professor Reymond has for many years given lectures on the history of science in the University of Neuchatel. These lectures have been attended both by the students of the Faculté des Lettres and by those of the Faculté des Sciences, and similar provision has of late been made in the University of Lausanne. These facts, while having little to do with the merits of the book under review, are significant as regards the interest being shown in European universities in the history of science as a culture subject.

The work treats of the mathematical, astronomical, physical, and natural sciences, and what will here be said concerning the treatment of the first of these will probably allow for a fair estimate of the treatment of the others.

It is apparent from the first chapter that M. Reymond writes as a philosopher rather than as a scientist or a historian. He has gone to few original sources but has depended chiefly upon Zeuthen, Loria, Tannery, and Ball, with four references to Cantor and two to Heath and a like number to Montucla's work of more than a century and a quarter ago. Ball, who is hidden in the index under "Rouse Ball," is referred to more often than Zeuthen or Loria, while Tropfke is unknown. Even the frequent references to such men as Euclid and Archimedes show that the author's knowledge comes rather from writers like Boyer and Ball than from any study of the classics which these men wrote.

As a result of this dependence upon secondary sources, often of a rather inferior type, the work has many errors that will at once occur to anyone who is at all familiar with the history of mathematics. For example, it is not likely that any historian of the subject would care to subscribe to statements such as these:

"As to the information furnished by hieroglyphics and cuneiforms, it amounts to little." To be sure M. Reymond could assert that the Rhind Papyrus is in hieratic and not hieroglyphic, but the impression given by the statement is unfortunate, particularly in view of what the cuneiform tablets are revealing as to the mathematics of Babylon.

"We are reduced for the most part to conjectures concerning the scientific knowledge of the Egyptians and Chaldeans." On the contrary, we have very positive acquaintance with a considerable range of such knowledge as witness, in the case of Egypt, Professor Archibald's extensive bibliography in Dr. Chace's edition of the Rhind Papyrus, now in press.

"In practice and for reckoning they (the Egyptians and Chaldeans) made use of abacuses the arrangement of which calls to mind the ball-

frame formerly used in infants' schools." Aside from a single reference in Herodotus we have no direct reference to the Egyptian use of the abacus, and that goes back only to the fifth century B.C., more than a thousand years after Ahmes wrote—and as to the Chaldeans we have no direct evidence whatever.

"It would appear also that the Egyptians, as well as the Hindoos, had discovered, before Pythagoras, the relation between the surfaces of squares constructed on the sides of a right-angled triangle." It would be most interesting to have some proof for such a sweeping assertion as this.

These statements are merely typical. In the space allowed to this review it is not feasible to mention others.

As to dates, if the reader places any dependence upon those given, he will be led into difficulties. When M. Reymond states that Thales lived "about 624–528 B.C.," he is on safe ground; that is, he asserts that the dates are merely approximate. On the other hand there is no satisfactory evidence for saying that Pythagoras "died in the year 500 B.C.," that Zeno was "twenty-five years younger than his master" (Parmenides), the dates for neither being known with any degree of precision; that Hippocrates was born in 470 B.C., or Eudoxus in 408 B.C., or Euclid in 330 B.C., and so for various other such positive assertions. One of the most certain dates of this kind in Greek history is that of the birth of Archimedes, but this is here given as 257 B.C., when it should be 287.

There are various other types of error. For example, the assertion that Bryson contributed notably to the method of exhaustion (p. 56) is now considered very uncertain; that "the quadrature of the circle is, as we know, an insoluble geometrical problem" (p. 58) is a misleading assertion unless "geometrical" is defined before this statement is made; that the *Method* of Archimedes was discovered on a palimpsest "of Jerusalem" (pp. 75, 76) instead of "in Constantinople"; that the "writings" of Diophantus "became known" to the scientific world through Regiomontanus (p. 98), especially as M. Reymond himself says later (pp. 126, 127) that "it was only in the year 1575 that he (Diophantus) became known to the western world."

The bibliography will be helpful to students of the history of science in spite of such misprints as *uber* for *über*, "Kultur *de* Gegenwart," and J. H. Heiberg for J. L. Heiberg.

The index contains merely names. It is of no value to anyone wishing to look up topics. Even under names the reader will look in vain for Peet, Rhind, Pistelli, Festa, Friedlein, Hoche, Ilberg, and various others mentioned in the text or bibliography. He will, however, find both men bearing the name of Diocles listed under the same item and, as elsewhere stated, he will look unsuccessfully for Ball unless he happens to think of W. W. Rouse Ball.

On the whole the work has merits as a philosophical dissertation but none as a source of accurate historical information in the field of mathematics.

David Eugene Smith

REVISION OF PASCH'S GEOMETRY

Vorlesungen über neuere Geometrie. By Moritz Pasch. Second edition. With an appendix: *Die Grundlegung der Geometrie in historischer Entwicklung* by Max Dehn. (Die Grundlehren der mathematischen Wissenschaften, Band 23.) Berlin, Springer, 1926. x+275 pp.

In 1882, Pasch's *Vorlesungen über neuere Geometrie* inaugurated a new era in the development of projective geometry. For the first time, projective geometry was made a purely deductive science founded upon and logically developed from a complete system of postulates. At the same time, since the postulates completely ignored the concept of parallelism, it was definitely made independent of euclidean or any metric geometry. This was in accord with a previous discovery by Klein that the theorems of projective geometry are independent of the parallel postulate.

The book was reprinted in 1912 with a short appendix and in 1913 it was translated into Spanish and the appendix of the 1912 edition was distributed over the book as smaller appendices at the ends of different sections.

The second edition just published follows the plan of the Spanish edition. The portion revised is not extensive. The rewritten material totals about two pages in the second edition and takes the place of about five pages of the first edition. It consists almost entirely of simplified proofs. In one place (page 47, second paragraph) a proof that had covered two pages is now given in one short paragraph.

The second edition contains a total of eight pages of added material. This is inserted piecemeal as supplements to the various sections or as isolated paragraphs between two formerly adjacent paragraphs. Some new theorems are added, for example: p. 7, Kernsatz IX; p. 25, Lehrsätze 13, 14, 15; p. 39, theorem by Ventura Reyes y Prosper; p. 76, Sätze J and P. Aside from these, the supplementary material consists mainly of comments and references.

The typographical errors of the first edition are corrected. This was not done in the reprint of 1912.

The evident desire to supplement rather than revise this important pioneer work is best explained by the author himself in his preface to the second edition. After discussing some of the more recent methods and results, he continues, "I do not desire to incorporate these new geometric concepts in the new edition, but prefer to keep the point of view of the first edition. If the reader will allow himself to be carried back in imagination for half a century, he may wish at least to acquaint himself with the geometry of that period and with the position this book had in its time in the foundations of geometry."

As the preferable alternative to enlarging and revising his own book so as to include the new concepts, Pasch requested Dehn to write an appendix covering this field. The last eighty-seven pages of the book

comprise this appendix which, as stated in the preface, corresponds to a two-hour semester course of lectures given by the author.

In the introduction, the development of the investigations in the foundations of geometry is divided into four periods:

1. The parallel postulate of Euclid.
2. The continuity postulate of Archimedes.
3. Projective geometry.
4. The modern period in which the foundations of geometry have been studied as such. This period began with the work of Lobatchewsky in 1826.

The first chapter gives a rapid survey of the attempts to prove the euclidean parallel postulate, the new geometrical concepts that were initiated by these attempts, and especially the development of these concepts from the time of Lobatchewsky to that of Pasch.

The four remaining chapters follow closely in content and method the work of Hilbert contained in his *Grundlagen der Geometrie* (1899) and in a later memoir in the Mathematische Annalen (1903). Many references to Hilbert assign to him the authorship of the major portion of these chapters. More recent authors are also frequently cited, but most of these citations refer to new derivations or extensions of Hilbert's results. Some of these extensions were made by Dehn himself, notably his work in pseudo-geometry and non-Archimedean geometry.

The contributions of certain other geometers, particularly E. H. Moore and Veblen, should have had a place in this supplement. The findings of E. H. Moore that some of Hilbert's axioms are redundant and that Schur's statement concerning the redundance of certain others is incorrect are especially pertinent.

Some interesting new features are: The serviceable table on page 229 shows distinctly the interrelations of various basic theorems of projective geometry. The extended principle of duality is proved (pp. 237–239) to be a direct consequence of the commutative law for the multiplication of line segments. The euclidean parallel postulate and the Archimedean postulate of continuity are shown to have a projective property in common (pp. 247–248).

The appendix is ably written and well worth reading. Only the work of Hilbert is given in detail, but most of the principal European geometers are cited. The author has succeeded remarkably well in presenting such an extensive subject so briefly.

It is peculiarly fitting that these two contributions to the foundations of geometry should be bound in one volume—the classical work of Pasch, reëdited after a half century by himself and the great work of Hilbert, rewritten, extended at times and revised somewhat in methods of presentation and proof by one of his former students who was with him when these concepts were taking form.

T. R. HOLLCROFT

FOUR BOOKS ON VECTOR ANALYSIS

1. *Vectorrechnung.* By Jean Spielrein. 2d edition. Stuttgart, Konrad Wittwer, 1926. xvi+434 pp.
2. *Die Vektoranalysis und ihre Anwendung in der theoretischen Physik*, Teil 1: *Die Vektoranalysis.* Teil 2: *Anwendung der Vektoranalysis in der theoretischen Physik.* By W. v. Ignatowsky. 3d edition. Leipzig and Berlin, Teubner, 1926. x+110 pp.+iv+123 pp.
3. *Die ebene Vectorrechnung und ihre Anwendungen in der Wechselstromtechnik*; Teil 1, *Grundlagen.* By Heinrich Kafka. Leipzig and Berlin, Teubner, 1926. viii+132 pp.
4. *Initiation aux Méthodes Vectorielles et aux Applications Géométriques de l'Analyse.* By G. Bouligand and G. Rabaté. Paris, Librairie Vuibert, 1926. viii+215 pp.

A course in vector analysis is now regarded as a necessary part of the training of any student of physics or of applied mathematics. Indeed no one can intelligently study any branch of mathematics without soon meeting the concept which is at the foundation of the whole subject of vector analysis, namely the concept of invariance under certain specified types of transformations. It is the great merit of the modern treatment of tensor analysis that the allowable transformations are kept as general as possible; for most purposes differentiability and reversibility being sufficient. Looked at from this point of view tensor analysis is a *general* vector analysis in the sense of E. H. Moore. It seems, then, to the present reviewer a little unfortunate that the writers of texts on vector analysis still treat in such detail the special vector analysis where the allowable transformations are those from one set of rectangular Cartesian coordinates to another. No doubt the reason given would be a pedagogic one but very often a view from a higher standpoint gives a real understanding that is almost impossible to secure from a lower level.

The first of the books under review is encyclopaedic in character, and can be highly recommended as a reference work rather than as a text. Some idea of its completeness may be gathered from the fact that appended to the book is a "Formelsammlung" in which are listed no less than 881 of the principal results. A rather severe attack of mental indigestion would threaten an immature student who tried to learn his subject from this book. A very valuable feature of the book, which is carefully printed and bound, is a collection of 190 problems with detailed solutions. The following chapter headings will give an idea of the contents.

Elementare Vektoroperationen. Besondere Vektoren. Bezeichnungen. Funktionen skalarer Veränderlichen. Ortsfunktionen. Geometrie der Vektorfelder. Lineare Vektorfunktionen. Affinoralgebra. Affinoranalysis.

The books 2 and 3 are, respectively, nos. 6 and 22 of the valuable series of monographs, known as the "Sammlung Mathematisch-Physikali-

scher Lehrbücher," which is being published under the general editorship of E. Trefftz. Ignatowsky's work is now in its third edition and is well and favorably known. It is not intended to be nearly as complete as Spielrein's work to which it refers for more advanced parts of the subject. The first edition (1909–1910) was reviewed in this Bulletin (vol. 17, pp. 102–104). The changes in the third edition are of a minor character. For a student interested mainly in physical applications this book is one of the best on the subject. Kafka's book is intended principally for students of electrical engineering. Here consideration is restricted to plane vectors and the subject is merely a geometrical consideration of complex variable theory. The electrical engineer has by this time a pretty definitely established mode of treatment; this is very different, at least in matters of detail, from the established method of the mathematician. The present work should be very useful and easy reading for a student beginning the study of functions of a complex variable.

The book by Bouligand and Rabaté is very different from the three already mentioned, reflecting a national difference of emphasis. Here the interest is almost entirely geometrical and the treatment is clear and elementary. We would recommend the work to a student who contemplates a study of Darboux's monumental work on the theory of surfaces, or of Appell's treatise on mechanics. The following chapter headings give a good idea of the contents.

Vecteurs. Produit scalaire et applications. Produit vectoriel. Théorie des moments; systèmes de vecteurs glissants. La dérivation géométrique et la théorie des courbes. Étude intrinsèque des fonctions de point. Problèmes d'intégration aux point de vue géométrique. Théorie des surfaces.

F. D. MURNAGHAN

SHORTER NOTICES

Die Welt der Atome. By Arthur Haas. Berlin, Walter de Gruyter & Co., 1926. xii+130 pp.

The evolution of our ideas with respect to physical phenomena has been so rapid during the last few years that one who does not devote a fair amount of time to following the development of the subject is likely to feel confused and fail to realize the intimate connection between the new results of experiment. Dr. Haas gave, during the summer of 1926, a series of ten lectures at the University of Vienna, which were intended to give the general public a comprehensive view of the modern physics of the atom, and these lectures have now been published. The author has succeeded admirably in giving a very clear presentation of the modern conception of the atom and of atomic phenomena; and even many who have followed more or less closely the recent developments will find a reading of this book to be both interesting and profitable. In view of the purpose of the author, it is natural that he should have described a very definite model of an atom, and perhaps not laid sufficient emphasis upon the provisional character of the model. But most of the discoveries that have recently been made have resulted from a very definite picture of the atom and so this can hardly be considered a serious fault.

E. P. Adams

Mathematical Investigations in the Theory of Value and Prices. By Irving Fisher. New Haven, Yale University Press, 1926. xii+126 pages.

This interesting little book is a photo-engraved reprint of the memoir by Irving Fisher on the subject published in 1892 and long since out of print. An eight page review of the book was published by Professor Fiske in this Bulletin (vol. 2 (1893), pp. 204–211). The impressions of the present reviewer in relation to the book were well expressed by Professor Fiske. In spite of the fact that certain theorists are at present inclined to the view that the students of economics have very commonly been on the wrong track in their efforts to adopt the methods of the physical sciences, it seems fairly obvious that Professor Fisher's book clarifies certain economic theories by the use of mathematics and physical analogies. Thus, clearness is added to the idea of marginal utility by introducing a unit of measurement for utility and by establishing relations among marginal utility, total utility, utility-value, and gain at a given time.

That the book has been recognized by mathematical economists as an enduring contribution to the theory of value and prices is fairly obvious in the light of the facts that it was translated into French by Jacques Moret twenty-five years after the first publication, and that there existed a demand that it be reprinted a third of a century after its first publication.

H. L. Rietz

The Byzantine Astrolabe at Brescia. By O. M. Dalton. New York, Oxford University Press (American Branch). 14 pp.+3 plates. Price 70 cents.

This monograph was communicated to the British Academy on July 7, 1926, and is now reprinted from the Proceedings. The astrolabe described in the paper has an unusual interest because it is of Byzantine workmanship and dates from a period of which, as regards the science of astronomy on the banks of the Bosphorus, we know but little. The instrument is larger than most of those which have come down to us, being nearly fifteen inches in diameter, but the workmanship is crude in comparison with the neat work of European craftsmen of the Middle Ages or with the Arabic pieces. A special interest, however, lies in the fact that the inscriptions are all in Greek, the instrument being, as the author states, "perhaps the sole survival of an astrolabe representing a Greek type apparently unmodified by foreign ideas." On the arachne are some iambic verses which state that the piece,

> "being an intricate work, with ardent mind fashioned Sergius the Persian, holding a consul's rank";

and another inscription reads

> "Decree and command of Sergius, protospatharios, consul, and man of science [?], in the month of July, fifteenth indiction, year 6570,"

being the year 1062 of our era.

The author prefaces his monograph with a brief statement as to the general nature of an astrolabe and then gives description of the one which is the subject of his researches. The instrument is now in the Museo dell'Età Cristiana at Brescia, having been presented in 1844. The essay is a genuine contribution to our knowledge not only of the history of the astrolabe but of the state of science in the eleventh century.

DAVID EUGENE SMITH

Les Groupes Abéliens Finis et les Modules de Points Entiers. By Albert Chatelet. Paris and Lille, 1925. 221 pp.

As the author states in the introductory chapter, this book was the outgrowth of a much more limited treatment of abelian groups that he had originally planned as the first part of a treatise on abelian algebraic fields. While much more narrow in scope than the existing texts on groups, which treat those of the commutative sort more or less incidentally, it neverthe less embraces considerable material that is not found in such texts and, while following in large part the developments of the subject as given in papers by various authors, appears nevertheless to have considerable novelty in its treatment.

When the elements of an abelian group are represented in all possible ways as products of powers of a suitably chosen set, the set of exponents of such powers may be regarded as the coordinates of points in a certain number of dimensions. The set of points that thus correspond to the identical element of the abelian group form a "module" of a particular type in the same space, i.e., the "sum" or "difference" of any two points

in the set is itself in the set. More generally, any element of the abelian group corresponds to a set of points with integral coordinates that are "congruent" with respect to this module. Following the method employed by Frobenius and Stickelberger in their well known paper on abelian groups, the author therefore develops their properties very largely from the corresponding properties of modules.

The problem of making a change in the "basis" of a module, i.e., a finite number of points in terms of which all the points of the module may be represented by means of integral multipliers, involves naturally the question of the equivalence of matrices of a somewhat more restricted sort than that employed in the usual algebraic theory. The proof of the existence of a "reduced" basis is omitted (rather unfortunately, it seems to the reviewer), reference being made to another text by the same author. The questions of "divisibility," "greatest common divisor," and "least common multiple" of modules and matrices are treated quite fully. The same is true of the subject of the "invariants" of matrices with integral elements, which bears rather a close analogy to the more familiar theory of elementary divisors.

Following the rather extended treatment of modules and matrices in the second chapter, the remainder of the book is devoted to abelian groups. The subsequent chapters vary considerably in interest and importance, the subject of the automorphisms of an abelian group being treated quite comprehensively, while the theory of group characters, being included in the corresponding theory for finite groups in general, seems too special to attract the reader particularly. This would have been unavoidable, however, unless the scope of the book had been considerably widened.

The book should prove valuable as a reference text on the subject of which it treats, although too specialized probably to have a very wide use. It has evidently been very carefully prepared, including as it does a complete index of definitions and a fairly comprehensive bibliography. In the text definitions are emphasized by bold-faced type and theorems are in italics. Some use of numbered formulas might have been an improvement.

H. H. Mitchell

Höhere Mathematik. Teil I. *Differentialrechnung und Grundformeln der Integralrechnung, nebst Anwendungen.* By Rudolf Rothe. 2d edition. Leipzig, B. G. Teubner, 1927. vii+186 pp.

The first edition of this volume was reviewed in this Bulletin in vol. 31 (1925), pp. 566–67. The second edition differs from the first only in unessential details, mainly matters of rewording in a number of places. One error which crept into the first edition might well have been corrected in the second; viz., on p. 116 the author states that the equations $x-e^x\cos y=0$, $y-e^x\sin y=0$ have but one common solution. By sketching the curves it is easily evident that they have an infinity of solutions, which fact one might easily guess by considering the equivalent equation $e^z=z$ in the complex variable.

T. H. Hildebrandt

Cambridge Readings in the Literature of Science. Being Extracts from the Writings of Men of Science to Illustrate the Development of Scientific Thought. Arranged by W. C. D. and M. D. Whetham. Cambridge, The University Press, 1924. x+275 pp.

One of the most difficult things for a student of history to do is to reconstruct the "mental furniture" of other times and civilizations—to comprehend feelings and mental attitudes which were natural products of an environment containing elements and circumstances so foreign to the present-day student as to be entirely unsuspected by him.

It is in this respect that the reading of source material is most helpful. By a study of the exact language of original documents one gains, not only more precise and accurate information, but also something of the spirit and intellectual perspective of the time, which summaries and secondary accounts can not supply.

In the last twenty-five years, especially, these facts have become well recognized and many collections of "Select Documents," "Select Readings from the Original Sources," "Reprints and Translations," etc., have appeared. Most of these collections have been in the field of social sciences. The book under review is a welcome addition to the relatively few which have been published in the field of natural sciences.

Mr. Whetham and his daughter have arranged a selection of extracts from the writings of men of science to illustrate the development of scientific thought along three lines:

(1) the structure of the universe—cosmogony;
(2) the nature of matter—atomic theories;
(3) the development of life—evolution.

Along these lines the authors "try to trace the thoughts from the inspired poetry of the Book of Genesis to the latest revelations of the telescope and the laboratory."

There is a satisfaction in reading Newton's ideas about the *System of the World*, and John Dalton's *On Chemical Synthesis* exactly as they formulated them. Translations of extracts from the writings of Aristotle, Archimedes, Copernicus, Laplace, Lamarck, Bergson, and more than a dozen others, are a bit less satisfying than if given in the original language, but are, of course, much more widely available. The insertion of explanatory and historical material between extracts aids greatly in carrying the thread of the story and these insertions seem to the reviewer to be very well written.

In looking over any such collection one is likely to feel that there are omissions more notable than some of the selections given and it is quite probable that many readers of the book under consideration will have that impression. For example, how many will feel that a worthy attempt at tracing the development of human thought concerning the structure of the universe can afford to give no quotation from the great *Almagest* of Claudius Ptolemaeus, the greatest existing authority on ancient astronomy, or can afford, in tracing the story of the theory of evolution, to omit some quotation from Alfred Russell Wallace's famous essay *On the Tendency*

of Varieties to Depart Indefinitely from the Original Type, which led to the publication of Darwin's preliminary essay in 1858? However, the matter of selection is always a difficult one and largely dependent upon individual judgments.

The value of the book, in the opinion of the reviewer, would have been greatly enhanced by incorporating with each extract a paragraph* indicating various other publications, if such exist, where the writing from which the extract was taken (or the original text in case of a translation) may be found. To quote an extract as "From *The Works of Archimedes*, edited by Sir Thomas Heath," with no mention of publishers or date or place of publication or reference as to where the original Greek text may be found, and no listing of cross-references to other publications containing the same writing, offers little help and encouragement to one who may be interested in investigating the subject more fully.

U. G. Mitchell

Les Nouveaux Axiomes de l'Electronique. By R. Ferrier. Paris, A. Blanchard, 1925. 61 pp.

This is an attempt to modify the laws of electrodynamics to fit the quantum theory of Bohr. The author considers two point charges and puts the energy of the system equal to the quotient of a function of the relative velocity by the distance between the charges. This leads to a fundamental dynamical frequency proportional to the energy instead of proportional to the three-halves power of the energy as in the classical dynamics. While the potential energy on the classical theory is inversely proportional to the distance between charges, it is difficult to see how any theory which hoped to obtain macroscopic results at all comparable with experiment can proceed from an expression for the total energy of the form used in this monograph.

Leigh Page

The Quantum Theory. By N. M. Bligh. New York, Longmans, Green & Co., 1926. 112 pp. $3.00.

Beginning with the general problem of radiation, the author has given a very brief sketch of the development of the quantum theory leading to Planck's formula. He next introduces the light-quantum hypothesis and its application to the photo-electric effect. The subject of atomic heats is briefly considered, and finally Bohr's theory of spectra and atomic constitution is outlined. The treatment throughout is too little critical and too superficial to be of much value, but it may serve a useful purpose as an outline, with references to the original sources, of the older form of the quantum theory.

E. P. Adams

* Such, for example, as the excellent bibliographical paragraphs in William MacDonald's *Select Documents Illustrative of American History*, New York, The Macmillan Company, 1924.

Traité de Physique. Tome supplémentaire: *La Physique de* 1914 *à* 1926, première partie. By O. D. Chwolson. Translated by A. Corvisy. Paris, Hermann, 1927. 339 pp.

Investigations on the Theory of the Brownian Movement. By Albert Einstein. Translated by A. D. Cowper. New York, Dutton. viii+124 pp.

Three Lectures on Atomic Physics. By A. Sommerfeld. Translated by H. L. Brose. New York, Dutton. 70 pp.

The first of these three volumes (translated from the Russian) contains a rather elementary account of the progress in physics from 1914 to 1926. It is divided into nine chapters as follows: the charge and the mass of the electron (chapter I), theory of quanta (II), the structure of the atom (III and IV), line spectra (V), X-rays (VI), band spectra (VII), ultra-violet and infra-red rays (VIII), excitation and ionization of gases by electrons (IX). The treatment is mainly physical in character, and the mathematical analysis is reduced to a minimum. Each chapter is followed by a bibliography indicating the sources of the material and giving references to further related matters.

During the period from 1905 to 1911 Einstein set forth the results of his investigations on the theory of the Brownian movement. Concerning this work of Einstein, F. A. Lindemann (in his article on Einstein, Enc. Brit., 13th edition) says: "One of his [Einstein's] earliest publications gave the complete theory and formulae of the phenomenon known as Brownian motion, which had puzzled physicists for nearly 80 years. He showed that the heat motion of particles, which is too small to be perceptible when these particles are large, and which can not be observed in molecules since these themselves are too small, must be perceptible when the particles are just large enough to be visible, and gave complete equations which enable the masses themselves to be deduced from the motions of these particles." The second volume in the foregoing list contains five of Einstein's most important papers on the Brownian movement together with a substantial appendix of notes by R. Fürth, containing corrections, later references, and some expansions of the original text. The book will serve as an introduction to the study of the Brownian movement.

In his London University lectures on atomic physics (third volume in the foregoing list), Sommerfeld undertakes to set forth his "present opinions" on current problems of atomic physics. He deals mainly with recent developments of the quantum theory. In the first lecture he presents among other things a new theory of the hydrogen atom according to which it has a structure analogous to that of the alkali metals. The second lecture is entitled "The general system of the complex terms." In the third lecture the author deals with the periodic system of the elements and the general problem of chemical linkage. The lectures are of interest to students of physics rather than of mathematics.

R. D. Carmichael

NOTES

The April, 1927, number of the American Journal of Mathematics (volume 49, No. 2) contains the following papers: *A class of functions harmonic within the sphere*, by H. E. Bray and G. C. Evans; *Notes on formal modular protomorphs*, by O. C. Hazlett; *Irreducible continuous curves*, by H. M. Gehman; *The plane quintic with five cusps*, by M. Lehr; *The convergence of general means and the invariance of form of certain frequency functions*, by E. L. Dodd; *On the interpolatory properties of a linear combination of continuous functions*, by D. V. Widder; *Note on Tschebycheff approximation*, by D. V. Widder; *Generalization of Waring's theorem on fourth, sixth, and eighth powers*, by L. E. Dickson; *On complete systems of irrational invariants of associated point sets*, by C. M. Huber; *Regular maps and their groups*, by H. R. Brahana; *The groups belonging to a linear associative algebra*, by A. Ranum.

The July, 1927, number of the Annals of Mathematics (series 2, volume 28, No. 3) contains: *Sextic surfaces with a double septimic curve*, by B. C. Wong; *Determination of a basis for the integral elements of certain generalized quaternion algebras*, by M. D. Darkow; *Expansions in Bessel functions*, by M. H. Stone; *On the Green's function for systems of differential equations*, by W. M. Whyburn; *On Lamé families of surfaces*, by C. E. Weatherburn; *Derivation of the Fredholm theory from a differential equation of infinite order*, by H. T. Davis; *Asymptotic expression for the probability of trials connected in a chain*, by E. D. Pepper; *On certain indefinite quaternary forms representing all integers*, by C. G. Latimer; *Application of algebraic number theory to congruences involving binomial coefficients*, by H. S. Vandiver; *Ternary quadratic forms and congruences*, by L. E. Dickson; *Correspondences between algebraic curves*, by S. Lefschetz; *On the existence and properties of the solutions of a certain differential equation of the second order*, by T. H. Gronwall; *A new definition of almost periodic functions*, by N. Wiener; *A theory of integers, in relation to the iteration of algebraic functions*, by O. E. Glenn; *On systems of total differential equations*, by J. M. Thomas; *On positive solutions of a system of linear equations*, by L. L. Dines; *On completely signed sets of functions*, by L. L. Dines; *Concerning continuous curves and correspondences*, by W. L. Ayres.

The British Association for the Advancement of Science met at Leeds, August 31–September 7, 1927. Professor E. T. Whittaker presided over Section A (mathematical and physical sciences); his address was entitled *The outstanding problems of relativity*.

Cambridge University has awarded its Mayhew prize in applied mathematics to J. Hargreaves, of Clare College, and its Ricardo prize in thermodynamics to J. N. Goodier, of Downing College.

The James Scott prize of the Royal Society of Edinburgh, for the period 1922–1926, "for a lecture or essay on the fundamental concepts of natural philosophy," has been awarded to Sir Joseph Larmor.

Professors A. Einstein, of Berlin, O. Hölder, of Leipzig, F. Schur, of Breslau, and E. Study, of Bonn, have been elected corresponding members of the Bavarian Academy of Sciences.

Professor P. Koebe, of Leipzig, has been elected a member of the Saxon Academy of Sciences.

Dr. Fridjof Nansen, professor of oceanography at the University of Oslo, has been elected a corresponding member of the Prussian Academy of Sciences in the section of mathematical physics.

The Technical School at Darmstadt has conferred an honorary doctorate of engineering on Professor G. Scheffers, of the Berlin Technical School, on the occasion of his sixtieth birthday.

Professor Charles Fabry, director of the institute of optics of the University of Paris, has been elected a member of the Paris Academy of Sciences in the section of physics, as successor to the late Daniel Berthelot.

Professors N. Bohr and A. Einstein have been elected foreign honorary fellows of the Royal Society of Edinburgh.

On the occasion of the meeting of the British Association at Leeds, the University of Leeds conferred an honorary doctorate of science on Professor R. A. Millikan, of the California Institute of Technology.

Professor G. H. Hardy, of Oxford University, has been elected foreign associate of the National Academy of Sciences.

Professor J. S. Ames, of Johns Hopkins University, has been elected chairman of the National Advisory Committee on Aeronautics.

Professor F. R. Moulton has been elected a member of the executive board of the National Research Council.

Dr. Irwin Schrödinger, of the University of Zurich, has been called to the University of Berlin, as successor to Professor Max Planck.

The following have been admitted as private docents in German universities: F. Lettenmeyer, at Munich; B. L. van der Waerden, at Göttingen.

M. Bachelier, of the University of Rennes, has been appointed professor of the differential and integral calculus at the University of Besançon.

Dr. T. M. MacRobert, of the University of Glasgow, has been promoted to a professorship of mathematics.

Mr. L. A. Pars, of Jesus College, and Mr. H. A. Newman, of St. John's College, have been elected university lecturers in mathematics at Cambridge.

Mr. H. E. Arnold has been promoted to an assistant professorship at Wesleyan University.

Dr. Walter Bartke has been appointed assistant professor in mathematical astronomy at the University of Chicago.

Dr. R. W. Barnard has been appointed assistant professor of mathematics at the University of Chicago.

Dr. H. Y. Benedict, professor of applied mathematics and dean of the College of Arts and Sciences at the University of Texas, has been elected president of the University.

Mr. A. H. Blue, of the University of Iowa, has been elected professor of mathematics at West Union College.

Dr. C. C. Camp, of the University of Illinois, has been appointed associate professor of mathematics at the University of Nebraska.

Professor D. F. Campbell has resigned, and associate professor C. I. Palmer has been made professor of mathematics at the Armour Institute of Technology. Professor Palmer is also acting dean of students.

Mr. W. B. Campbell, of Cornell University, has been appointed assistant professor of mathematics at Colgate University.

Associate Professor Arnold Emch has been appointed professor of mathematics at the University of Illinois.

Assistant Professor W. L. Crum, of Harvard University, has been appointed professor of statistics in the Graduate School of Business, Stanford University.

Associate Professor H. J. Ettlinger has been appointed professor of mathematics at the University of Texas.

Professor H. S. Everett, of Bucknell University, has been appointed extension professor of mathematics at the University of Chicago.

Assistant Professor M. C. Foster, of Williams College, has been appointed associate professor of mathematics at Wesleyan University.

Assistant Professor W. V. N. Garretson, of Rutgers University, has been appointed head of the department of mathematics at Ouachita College, Arkadelphia, Arkansas.

Assistant Professor Jewell C. Hughes, of the University of Arkansas, has been appointed associate professor of mathematics there.

Dr. C. H. Langford, of Harvard University, has been appointed assistant professor of philosophy at the University of Washington.

Mr. W. T. MacCreadie, of Cornell University, has been appointed assistant professor of mathematics at Bucknell University.

Dr. L. H. MacFarlan has been appointed assistant professor of mathematics at the University of Washington.

Dr. Florence M. Mears, of Cornell University, has been appointed professor of mathematics at Alabama College.

Dr. E. B. Miller, of the University of Wisconsin, has been appointed professor of mathematics at Illinois College, Jacksonville.

Mr. H. C. Schaub, of Cornell University, has been appointed assistant professor of mathematics at Washington and Jefferson College.

Dr. Marion E. Stark, of Wellesley College, has been appointed assistant professor of mathematics there.

Dr. J. M. Thomas has been appointed assistant professor of mathematics at the University of Pennsylvania.

Dr. V. A. Tam has been appointed professor of mathematics at the University of the Phillipines.

Dr. W. J. Trjitzinsky has been appointed assistant professor of mathematics at the University of Valparaiso, Valparaiso, Indiana.

Assistant Professor L. A. H. Warren, of the University of Manitoba, has been appointed full professor of mathematics there.

Dr. Louis Weisner has been appointed assistant professor of mathematics at Hunter College.

Mr. P. D. Wilkins of Case School of Applied Science has been appointed assistant professor of mathematics at Bates College.

Dr. W. H. Wilson, of the University of Iowa, has been appointed associate professor of mathematics at the University of Florida.

The following appointments to instructorships in mathematics are announced:

University of Akron, Ohio, Mr. Samuel Silberfarb;

University of Arkansas, Mr. C. R. Worth;

University of Chicago, Dr. F. R. Bamforth;

Cornell University, Messrs. E. H. Hadlock, R. L. Jeffery, P. W. Swingle, C. C. Torrance, and F. G. Williams;

University of Florida, Mr. C. A. Messick;

Hunter College, Miss Mary V. Kenny, Dr. Louis Weisner;

Illinois State Teachers College, Miss Gladys H. Freeman;

Johns Hopkins University, Mr. Howard Pixley;

Northwestern University, Dr. Lois W. Griffiths;

Oklahoma Agricultural and Mechanical College, Mr. H. C. Billings;

Pennsylvaina State College, Dr. Marguerite Darkow, and Miss Marie Johnson;

University of Rochester, Dr. Raymond Garver;

Rutgers College, Messrs. M. G. Boyce and C. R. Wilson:

Stanford University, Miss Rosa L. Jackson;

Wells College, Miss Ethel I. Moody;

Junior College at Jefferson City, Miss Frances Baker.

Professor G. Mittag-Leffler, of the University of Stockholm, founder and editor of Acta Mathematica, died July 12, 1927, at the age of eighty-one.

Dr. Albert Gockel, professor of cosmic physics at the University of Freiburg, Switzerland, died March 4, 1927.

Dr. Anton Wassmuth, professor of mathematics at the University of Graz, has died, at the age of eighty-two.

Mr. A. C. Bose, controller of examinations at Calcutta University, died December 11, 1926. Mr. Bose had been a member of the American Mathematical Society since 1916.

Dr. William Burnside, formerly professor of mathematics at the Royal Naval College, Greenwich, died August 21, 1927, at the age of seventy-five.

Dr. I. B. Crandall, of the Bell Telephone Laboratories, died April 22, 1927, at the age of thirty-six.

Dr. G. A. Hill, senior astronomer of the Naval Observatory, died August 29, 1927, at the age of sixty-nine.

Professor H. D. Thompson, of Princeton University, has died, at the age of sixty-three. Professor Thompson had been a member of the American Mathematical Society since 1897.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

ARISTOTLE. The works of Aristotle. Translated into English under the editorship of W. D. Ross. Volume 7: Problemata. By E. S. Foster. Oxford, Clarendon Press, 1927. 9+384 pp.

BARRAU (J. A.). Analytische meetkunde. Tweede deel: De ruimte. Groningen, Noordhoff, 1927.

BEHMANN (H.). Mathematik und Logik. (Mathematisch-Physikalische Bibliothek.) Leipzig, Teubner, 1927. 59 pp.

BERTI (G.). Teoria dei logaritmali. Bologna, Coop. Tip. Azzognidi, 1926.

BOUCHER (M.). Essai sur l'hyperespace. Le temps, la matière, et l'énergie. 3e édition. Paris, Gauthier-Villars, 1927. 264 pp.

BRUN (V.). See NETTO (E.).

CAREY (F. S.) and GRACE (S. F.). Four-place mathematical tables with forced decimals. London, Longmans, 1927. 40 pp.

CASSINIS (G.). Calcolo numerici, grafici e meccanici. Fascicolo I. Pisa, Mariotti-Pacini, 1927. 6+280 pp.

COURANT (R.). Vorlesungen über Differential- und Integralrechnung. Band 1: Funktionen einer Veränderlichen. Berlin, Springer, 1927. 14+410 pp.

DECKERT (A.) und ROTHER (E). Mathematische Hilfsmittel für Techniker. Eine Sammlung von Formeln und anderen Gesetzmässigkeiten der analytischen Geometrie. Halle, Ziemsen Verlag, 1927.

DIMIER (L.). La vie raisonnable de Descartes. Paris, Plon, 1926.

DOWSETT (J. F.). Advanced constructive geometry. London and New York, Oxford University Press, 1927. 8+340 pp.

DURELL (C. V.). A concise geometrical conics. London, Macmillan, 1927. 16+99 pp.

EMGE (C. A.). Hegels Logik und die Gegenwart. Karlsruhe, Braun, 1927. 4+68 pp.

EMMING (A.). Eine Umwälzung in der Mathematik und ihren Anwendungen. München, Pflaum, 1927. 76 pp.

ENRIQUES (F.). Questions relatives aux mathématiques élémentaires. I: L'évolution des idées géométriques dans la pensée grecque. Point, ligne, surface. Paris, Gauthier-Villars, 1927.

FETTWEIS (E.). Das Rechnen der Naturvölker. Leipzig, Teubner, 1927. 96 pp.

FOSTER (E. S.). See ARISTOTLE.

GODEAUX (L.). Cours de géométrie projective. Liége, Imprimerie A. Pholien, 1927. 219 pp.

—— Les transformations birationnelles du plan. (Mémoriel des Sciences Mathématiques, No. 22.) Paris, Gauthier-Villars, 1927.

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GREENSTREET (W. J.). Isaac Newton, 1642–1727. A memorial volume edited for the Mathematical Association. London, Bell, 1927. 8+181 pp.

HAALMEIJER (B. P.) en SCHOGT (J. H.). Inleiding tot de leer der verzamelingen. Groningen, Noordhoff, 1927.

HOFMANN (L.). Über einige spezielle Strahlenkongruenzen die mit analytischen Funktionen zusammenhängen. (Dissertation, Zürich.) Zürich, 1927. 68 pp.

HUDSON (H. P.). Cremona transformations in plane and space. Cambridge, University Press, 1927. 20+454 pp.

INCE (E. L.). Ordinary differential equations. London, Longmans, 1927. 8+558 pp.

JULIA (G.). Eléments de géométrie infinitésimale. Paris, Gauthier-Villars, 1927. 6+242 pp.

KLIEM (F.) und WOLFF (G.). Archimedes. Berlin, Salle, 1927. 143 pp.

LECAT (M.). Coup d'oeil sur la théorie des déterminants supérieurs dans son etat actuel. Bruxelles, Lamertin, 1927. 8+100 pp.

LEMAIRE (J.). Etude élémentaire de l'hyperbole équilatere et de quelques courbes dérivées. Paris, Vuibert, 1927. 172 pp.

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NETTO (E.). Lehrbuch der Combinatorik. 2te Auflage erweitert und mit Anmerkungen versehen von V. Brun und T. Skolem. Leipzig, Teubner, 1927. 8+341 pp.

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REEVE (W. D.). See SMITH (D. E.).

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ROTHER (E.). See DECKERT (A.).

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SCHOGT (J. H.). See HAALMEIJER (B. P.).

SCHOY (C.). Die trigonometrischen Lehren des persischen Astronomen Abu'l-Reihân Muh. Ibn Ahmad Al-Bîrûni. Hannover, Heinz Lafaire, 1297. 8+108 pp.

SKOLEM (T.). See NETTO (E.).

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SPAIER (A.). La pensée et la quantité. Paris, Alcan, 1927. 408 pp.

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TANNERY (P.). Mémoires scientifiques. Tome 8: Philosophie moderne, 1876–1903. Toulouse, Privat, et Paris, Gauthier-Villars, 1927. 15+413 pp.

VER EECKE (P.). Les spheriques de Théodose de Tripoli. Bruges, Desclée, de Brouwer et Cie., 1927.

WOLFF (G.). See KLIEM (F.).

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Bateman (H.). See Lorentz (H. A.).

Baur (C.). Die Elektrizität als Aetherströmung. Versuch einer Mechanik der Elektrizität. Halle, Ziemsen Verlag, 1927. 92 pp.

Biggs (H. F.). Wave mechanics. An introductory sketch. London, Oxford University Press, 1927. 77 pp.

Bilau (K.). Die Windkraft in Theorie und Praxis. Gemeinverständliche Aerodynamik. Berlin, Parey, 1927. 158 pp.

Born (M.). The mechanics of the atom. Translated by J. W. Fisher and revised by D. R. Hartree. London, Bell, 1927.

Boutaric (A.). La chaleur et le froid. Paris, Flammarion, 1927. 279 pp.

Boyle (R.). Electricity and magnetism. 1675–1676. (Old Ashmolean Reprints, 7.) Oxford, R. T. Gunther, 1927. 4+38+3+20 pp.

Bridgman (P. W.). The logic of modern physics. New York, Macmillan, 1927. 14+228 pp.

Carnap (R.). Physikalische Begriffsbildung. Karlsruhe, Braun, 1926.

Carot (P.). See Riéger (J.).

Charbonnier (P.). Traité de balistique extérieure. Tome II: Balistique extérieure rationnelle. Les théories balistiques. Paris, Gaston Doin, et Gauthier-Villars, 1927. 797 pp.

Chwolson (O. D.). Die Physik 1914-1926. Übersetzt von G. Kluge. Braunschweig, Vieweg, 1927. 9+696 pp.

Darwin (C. G.). Recent developments in atomic theory. London, Oxford University Press, 1927. 15 pp.

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Fellenius (W.). Erdstatische Berechungen mit Reibung und Kohäsion (Adhäsion) und unter Annahme kreiszylinderscher Gleitflächen. Berlin, Wilhelm Ernst Sohn, 1927. 4+40 pp.

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Gifford (—.). Lens computing by trigonometrical trace. London, Macmillan, 1927. 81 pp.

Grassmann (R.). Geometrie und Massbestimmung der Kulissensteuerungen. 2te Auflage. Berlin, Springer, 1927.

Haas (A.). Atomic theory. An elementary exposition. Translated by T. Verschoyle. London, Constable, 1927. 14+222 pp.

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von Hevesy (G.). Die seltenen Erden vom Standpunkte des Atombaues. Berlin, Springer, 1927. 140 pp.

Hotopp (L.). See Kaufmann (W.).

Hund (F.). Linienspektren und periodisches System der Elemente. (Struktur der Materie, IV.) Berlin, Springer, 1927. 6+220 pp.

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Kaufmann (W.). Vorträge über Mechanik als Grundlage für das Bau- und Maschinenwesen. Teil 1. Als 8te Auflage des gleichnamigen Lehrbuchs von W. Keck und L. Hotopp. Hannover, Helwingsche Verlagsbuchhandlung, 1927. 11+632 pp.

Keck (W.). See Kaufmann (W.).

Kemp (P.). See Maycock (W. P.).

Kluge (G.). See Chwolson (O. D.).

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Le Besnerais (M.). Théorie du navire. Tome 2. Paris, Armand Colin, 1927. 167 pp.

Lipsius (F. R.). Wahrheit und Irrtum in der Relativitätstheorie. Tübingen, Mohr, 1927. 7+154 pp.

Loewe (H.). Theorie des Wechselstromes in Einzeldarstellungen. Teil 1. Leipzig, Hachmeister, 1927. 78 pp.

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On page 631 change Edmund B. Wilson, in line 23, to Edwin B. Wilson.

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